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STRONG CONVERSE RESULT FOR BASKAKOV OPERATOR

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Abstract. We give a strong converse inequality of type A in terms of K-functional $K_\psi(f,t)$ for classical Baskakov operator. In order to establish it, we use the iterated Baskakov operator and, in particular, we prove the Voronovskaya and Bernstein-type inequalities for it.

1. Introduction. The approximation by positive linear operators has been a widely discussed topic. To deal with this operators, it is useful to employ K-functionals, which in many cases are equivalent to the appropriate modulus of smoothness.

For the most important operators the direct theorems can be found in [3]. For the converse theorems, Ditzian and Ivanov suggested in [2] a classification and defined four types of strong converse inequalities. The strongest are the inequalities of type A.

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For the Bernstein operator (see [6]), given by

$$B_n(f,x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k},$$

where $x\in[0,1]$ and f is a continuous on $[0,1]$ function, they proved a strong converse inequality of type B [2, Theorem 8.1] or more precisely, that there exist constants C and $R\geq 19$ such that for $n\geq 12$ and $k\geq Rn$

$$K_\varphi^2\left(f,\frac{1}{n}\right)\leq C\frac{k}{n}\Big(\|B_n(f)-f\|+\|B_k(f)-f\|\Big),$$

where $\varphi^2(x)=x(1-x)$, $\|f\|=\sup\limits_{x\in[0,1]}|f(x)|$, and

$$K_\varphi^r(f,t^r)=\inf_{g\in C^2[0,1]}\Big\{\|f-g\|+t^r\|\varphi^r g^{(r)}\|\Big\}.$$

Totik [8] extended this result to a large family of operators. Guo and Qi [4] gave a strong converse inequality of type B using a K-functional for Baskakov operators. Regarding strong converse inequalities of type A for the Bernstein operator, Totik [7] and independently Knoop and Zhou [5] proved that there exists an absolute constant C, such that

$$K_\varphi^2\left(f,\frac{1}{n}\right)\leq C\|B_n(f)-f\|.$$

Combining this result with the direct theorem we have a full equivalence $K_\varphi^2\left(f,\frac{1}{n}\right)\sim\|B_n(f)-f\|$, i.e. there exist absolute constants C^*,C_* such that

$$C_*\|B_n(f)-f\|\leq K_\varphi^2\left(f,\frac{1}{n}\right)\leq C^*\|B_n(f)-f\|.$$

Also, in [7] Totik proved the strong converse inequality of type A for the Szasz-Mirakjan operator and stated that the method could be used to prove the strong converse inequality of type A for the Baskakov operator [7, Theorem 3.2]. But, to our knowledge, the strong converse inequality of type A for the Baskakov operator has not been proven yet, and we were unable to prove it by the method of [7].

Our goal in this paper is establishing in Theorem 1.1 an equivalence theorem about classical Baskakov operator, similar to the ones for the Bernstein and Szasz-Mirakjan operators.

For functions $f \in C[0;\infty)$ the Baskakov operator is given by (see [1])

$$V_n f(x) = (V_n f, x) = V_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) V_{n,k}(x) \quad \text{for} \quad 0 \le x < \infty, \tag{1.1}$$

where

$$V_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}. \tag{1.2}$$

We formulate our main result as Theorem 1.1. It consists of a direct inequality (the first inequality in Theorem 1.1) and a strong converse inequality of type A in the terminology in [2] (the second inequality in Theorem 1.1).

Before stating it we will introduce some notations. The first derivative operator is denoted by $D = \dfrac{d}{dx}$. Thus, $Dg(x) = g'(x)$ and $D^2 g(x) = g''(x)$. By $\psi^2(x) = x(1+x)$ we denote the weight which is naturally connected with the second derivative of Baskakov operator (1.1).

By $C[0,\infty)$ we denote the space of all continuous on $[0,\infty)$ functions, by $L_\infty[0,\infty)$ the space of all Lebesgue measurable and essentially bounded in $[0,\infty)$ functions equipped with the uniform norm $\|.\|$ and by

$$CB[0,\infty) = C[0,\infty) \cap L_\infty[0,\infty)$$

the space of all continuous and bounded in $[0,\infty)$ functions.

Also, we define

$$W^r_\infty(\psi) = \{g : D^{r-1} g \in AC_{\text{loc}}(0,\infty) \quad \text{and} \quad \psi^r D^r g \in L_\infty[0,\infty)\}$$

where $AC_{\text{loc}}(0,\infty)$ consists of the functions which are absolutely continuous in $[a,b]$ for every interval $[a,b] \subset (0,\infty)$.

Finally, we denote $\mathcal{F} = CB[0,\infty) + W^2_\infty(\psi)$, i.e. all functions f which can be represented as $f = f_1 + f_2$ where $f_1 \in CB[0,\infty)$ and $f_2 \in W^2_\infty(\psi)$.

To estimate the approximation $f \approx V_n(f)$ we will use the K-functional, defined by

$$K_\psi(f,t) = \inf\left\{\|f-g\| + t\|\psi^2 D^2 g\| : g \in W^2_\infty(\psi), f - g \in CB[0,\infty)\right\}$$

for every function $f \in \mathcal{F}$.

Our main result is the following theorem, establishing a full equivalence between the K-functional $K_\psi\left(f, \dfrac{1}{n}\right)$ and $\|V_n f - f\|$.

Theorem 1.1. *For V_n defined by* (1.1) *there exist absolute constants $L, C_1, C_2 > 0$ such that for every $n > L$*

$$C_1\|V_n f - f\| \le K_\psi\left(f, \frac{1}{n}\right) \le C_2\|V_n f - f\|$$

holds for all $f \in \mathcal{F}$.

The left inequality can be found in [3, Theorem 9.3.2, page 117]. We will prove the right inequality in Section 3.

Although Theorem 1.1 is formulated and proved for integer n it also holds true if n is assumed to be a continuous positive parameter. In this case

$$V_{n,k}(x) = \frac{\Gamma(n+k)}{k!\Gamma(n)} x^k (1+x)^{-n-k},$$

where Γ stands for the Gamma function, V_n is defined again by (1.1).

The paper is organized as follows. Some auxiliary results are proved in Section 2. We prove the main result in Section 3. In order to prove it, we will need an estimation, formulated as Theorem 2.2, which is proved in Section 4. And the proof of Theorem 2.2 is based on two lemmas, which are proved in Sections 5 and 6.

For the rest of this paper we will use some notations and conventions. The constants $C, C_1, C_2, \ldots$ will always be absolute constants which in this case means that they do not depend on n, N and the function f. They are not the best possible constants by any means. Also, the indexed constants $C_1, C_2, \ldots$ will always be the same while the constant C may be different on each occurrence. By A, B we will denote expressions defined in the proofs of some statements with no meaning outside them.

2. Auxiliary results. We will mention some properties of Baskakov operator, which can be found in [1]

$$V_n \quad \text{is a linear, positive operator with} \quad \|V_n f\| \le \|f\| \tag{2.1}$$

$$V_n(1,x) = 1, \qquad V_n\big((t-x),x\big) = 0, \qquad V_n\big((t-x)^2,x\big) = \frac{1}{n}\,\psi^2(x). \tag{2.2}$$

For $k \ge 0$ we have the next easily verified identities [3]

$$(DV_{n,k})(x) = n\Big(V_{n+1,k-1}(x) - V_{n+1,k}(x)\Big), \tag{2.3}$$

$$(DV_{n,k})(x) = \frac{n}{\psi^2(x)}\left(\frac{k}{n} - x\right) V_{n,k}(x) \tag{2.4}$$

where $V_{n,-1}(x) = 0$.

We will need the second derivative of Baskakov operator as well.

For $f \in CB[0,\infty)$ we have

$$(D^2V_nf)(x) = n(n+1)\sum_{k=0}^{\infty}\Delta^2_{\frac{1}{n}}f\left(\frac{k}{n}\right)V_{n+2,k}(x),$$

where, as usual,

$$\Delta^r_h f\left(\frac{k}{n}\right) = \sum_{k=0}^{r}(-1)^k\binom{r}{k}f\big(x+(r-k)h\big).$$

D^2V_n is well defined because

$$\sum_{k=0}^{\infty}\Delta^2_{\frac{1}{n}}f\left(\frac{k}{n}\right)V_{n+2,k}(x) \le 4\|f\|\sum_{k=0}^{\infty}V_{n+2,k}(x) = 4\|f\| < \infty.$$

For $f \in W^2_\infty(\psi)$ we will use the representation

$$(D^2V_nf)(x) = \sum_{k=0}^{\infty}n(n+1)\int_0^{1/n}\!\!\int_0^{1/n}(D^2f)\left(\frac{k}{n}+u+v\right)dudv\cdot V_{n+2,k}(x). \tag{2.5}$$

It is also well defined. Indeed,

$$\begin{aligned}&\left|\sum_{k=0}^{\infty}\int_0^{1/n}\!\!\int_0^{1/n}(D^2f)\left(\frac{k}{n}+u+v\right)dudv\cdot V_{n+2,k}(x)\right|\\
&\le \sum_{k=0}^{\infty}\int_0^{1/n}\!\!\int_0^{1/n}\left|(\psi^2D^2f)\left(\frac{k}{n}+u+v\right)\right|\psi^{-2}\left(\frac{k}{n}+u+v\right)dudv\cdot V_{n+2,k}(x)\\
&\le \|\psi^2D^2f\|\sum_{k=0}^{\infty}\int_0^{1/n}\!\!\int_0^{1/n}\psi^{-2}\left(\frac{k}{n}+u+v\right)dudv\cdot V_{n+2,k}(x).\end{aligned}$$

But

$$\int_0^{1/n}\!\!\int_0^{1/n}\psi^{-2}\left(\frac{k}{n}+u+v\right)dudv \le \int_0^{1/n}\!\!\int_0^{1/n}\frac{dudv}{u+v} = \int_0^{1/n}\!\!\int_0^{1/n}\frac{dudv}{\sqrt{u+v}\sqrt{u+v}}$$

$$\leq \int_0^{1/n}\int_0^{1/n} \frac{dudv}{\sqrt{u}\sqrt{v}} = \left(\int_0^{1/n} \frac{du}{\sqrt{u}}\right)^2 = \frac{4}{n}$$

i.e.

$$\left|\sum_{k=0}^{\infty}\int_0^{1/n}\int_0^{1/n}(D^2f)\left(\frac{k}{n}+u+v\right)dudv\cdot V_{n+2,k}(x)\right|$$
$$\leq \frac{4\|\psi^2D^2f\|}{n}\sum_{k=0}^{\infty}V_{n+2,k}(x) = \frac{4\|\psi^2D^2f\|}{n}.$$

From all this it follows $(D^2V_nf)(x)$ is well defined for all $f\in\mathcal{F}$ and consequently $D^2(V_n^Nf)(x)$ where as usual we set $V_n^0f=f$, $V_n^1=V_n$ and recurrently $V_n^{k+1}=V_n(V_n^k)$.

We also need several additional results.

We will use a Bernstein type inequality for the Baskakov operator [3, Theorem 9.4.1, page 125]

$$\|\psi^2D^2(V_nf)\| \leq C_3n\|f\| \tag{2.6}$$

and the next two computational results from [5, Lemma 4.1, page 324]:

$$\int_0^1\int_0^1\int_0^1 e^{(u_2+u_3-1)^2/(1+u_1)}\,du_1du_2du_3 - \frac{e}{4}(1-e^{-1})(1-e^{-2})^2+(1-e^{-1})^4 \leq 1, \tag{2.7}$$

$$\int_0^1\int_0^1\int_0^1 \frac{(j+u_1+u_2)}{(j+u_1+u_2+u_3)^2}\,e^{u_3^2/(j+u_1+u_2)}\,du_1du_2du_3 \leq \frac{1}{j+1},\ j\geq 1. \tag{2.8}$$

The proof of Theorem 1.1 is based on two additional theorems.

Theorem 2.1. *There exists an absolute constant $C_4>0$ such that for all functions $f\in W^3_\infty(\psi)$ the estimation*

$$\|V_nf-f-\frac{1}{2n}\psi^2D^2f\| \leq C_4n^{-3/2}\|\psi^3D^3f\|$$

holds true.

P r o o f. Applying Taylor's formula we have

$$f\left(\frac{k}{n}\right) = f(x)+\left(\frac{k}{n}-x\right)f'(x)+\frac{1}{2}\left(\frac{k}{n}-x\right)^2 f''(x)+\frac{1}{2}\int_x^{k/n}\left(\frac{k}{n}-v\right)^2 f'''(v)\,dv.$$

Multiplying both sides by $V_{n,k}(x)$, summing with respect to k and using the identities (2.2) we obtain

$$\begin{aligned}\Big|V_n(f,x)-f(x)-\frac{1}{2n}\,\psi^2(x)f''(x)\Big| &= \Big|\frac{1}{2}\sum_{k=0}^{\infty}V_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n}-v\right)^2 f'''(v)\,dv\Big|\\ &\le \frac{1}{2}\,\|\psi^3 f'''\|\Big|\sum_{k=0}^{\infty}V_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n}-v\right)^2\psi^{-3}(v)\,dv\Big|.\end{aligned}$$

We will show that

$$A_n(x) := \Big|\sum_{k=0}^{\infty}V_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n}-v\right)^2\psi^{-3}(v)\,dv\Big| \le Cn^{-3/2}.$$

For $x \ge \dfrac{1}{n}$ the above inequality is a part of the proof of Lemma 2.5, [4, (2.13), page 226].

Now let us consider the case $x < \dfrac{1}{n}$. Analogously to [2, Lemma 8.3], we will estimate terms in the sum of $A_n(x)$ separately for $k=0,1,2$ and $k \ge 3$. We have

$$\begin{aligned}V_{n,0}(x)\int_0^x v^2\psi^{-3}(v)\,dv &= \frac{1}{(1+x)^n}\int_0^x \frac{v^2\,dv}{\big(v(1+v)\big)^{3/2}} \le \int_0^x v^{1/2}\,dv\\ &= \frac{2}{3}\,x^{3/2} \le \frac{2}{3}\,n^{-3/2},\end{aligned}$$

$$\begin{aligned}V_{n,1}(x)\int_x^{1/n}\left(\frac{1}{n}-v\right)^2\psi^{-3}(v)\,dv &= \frac{nx}{(1+x)^{n+1}}\int_x^{1/n}\frac{\left(\frac{1}{n}-v\right)^2dv}{\big(v(1+v)\big)^{3/2}}\\ &\le \frac{x}{n}\int_x^{\infty}v^{-3/2}\,dv = \frac{2x^{1/2}}{n} \le 2n^{-3/2},\end{aligned}$$

$$V_{n,2}(x)\int_x^{2/n}\left(\frac{2}{n}-v\right)^2\psi^{-3}(v)\,dv = \frac{(n+1)nx^2}{2(1+x)^{n+2}}\int_x^{2/n}\frac{\left(\frac{2}{n}-v\right)^2dv}{\big(v(1+v)\big)^{3/2}}$$

$$\begin{aligned}&\le \frac{4(n+1)nx^2}{2n^2}\int_x^\infty v^{-3/2}\,dv = \frac{4(n+1)}{n}x^{3/2}\\&\le 8n^{-3/2},\end{aligned}$$

$$\begin{aligned}\Big|\sum_{k=3}^\infty V_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n}-v\right)^2\psi^{-3}(v)\,dv\Big| &\le \sum_{k=3}^\infty V_{n,k}(x)\int_x^{k/n}\left(\frac{k}{n}-v\right)^2\psi^{-3}(x)\,dv\\&= \frac{1}{3}\psi^{-3}(x)\sum_{k=3}^\infty V_{n,k}(x)\left(\frac{k}{n}-x\right)^3\\&\le \frac{1}{3}\psi^{-3}(x)\sum_{k=3}^\infty \binom{n+k-1}{k}\frac{k^3x^k}{n^3(1+x)^{n+k}}\\&\le \frac{3}{2}\frac{(n+1)(n+2)x^3}{n^2\psi^3(x)}\sum_{k=3}^\infty\binom{n+k-1}{k-3}\frac{x^{k-3}}{(1+x)^{n+k}}\\&\le 9x^{3/2}V_{n+3}(1,x)\le 9n^{-3/2}.\end{aligned}$$

Therefore in the case of $x < \dfrac{1}{n}$ we obtain $A_n(x) \le \dfrac{59}{3}n^{-3/2}$ and Theorem 2.1 follows. □

Theorem 2.2. *Let* $2 \le N \le \dfrac{n-2}{2}, n \ge 10$. *There exists an absolute constant* C *such that for all functions* $f \in W^2_\infty(\psi)$

$$\|\psi^3 D^3(V_n^N f)\| \le K(N)\sqrt{n}\|\psi^2 D^2 f\| \quad \textit{where} \quad K(N) \le CN^{-1/4}\ln N.$$

Complete proof of this theorem is given in Section 4.

Now we will prove several lemmas, which we will need in the last two sections. In the next two lemmas we collect some inequalities about Baskakov polynomials $V_{n,k}$.

Lemma 2.1. *For* $V_{n,k}$ *defined by* (1.2) *and for every real numbers* $n \ge 3, z > 0, 0 \le a \le 1$ *we have*

$$\sum_{k=0}^\infty \frac{V_{n+3,k}(z)}{(k+1)(n+k+2)} \le \frac{1}{n^2z(1+z)}, \tag{2.9}$$

$$\sum_{k=0}^\infty \frac{a^k}{\left(1+\frac{k}{n}\right)^4}V_{n+2,k}(z) \le \frac{n^3}{(n+1)(n-1)(n-2)}\frac{(1+z-az)^{-(n-2)}}{(1+z)^4}, \tag{2.10}$$

$$\sum_{k=0}^{\infty} \frac{a^k}{\left(1+\frac{k}{n}\right)^4} V_{n+3,k}(z) \le \frac{n^2}{(n-1)(n+1)} \frac{(1+z-az)^{-(n-1)}}{(1+z)^4}. \tag{2.11}$$

P r o o f. Inequality (2.9) is evident as

$$\frac{V_{n+3,k}(z)}{(k+1)(n+k+2)} = \frac{V_{n+1,k+1}(z)}{(n+1)(n+2)z(1+z)} \le \frac{V_{n+1,k+1}(z)}{n^2 z(1+z)}$$

and

$$\sum_{k=0}^{\infty} V_{n+1,k+1}(z) = \sum_{k=1}^{\infty} V_{n+1,k}(z) \le \sum_{k=0}^{\infty} V_{n+1,k}(z) = 1.$$

For the second one we have

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{a^k}{(n+k)^4} V_{n+2,k}(z) &= \frac{1}{(1+z)^{n+2}} \sum_{k=0}^{\infty} \frac{(n+k+1)\cdots(k+1)}{(n+1)!} \frac{1}{(n+k)^4} \left(\frac{az}{1+z}\right)^k \\ &\le \frac{1}{(1+z)^{n+2}} \sum_{k=0}^{\infty} \frac{(n+k-3)\cdots(k+1)}{(n+1)!} \left(\frac{az}{1+z}\right)^k \\ &= \frac{1}{(n+1)n(n-1)(n-2)} \frac{1}{(1+z)^{n+2}} \sum_{k=0}^{\infty} \binom{n+k-3}{k} \left(\frac{az}{1+z}\right)^k.\end{aligned}$$

But

$$\sum_{k=0}^{\infty} \binom{n+k-3}{k} \left(\frac{az}{1+z}\right)^k = \left(\frac{1+z}{1+z-az}\right)^{n-2}$$

and consequently (2.10) follows. Analogously for (2.11). □

Later in the paper we will apply (2.11) only for $n \ge 10$. In that case

$$\frac{n^2}{(n-1)(n+1)} \le 1 + \frac{4}{n},$$

so

$$\sum_{k=0}^{\infty} \frac{a^k}{\left(1+\frac{k}{n}\right)^4} V_{n+3,k}(z) \le \left(1+\frac{4}{n}\right) \frac{(1+z-az)^{-(n-1)}}{(1+z)^4}.$$

Setting $a = 1$ we obtain

$$\sum_{k=0}^{\infty} \frac{V_{n+3,k}(z)}{\left(1+\frac{k}{n}\right)^4} \le \frac{1+\frac{4}{n}}{(1+z)^4}. \tag{2.12}$$

Lemma 2.2. *For $V_{n,k}$ defined by (1.2) and $n \geq 10$, $j = 0, 1, 2, \ldots$ there exists an absolute constant C_5 such that*

$$\sum_{k=0}^{\infty} \frac{\left[\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}(DV_{n+2,k})\left(\frac{j}{n}+u_1+u_2+u_3\right)\,du_1du_2du_3\right]^2}{\int_0^{1/n}\int_0^{1/n} V_{n+2,k}\left(\frac{j}{n}+u_1+u_2\right)\,du_1du_2} \leq \frac{C_5}{n^3}\,\psi^{-2}\left(\frac{j+1}{n+1}\right).$$

P r o o f. Let us denote

$$I = \sum_{k=0}^{\infty} I_k,$$

$$I_k = \frac{\left[\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}(DV_{n+2,k})\left(\frac{j}{n}+u_1+u_2+u_3\right)\,du_1du_2du_3\right]^2}{\int_0^{1/n}\int_0^{1/n} V_{n+2,k}\left(\frac{j}{n}+u_1+u_2\right)\,du_1du_2}, \quad k = 0, 1, \ldots.$$

We consider two cases.

Case 1. $j = 0$.

For $k = 0$ we have

$$\begin{aligned} I_0 &= \frac{\left[(n+2)\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}(1+u_1+u_2+u_3)^{-(n+3)}du_1du_2du_3\right]^2}{\int_0^{1/n}\int_0^{1/n}(1+u_1+u_2)^{-(n+2)}du_1du_2} \\ &\leq \frac{(n+2)^2\left(\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}du_1du_2du_3\right)^2}{\int_0^{1/n}\int_0^{1/n}\left(1+\frac{2}{n}\right)^{-(n+2)}\,du_1du_2} = \frac{(n+2)^2}{n^4}\left(1+\frac{2}{n}\right)^{n+2} \leq \frac{C}{n^2}. \end{aligned}$$

For $k \geq 1$ we apply (2.3) for $(n+2)$ to obtain

$$\begin{aligned} &\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}(DV_{n+2,k})(u_1+u_2+u_3)\,du_1du_2du_3 \\ &\qquad \leq \frac{n+2}{n^3}\left[\binom{n+k+1}{k-1}\left(\frac{3}{n}\right)^{k-1} + \binom{n+k+2}{k}\left(\frac{3}{n}\right)^{k}\right]. \end{aligned}$$

For the denominator we have

$$\int_0^{1/n}\int_0^{1/n} V_{n+2,k}(u_1+u_2)\,du_1du_2 \geq \binom{n+k+1}{k}\int_0^{1/n}\int_0^{1/n}\frac{(u_1+u_2)^k du_1du_2}{\left(1+u_1+\frac{1}{n}\right)^{n+k+2}}$$
$$= \binom{n+k+1}{k}\int_0^{1/n}\frac{(u_1+\frac{1}{n})^{k+1}-u_1^{k+1}}{(k+1)\left(1+u_1+\frac{1}{n}\right)^{n+k+2}}\,du_1.$$

But

$$\left(u_1+\frac{1}{n}\right)^{k+1}-u_1^{k+1} = u_1\left(u_1+\frac{1}{n}\right)^k+\frac{1}{n}\left(u_1+\frac{1}{n}\right)^k - u_1^{k+1} \geq \frac{1}{n}\left(u_1+\frac{1}{n}\right)^k.$$

Consequently,

$$\int_0^{1/n}\int_0^{1/n} V_{n+2,k}(u_1+u_2)\,du_1du_2$$
$$\geq \binom{n+k+1}{k}\frac{1}{(k+1)n}\int_0^{1/n}\left(\frac{u_1+\frac{1}{n}}{1+u_1+\frac{1}{n}}\right)^k\frac{du_1}{\left(1+u_1+\frac{1}{n}\right)^{n+2}}.$$

And because of

$$\left(1+u_1+\frac{1}{n}\right)^{n+2} \leq \left(1+\frac{2}{n}\right)^{n+2} \leq C$$

we have

$$\int_0^{1/n}\int_0^{1/n} V_{n+2,k}(u_1+u_2)\,du_1du_2 \geq \frac{C}{(k+1)n}\binom{n+k+1}{k}\int_0^{1/n}\left(\frac{u_1+\frac{1}{n}}{1+u_1+\frac{1}{n}}\right)^k du_1$$
$$= \frac{C}{(k+1)n}\binom{n+k+1}{k}\int_{1/n}^{2/n}\left(\frac{u}{1+u}\right)^k du.$$

The function $\dfrac{u}{1+u}$ is monotonically increasing, so $\dfrac{u}{1+u} \geq \dfrac{\frac{1}{n}}{1+\frac{1}{n}} = \dfrac{1}{n+1}$ for $u \geq \dfrac{1}{n}$, which yields

$$\int_0^{1/n}\int_0^{1/n} V_{n+2,k}(u_1+u_2)\,du_1du_2 \geq \frac{C}{(k+1)n^2}\binom{n+k+1}{k}\frac{1}{(n+1)^k}.$$

Then

(2.13)
$$\sum_{k=1}^{\infty} I_k \leq \frac{C(n+2)^2}{n^4}\left[\sum_{k=1}^{\infty}\frac{(k+1)\binom{n+k+1}{k-1}^2\left(\frac{3}{n}\right)^{2k-2}}{\binom{n+k+1}{k}\frac{1}{(n+1)^k}}+\sum_{k=1}^{\infty}\frac{(k+1)\binom{n+k+2}{k}^2\left(\frac{3}{n}\right)^{2k}}{\binom{n+k+1}{k}\frac{1}{(n+1)^k}}\right].$$

Next we estimate each of the sums in (2.13). For the first one we have

$$\sum_{k=1}^{\infty} \frac{(k+1)\binom{n+k+1}{k-1}^2 \left(\frac{3}{n}\right)^{2k-2}}{\binom{n+k+1}{k}\frac{1}{(n+1)^k}} = \sum_{k=1}^{\infty} \frac{k(k+1)(n+1)}{n+2}\binom{n+k+1}{k-1}\left(\frac{9n+9}{n^2}\right)^{k-1}$$
$$\le \sum_{k=1}^{\infty} k(k+1)\binom{n+k+1}{k-1}\left(\frac{9n+9}{n^2}\right)^{k-1}.$$

Computing the last sum we get

$$\sum_{k=0}^{\infty}(k+1)(k+2)\binom{n+k+2}{k}x^k = \frac{2}{(1-x)^{n+3}} + \frac{4(n+3)x}{(1-x)^{n+4}} + \frac{(n+3)(n+4)x^2}{(1-x)^{n+5}}.$$

Now, for $n \ge 10$ we have $x = \dfrac{9n+9}{n^2} < 1$ and from $(1-x)^{-(n+i)} \le C$, $i = 3,4,5$, it follows

$$\sum_{k=1}^{\infty} \frac{(k+1)\binom{n+k+1}{k-1}^2 \left(\frac{3}{n}\right)^{2k-2}}{\binom{n+k+1}{k}\frac{1}{(n+1)^k}} \le C.$$

We estimate the second sum in (2.13) in a similar way.

Consequently,

$$I \le \frac{C}{n^2} \le \frac{C}{n^3\psi^2\left(\frac{1}{n+1}\right)}.$$

Case 2. $j \ge 1$.

We will use the second representation (2.4) of the first derivative of $V_{n+2,k}$. To shorten the following expressions we denote

$$\xi = \frac{j}{n} + u_1 + u_2 + u_3, \quad \eta = \frac{j}{n} + u_1 + u_2.$$

Now, by using Cauchy's inequality two times, we obtain

$$\left(\int_0^{1/n}\int_0^{1/n}\int_0^{1/n} (DV_{n+2,k})(\xi)\, du_1du_2du_3\right)^2$$
$$= \left[\int_0^{1/n}\left(\int_0^{1/n}\int_0^{1/n} (DV_{n+2,k})(\xi)\, du_1du_2\right) du_3\right]^2$$
$$\le \frac{1}{n}\int_0^{1/n}\left(\int_0^{1/n}\int_0^{1/n} (DV_{n+2,k})(\xi)\, du_1du_2\right)^2 du_3$$
$$= \frac{1}{n}\int_0^{1/n}\left(\int_0^{1/n}\int_0^{1/n} (DV_{n+2,k})(\xi)\frac{\sqrt{V_{n+2,k}(\eta)}}{\sqrt{V_{n+2,k}(\eta)}}\, du_1du_2\right)^2 du_3$$

$$\leq \frac{1}{n}\int_0^{1/n}\left(\int_0^{1/n}\int_0^{1/n}\frac{\Big(\big(DV_{n+2,k}\big)\big(\xi\big)\Big)^2 du_1du_2}{V_{n+2,k}\left(\eta\right)}\int_0^{1/n}\int_0^{1/n}V_{n+2,k}\left(\eta\right)\,du_1du_2\right)du_3$$

$$= \frac{1}{n}\int_0^{1/n}\int_0^{1/n}V_{n+2,k}\left(\eta\right)\,du_1du_2\cdot\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}\frac{\Big(\big(DV_{n+2,k}\big)\big(\xi\big)\Big)^2 du_1du_2du_3}{V_{n+2,k}\left(\eta\right)}.$$

Then

$$\begin{aligned} I_k &\leq \frac{1}{n}\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}\frac{\Big(\big(DV_{n+2,k}\big)\big(\xi\big)\Big)^2 du_1du_2du_3}{V_{n+2,k}\left(\eta\right)} \\ &= \frac{1}{n}\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}\frac{(n+2)^2}{\psi^4(\xi)}\left(\frac{k}{n+2}-\xi\right)^2\frac{V_{n+2,k}^2(\xi)\,du_1du_2du_3}{V_{n+2,k}\left(\eta\right)}. \end{aligned}$$

We have

$$\begin{aligned} \psi^2(\xi) &= \left(\frac{j}{n}+u_1+u_2+u_3\right)\left(1+\frac{j}{n}+u_1+u_2+u_3\right) \geq \frac{j}{n}\left(1+\frac{j}{n}\right) \\ &\geq \frac{j+1}{2(n+1)}\left(1+\frac{j+1}{2(n+1)}\right) \geq \frac{j+1}{4(n+1)}\left(1+\frac{j+1}{n+1}\right) = \frac{1}{4}\,\psi^2\left(\frac{j+1}{n+1}\right) \end{aligned}$$

and also

$$\Big|\frac{k}{n+2}-\xi\Big| \leq \Big|\frac{k}{n+2}-\frac{j}{n}\Big| + \Big|\xi-\frac{j}{n}\Big| \leq 3\left(\Big|\frac{k}{n+2}-\frac{j}{n}\Big|+\frac{1}{n}\right).$$

Let us denote

$$A = \frac{\xi^2}{\eta}, \qquad B = \frac{(1+\xi)^2}{1+\eta}.$$

Then

$$\begin{aligned} I_k &\leq \frac{144(n+2)^2}{n\psi^4\left(\frac{j+1}{n+1}\right)}\left(\Big|\frac{k}{n+2}-\frac{j}{n}\Big|+\frac{1}{n}\right)^2\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}\frac{V_{n+2,k}^2\left(\xi\right)}{V_{n+2,k}\left(\eta\right)}\,du_1du_2du_3 \\ &\leq \frac{Cn}{\psi^4\left(\frac{j+1}{n+1}\right)}\left(\Big|\frac{k}{n+2}-\frac{j}{n}\Big|+\frac{1}{n}\right)^2 \\ &\qquad\qquad \times\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}\binom{n+k+1}{k}A^kB^{-(n+k+2)}\,du_1du_2du_3. \end{aligned}$$

Now,

$$B - A = 1 - \frac{u_3^2}{\psi^2\left(\frac{j}{n} + u_1 + u_2\right)} \geq 1 - \frac{\frac{1}{n^2}}{\frac{j}{n}\left(1 + \frac{j}{n}\right)} = 1 - \frac{1}{j(n+j)} \geq 1 - \frac{1}{n+1},$$

i.e.

$$1 + A - B \leq \frac{1}{n+1} < \frac{1}{n}.$$

Further,

$$\left(\frac{k}{n+2} - \frac{j}{n}\right)^2 \leq 2\left[\left(\frac{k}{n+2} - A\right)^2 + \left(A - \frac{j}{n}\right)^2\right] \leq C\left[\left(\frac{k}{n+2} - A\right)^2 + \frac{1}{n^2}\right]$$

The last inequality is true because from the definition of A we have

$$\frac{j^2}{(j+2)n} \leq A \leq \frac{(j+3)^2}{jn}.$$

Then

$$\begin{aligned}\left(A - \frac{j}{n}\right)^2 &\leq \max\left\{\left(\frac{j}{n} - \frac{j^2}{(j+2)n}\right)^2, \left(\frac{(j+3)^2}{jn} - \frac{j}{n}\right)^2\right\} \\ &= \max\left\{\frac{4j^2}{(j+2)^2n^2}, \frac{(6j+9)^2}{j^2n^2}\right\} \leq \max\left\{\frac{4}{n^2}, \frac{225}{n^2}\right\} \leq \frac{225}{n^2}.\end{aligned}$$

In order to estimate I in this case we have

$$\begin{aligned}I &\leq \frac{Cn}{\psi^4(\frac{j+1}{n+1})}\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}\sum_{k=0}^{\infty}\left[\left(\frac{k}{n+2} - A\right)^2 + \frac{1}{n^2}\right] \\ &\qquad\qquad \times \binom{n+k+1}{k} A^k B^{-(n+k+2)} du_1 du_2 du_3 \\ &= \frac{Cn}{\psi^4(\frac{j+1}{n+1})}\int_0^{1/n}\int_0^{1/n}\int_0^{1/n} B^{-n-2}\left[\frac{1}{(n+2)^2}\sum_{k=0}^{\infty}k^2\binom{n+k+1}{k}\left(\frac{A}{B}\right)^k\right. \\ &\qquad - \frac{2A}{n+2}\sum_{k=0}^{\infty}k\binom{n+k+1}{k}\left(\frac{A}{B}\right)^k \\ &\qquad \left. + \left(A^2 + \frac{1}{n^2}\right)\sum_{k=0}^{\infty}\binom{n+k+1}{k}\left(\frac{A}{B}\right)^k\right] du_1 du_2 du_3.\end{aligned}$$

Since, for $|x| < 1$,

$$\sum_{k=0}^{\infty} k^i \binom{n+k+1}{k} x^k = \begin{cases} (1-x)^{-n-2}, & i=0, \\ \dfrac{(n+2)x}{(1-x)^{n+3}}, & i=1, \\ \dfrac{(n+2)x}{(1-x)^{n+3}} + \dfrac{(n+2)(n+3)x^2}{(1-x)^{n+4}}, & i=2, \end{cases}$$

and $A/B < 1$, we obtain

$$\begin{aligned} I &\le \frac{Cn}{\psi^4(\frac{j+1}{n+1})} \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} B^{-n-2} \Bigg[\frac{AB^{-1}}{(n+2)(1-AB^{-1})^{n+3}} \\ &\quad + \frac{(n+3)A^2B^{-2}}{(n+2)(1-AB^{-1})^{n+4}} - 2A\,\frac{AB^{-1}}{(1-AB^{-1})^{n+3}} + \frac{A^2+\frac{1}{n^2}}{(1-AB^{-1})^{n+2}} \Bigg] du_1du_2du_3 \\ &= \frac{Cn}{\psi^4(\frac{j+1}{n+1})} \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} (B-A)^{-n-4} \Bigg[(A-B+1)^2A^2 + \frac{AB}{n+2} \\ &\qquad\qquad + \frac{(A-B)^2}{n^2} \Bigg] du_1du_2du_3. \end{aligned}$$

From $B - A \ge 1 - \dfrac{1}{n+1}$ it follows $\dfrac{1}{(B-A)^{n+4}} \le C$. Moreover,

$$\begin{gathered} A = \frac{\left(\frac{j}{n}+u_1+u_2+u_3\right)^2}{\frac{j}{n}+u_1+u_2} \le \frac{\left(\frac{j}{n}+\frac{3}{n}\right)^2}{\frac{j}{n}} = \frac{(j+3)^2}{jn} \le C\frac{j+1}{n+1}, \\ B = \frac{\left(1+\frac{j}{n}+u_1+u_2+u_3\right)^2}{1+\frac{j}{n}+u_1+u_2} \le \frac{\left(1+\frac{j}{n}+\frac{3}{n}\right)^2}{1+\frac{j}{n}} \le C\left(1+\frac{j+1}{n+1}\right) \\ B - A = 1 - \frac{u_3^2}{\psi^2\left(\frac{j}{n}+u_1+u_2\right)} \le 1, \\ A - B + 1 \le \frac{1}{n+1}. \end{gathered}$$

Therefore

$$\frac{1}{(B-A)^{n+4}} \left[(A-B+1)^2A^2 + \frac{AB}{n+2} + \frac{(A-B)^2}{n^2} \right] \le \frac{C\psi^2\left(\frac{j+1}{n+1}\right)}{n},$$

and

$$I \le \frac{Cn}{\psi^4\left(\frac{j+1}{n+1}\right)} \cdot \frac{C\psi^2\left(\frac{j+1}{n+1}\right)}{n} \cdot \frac{1}{n^3} = \frac{C}{n^3\psi^2\left(\frac{j+1}{n+1}\right)}.$$

The proof of Lemma 2.2 is complete. □

The next lemma is a very important technical result, which we will need later.

Lemma 2.3. *There exists an absolute constant C_6 such that for all real numbers $z \ge 0, \alpha \ge 1$ there holds*

$$\frac{1}{(1+z)^\alpha} \le e^{-\alpha z\left(1+\frac{3}{4}\frac{\ln\alpha}{\alpha}\right)^{-1}} + C_6(\alpha+2)^{-9/4}. \tag{2.14}$$

Proof. We have for $|t| < 1$,

$$\ln(1+t) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}t^k}{k}, \quad -\ln(1-t) = \sum_{k=1}^{\infty} \frac{t^k}{k}.$$

Adding them we get

$$\ln\frac{1+t}{1-t} = 2t\sum_{k=0}^{\infty} \frac{t^{2k}}{2k+1} \ge 2t.$$

Let $\dfrac{1+t}{1-t} = 1+z$. Then we have $t = \dfrac{z}{z+2}$ and $\ln(1+z) \ge \dfrac{2z}{z+2}$.

1) $0 \le z \le \dfrac{3}{2}\dfrac{\ln\alpha}{\alpha}$

$$\frac{1}{(1+z)^\alpha} = e^{-\alpha\ln(1+z)} \le e^{-\frac{2\alpha z}{z+2}} = e^{-\alpha z\left(1+\frac{z}{2}\right)^{-1}} \le e^{-\alpha z\left(1+\frac{3}{4}\frac{\ln\alpha}{\alpha}\right)^{-1}}.$$

2) $\dfrac{3}{2}\dfrac{\ln\alpha}{\alpha} \le z \le \dfrac{3\ln\alpha}{\alpha}$

$$\frac{1}{(1+z)^\alpha} - e^{-\alpha z\left(1+\frac{3}{4}\frac{\ln\alpha}{\alpha}\right)^{-1}} \le e^{-\alpha\left(z-\frac{z^2}{2}\right)} - e^{-\alpha z} = e^{-\alpha z}\left(e^{\frac{\alpha z^2}{2}} - 1\right).$$

But in this interval

$$\frac{1}{2}\alpha z^2 \le \frac{9}{2}\frac{\ln^2\alpha}{\alpha}.$$

Then

$$e^{\frac{\alpha z^2}{2}} - 1 = \frac{\alpha z^2}{2}\left(1 + \frac{1}{2}\frac{\alpha z^2}{2} + \cdots\right) \le C\frac{\alpha z^2}{2} \le C\frac{\ln^2\alpha}{\alpha}.$$

And from

$$e^{-\alpha z} \le e^{-\alpha\frac{3}{2}\frac{\ln\alpha}{\alpha}} = \alpha^{-\frac{3}{2}}$$

it follows

$$\frac{1}{(1+z)^\alpha} - e^{-\alpha z\left(1+\frac{3}{4}\frac{\ln\alpha}{\alpha}\right)^{-1}} \le C\alpha^{-5/2}\ln^2\alpha \le C(\alpha+2)^{-9/4}.$$

3) $z \ge \dfrac{3\ln\alpha}{\alpha}$

$$\frac{1}{(1+z)^\alpha} \le \left(1+\frac{3\ln\alpha}{\alpha}\right)^{-\alpha} = e^{-\alpha\ln\left(1+\frac{3\ln\alpha}{\alpha}\right)} \le e^{-\alpha\left(\frac{3\ln\alpha}{\alpha}-\frac{9}{2}\frac{\ln^2\alpha}{\alpha^2}\right)}$$
$$= \alpha^{-3}e^{\frac{9}{2}\frac{\ln^2\alpha}{\alpha}} \le C(\alpha+2)^{-9/4}.$$

This completes the proof. $\square$

We remark here, that it is not difficult to prove, for instance, that the constant $C_6 < 2$, but for our purposes it is enough that C_6 is an absolute constant. Now, let us denote

$$\left(1+\frac{3}{4}\frac{\ln(n-2)}{n-2}\right)^{-1} = b. \tag{2.15}$$

Then the inequality (2.14) can be written as

$$\frac{1}{(1+z)^{n-2}} \le e^{-(n-2)bz} + C_6 n^{-9/4}. \tag{2.16}$$

For the next two lemmas we introduce the following notations:

$$T_{2,k}(x) = n(n+1)\int_0^{1/n}\int_0^{1/n} V_{n+2,k}(x+t_1+t_2)\,dt_1dt_2, \tag{2.17}$$

and

$$T_{3,k}(x) = n(n+1)(n+2)\int_0^{1/n}\int_0^{1/n}\int_0^{1/n} V_{n+3,k}(x+t_1+t_2+t_3)\,dt_1dt_2dt_3. \tag{2.18}$$

Lemma 2.4. *For function $T_{2,k}$ defined by (2.17), b by (2.15) and for every real numbers $n \geq 10$, $x \geq 0$, $0 \leq a \leq 1$ the next inequalities hold*

$$\sum_{k=0}^{\infty} \frac{a^k T_{2,k}(x)}{\left(1+\frac{k}{n}\right)^4} \tag{2.19}$$

$$\leq \frac{1+\frac{4}{n}}{(1+x)^4}\left(\frac{n}{b(n-2)}\right)^2 e^{-(n-2)b(1-a)x}\left(\frac{1-e^{-\frac{(n-2)}{n}b(1-a)}}{1-a}\right)^2 + \frac{\left(1+\frac{4}{n}\right)C_6}{n^{9/4}(1+x)^4},$$

$$\sum_{k=0}^{\infty} \frac{T_{2,k}(x)}{\left(1+\frac{k+1}{n}\right)^4} \leq \sum_{k=0}^{\infty} \frac{T_{2,k}(x)}{\left(1+\frac{k}{n}\right)^4} \leq \frac{1+\frac{4}{n}}{(1+x)^4}. \tag{2.20}$$

Proof. By Lemma 2.1 we have

$$\sum_{k=0}^{\infty} \frac{a^k T_{2,k}(x)}{\left(1+\frac{k}{n}\right)^4} = n^5(n+1)\int_0^{1/n}\int_0^{1/n}\sum_{k=0}^{\infty}\frac{a^k}{(n+k)^4}V_{n+2,k}(x+t_1+t_2)dt_1dt_2$$

$$\leq \frac{n^4}{(n-1)(n-2)}\int_0^{1/n}\int_0^{1/n}\frac{1}{\left(1+x+t_1+t_2\right)^4}\frac{1}{\left[1+(1-a)(x+t_1+t_2)\right]^{n-2}}dt_1dt_2$$

$$\leq \frac{n^4}{(n-1)(n-2)}\frac{1}{(1+x)^4}\int_0^{1/n}\int_0^{1/n}\frac{dt_1dt_2}{\left[1+(1-a)(x+t_1+t_2)\right]^{n-2}},$$

and by (2.16)

$$\sum_{k=0}^{\infty} \frac{a^k T_{2,k}(x)}{\left(1+\frac{k}{n}\right)^4}$$

$$\leq \frac{n^4}{(n-1)(n-2)}\frac{1}{(1+x)^4}\int_0^{1/n}\int_0^{1/n}\left(e^{-(n-2)b(1-a)(x+t_1+t_2)}+\frac{C_6}{n^{9/4}}\right)dt_1dt_2$$

$$= \frac{n^4}{(n-1)(n-2)}\frac{1}{(1+x)^4}e^{-(n-2)b(1-a)x}\left(\int_0^{1/n}e^{-(n-2)b(1-a)t}dt\right)^2$$

$$+ \frac{n^2}{(n-1)(n-2)}\frac{C_6}{n^{9/4}(1+x)^4}$$

$$= \frac{n^2}{(n-1)(n-2)} \frac{1}{(1+x)^4} \left(\frac{n}{b(n-2)}\right)^2 e^{-(n-2)b(1-a)x} \left(\frac{1-e^{-\frac{(n-2)}{n}b(1-a)}}{1-a}\right)^2$$

$$+ \frac{n^2}{(n-1)(n-2)} \frac{C_6}{n^{9/4}(1+x)^4}.$$

But for $n \geq 10$

$$\frac{n^2}{(n-1)(n-2)} \leq 1 + \frac{4}{n} \quad \text{and (2.19) follows.}$$

Setting $a = 1$ in (2.10) gives us

$$\sum_{k=0}^{\infty} \frac{V_{n+2,k}(x+t_1+t_2)}{\left(1+\frac{k+1}{n}\right)^4} \leq \sum_{k=0}^{\infty} \frac{V_{n+2,k}(x+t_1+t_2)}{\left(1+\frac{k}{n}\right)^4}$$

$$\leq \frac{n^3}{(n+1)(n-1)(n-2)} \frac{1}{(1+x+t_1+t_2)^4},$$

and because of

$$\int_0^{1/n} \int_0^{1/n} \frac{dt_1 dt_2}{(1+x+t_1+t_2)^4} \leq \frac{1}{n^2(1+x)^4}$$

we obtain

$$\sum_{k=0}^{\infty} \frac{T_{2,k}(x)}{\left(1+\frac{k+1}{n}\right)^4} \leq \frac{n^2}{(n-1)(n-2)} \frac{1}{(1+x)^4} \leq \frac{1+\frac{4}{n}}{(1+x)^4}.$$ □

Lemma 2.5. *For $T_{2,k}$ defined by (2.17), $T_{3,k}$ by (2.18) and $n \geq 1$, $j = 0, 1, 2, \dots$ there exists an absolute constant C_7 such that*

$$\sum_{k=0}^{\infty} \frac{T_{3,k}^2\left(\frac{j}{n}\right)}{(k+1)(n+k+2)T_{2,k}\left(\frac{j}{n}\right)} \leq \frac{1+\frac{C_7}{n}}{(j+1)(n+j+2)}. \tag{2.21}$$

P r o o f. Let us denote the left side in (2.21) with I and the terms in it with I_k.

Case 1. $j=0$

Now

$$I_k = \frac{\left[n(n+1)(n+2)\int_0^{1/n}\int_0^{1/n}\int_0^{1/n} V_{n+3,k}(u_1+u_2+u_3)du_1du_2du_3\right]^2}{(k+1)(n+k+2)n(n+1)\int_0^{1/n}\int_0^{1/n} V_{n+2,k}(u_1+u_2)du_1du_2}$$
$$= \frac{n(n+1)(n+2)^2}{(k+1)(n+k+2)}\frac{\left[\int_0^{1/n}\int_0^{1/n}\int_0^{1/n} V_{n+3,k}(u_1+u_2+u_3)du_1du_2du_3\right]^2}{\int_0^{1/n}\int_0^{1/n} V_{n+2,k}(u_1+u_2)du_1du_2}.$$

By using two times Cauchy's inequality we get

$$\left[\int_0^{1/n}\int_0^{1/n}\int_0^{1/n} V_{n+3,k}(u_1+u_2+u_3)du_1du_2du_3\right]^2$$
$$\le \frac{1}{n^2}\int_0^{1/n}\int_0^{1/n}\left[\int_0^{1/n} V_{n+3,k}(u_1+u_2+u_3)du_1\right]^2 du_2du_3$$
$$= \frac{1}{n^2}\int_0^{1/n}\int_0^{1/n}\left[\int_0^{1/n} \frac{V_{n+3,k}(u_1+u_2+u_3)}{\sqrt{V_{n+2,k}\left(u_1+\frac{1}{n}\right)}}\sqrt{V_{n+2,k}\left(u_1+\frac{1}{n}\right)}du_1\right]^2 du_2du_3$$
$$\le \frac{1}{n^2}\int_0^{1/n}\int_0^{1/n}\left[\int_0^{1/n} \frac{V_{n+3,k}^2(u_1+u_2+u_3)}{V_{n+2,k}\left(u_1+\frac{1}{n}\right)}du_1 \times \int_0^{1/n} V_{n+2,k}\left(u_1+\frac{1}{n}\right)du_1\right] du_2du_3.$$

But

$$\int_0^{1/n} V_{n+2,k}\left(u_1+\frac{1}{n}\right)du_1 \le n(k+1)\int_0^{1/n}\int_0^{1/n} V_{n+2,k}(u_1+u_2)du_1du_2.$$

Indeed, this is true because

$$\int_0^{1/n} V_{n+2,k}(u_1+u_2)du_2$$
$$\ge \binom{n+k+1}{k}\int_0^{1/n}(u_1+u_2)^k\left(1+u_1+\frac{1}{n}\right)^{-(n+k+2)}du_2$$
$$= \binom{n+k+1}{k}\left(1+u_1+\frac{1}{n}\right)^{-(n+k+2)}\frac{\left(u_1+\frac{1}{n}\right)^{k+1}-u_1^{k+1}}{k+1}$$

and it follows from

$$\left(u_1 + \frac{1}{n}\right)^{k+1} - u_1^{k+1} \geq \frac{1}{n}\left(u_1 + \frac{1}{n}\right)^k.$$

Then

$$\left[\int_0^{1/n}\int_0^{1/n}\int_0^{1/n} V_{n+3,k}(u_1+u_2+u_3)du_1du_2du_3\right]^2$$
$$\leq \frac{k+1}{n}\int_0^{1/n}\int_0^{1/n}\int_0^{1/n} \frac{V_{n+3,k}^2(u_1+u_2+u_3)}{V_{n+2,k}\left(u_1+\frac{1}{n}\right)}du_1du_2du_3$$
$$\times \int_0^{1/n}\int_0^{1/n} V_{n+2,k}(u_1+u_2)du_1du_2$$

and consequently

$$I_k \leq \frac{n(n+1)(n+2)^2}{(k+1)(n+k+2)}\frac{k+1}{n}\int_0^{1/n}\int_0^{1/n}\int_0^{1/n} \frac{V_{n+3,k}^2(u_1+u_2+u_3)}{V_{n+2,k}\left(\frac{1}{n}+u_1\right)}du_1du_2du_3$$
$$= (n+1)(n+2)^2\int_0^{1/n}\int_0^{1/n}\int_0^{1/n} \frac{V_{n+3,k}^2(u_1+u_2+u_3)}{(n+k+2)V_{n+2,k}\left(\frac{1}{n}+u_1\right)}du_1du_2du_3.$$

Let

$$A = \frac{(u_1+u_2+u_3)^2}{\frac{1}{n}+u_1}, \quad B = \frac{(1+u_1+u_2+u_3)^2}{1+\frac{1}{n}+u_1}.$$

Then

$$\frac{V_{n+3,k}^2(u_1+u_2+u_3)}{(n+k+2)V_{n+2,k}\left(\frac{1}{n}+u_1\right)}$$
$$= \frac{\binom{n+k+2}{k}^2(u_1+u_2+u_3)^{2k}(1+u_1+u_2+u_3)^{-2(n+k+3)}}{(n+k+2)\binom{n+k+1}{k}\left(\frac{1}{n}+u_1\right)^k\left(1+\frac{1}{n}+u_1\right)^{-(n+k+2)}}$$
$$= \frac{1}{n+2}\binom{n+k+2}{k}A^kB^{-(n+k+3)}\frac{1}{1+\frac{1}{n}+u_1}$$
$$\leq \frac{n}{(n+1)(n+2)}\binom{n+k+2}{k}A^kB^{-(n+k+3)},$$

and

$$\sum_{k=0}^{\infty} I_k \leq n(n+2)\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}\sum_{k=0}^{\infty}\binom{n+k+2}{k}A^kB^{-(n+k+3)}du_1du_2du_3.$$

We have

$$\sum_{k=1}^{\infty}\binom{n+k+2}{k}A^kB^{-(n+k+3)}=(B-A)^{-n-3}-B^{-n-3}$$

and

$$B-A=1-\frac{\left(u_2+u_3-\frac{1}{n}\right)^2}{\left(\frac{1}{n}+u_1\right)\left(1+\frac{1}{n}+u_1\right)}\geq 1-\frac{\left(u_2+u_3-\frac{1}{n}\right)^2}{u_1+\frac{1}{n}+\frac{1}{n^2}}.$$

Then

$$\begin{aligned}(B-A)^{-n-3}&\leq\left[1-\frac{\left(u_2+u_3-\frac{1}{n}\right)^2}{u_1+\frac{1}{n}+\frac{1}{n^2}}\right]^{-n-3}\\&=\left[\frac{u_1+\frac{1}{n}+\frac{1}{n^2}}{u_1+\frac{1}{n}+\frac{1}{n^2}-\left(u_2+u_3-\frac{1}{n}\right)^2}\right]^{n+3}\\&=\left[1+\frac{(u_2+u_3-\frac{1}{n})^2}{u_1+\frac{1}{n}+\frac{1}{n^2}-\left(u_2+u_3-\frac{1}{n}\right)^2}\right]^{n+3}\\&\leq\left[1+\frac{\left(u_2+u_3-\frac{1}{n}\right)^2}{u_1+\frac{1}{n}}\right]^{n+3}\leq e^{\frac{(n+3)\left(u_2+u_3-\frac{1}{n}\right)^2}{u_1+\frac{1}{n}}}.\end{aligned}$$

These inequalities are true for $0\leq u_i\leq\dfrac{1}{n}$.

Now

$$B=1+u_1+\frac{1}{n}+2\left(u_2+u_3-\frac{1}{n}\right)+\frac{\left(u_2+u_3-\frac{1}{n}\right)^2}{1+u_1+\frac{1}{n}},$$

and from

$$\left(u_2+u_3-\frac{1}{n}\right)^2\leq\frac{1}{n^2},\quad 1+u_1+\frac{1}{n}\geq 1+\frac{1}{n}$$

it follows

$$B\leq 1-\frac{1}{n+1}+u_1+2u_2+2u_3.$$

Then

$$B^{-n-3}\geq\left(1-\frac{1}{n+1}+u_1+2u_2+2u_3\right)^{-n-3}\geq e^{(n+3)\left(\frac{1}{n+1}-u_1-2u_2-2u_3\right)},$$

and consequently

$$\sum_{k=1}^{\infty} I_k$$
$$\leq n(n+2) \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} \left(e^{\frac{(n+3)\left(u_2+u_3-\frac{1}{n}\right)^2}{u_1+\frac{1}{n}}} - e^{(n+3)\left(\frac{1}{n+1}-u_1-2u_2-2u_3\right)} \right) du_1du_2du_3.$$

From

$$0 \leq \frac{\left(u_2+u_3-\frac{1}{n}\right)^2}{u_1+\frac{1}{n}} \leq \frac{1}{n} \quad \text{and} \quad -\frac{5}{n} \leq -u_1-2u_2-2u_3 \leq 0$$

it follows

$$e^{\frac{(n+3)\left(u_2+u_3-\frac{1}{n}\right)^2}{u_1+\frac{1}{n}}} \leq e^{\frac{n\left(u_2+u_3-\frac{1}{n}\right)^2}{u_1+\frac{1}{n}}} e^{3/n} \leq \left(1+\frac{C}{n}\right) e^{\frac{n\left(u_2+u_3-\frac{1}{n}\right)^2}{u_1+\frac{1}{n}}} \leq e^{\frac{n\left(u_2+u_3-\frac{1}{n}\right)^2}{u_1+\frac{1}{n}}} + \frac{C}{n},$$

and

$$e^{(n+3)\left(\frac{1}{n+1}-u_1-2u_2-2u_3\right)} \geq e^{n\left(\frac{1}{n}-u_1-2u_2-2u_3\right)} e^{-15/n} \geq \left(1-\frac{15}{n}\right) e^{n\left(\frac{1}{n}-u_1-2u_2-2u_3\right)}$$
$$\geq e^{n\left(\frac{1}{n}-u_1-2u_2-2u_3\right)} - \frac{C}{n}.$$

So,

$$\sum_{k=1}^{\infty} I_k$$
$$\leq n(n+2) \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} \left(e^{\frac{n\left(u_2+u_3-\frac{1}{n}\right)^2}{u_1+\frac{1}{n}}} - e^{n\left(\frac{1}{n}-u_1-2u_2-2u_3\right)} \right) du_1du_2du_3 + \frac{C}{n^2}$$
$$= \frac{n+2}{n^2} \int_0^1\int_0^1\int_0^1 \left(e^{\frac{(t_2+t_3-1)^2}{t_1+1}} - e^{1-t_1-2t_2-2t_3} \right) dt_1dt_2dt_3 + \frac{C}{n^2}$$
$$\leq \frac{1}{n} \int_0^1\int_0^1\int_0^1 \left(e^{\frac{(t_2+t_3-1)^2}{t_1+1}} - e^{1-t_1-2t_2-2t_3} \right) dt_1dt_2dt_3 + \frac{C}{n^2}$$
$$= \frac{1}{n} \left[\int_0^1\int_0^1\int_0^1 e^{\frac{(t_2+t_3-1)^2}{t_1+1}} dt_1dt_2dt_3 - \frac{e}{4}(1-e^{-1})(1-e^{-2})^2 \right] + \frac{C}{n^2}.$$

Using the inequality (2.7) we get

$$\sum_{k=1}^{\infty} I_k \leq \left(1-\left(1-e^{-1}\right)^4\right)\frac{1}{n}+\frac{C}{n^2}.$$

Now we will estimate I_0.

$$I_0 = \frac{\left[n(n+1)(n+2)\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}(1+u_1+u_2+u_3)^{-n-3}du_1du_2du_3\right]^2}{(n+2)n(n+1)\int_0^{1/n}\int_0^{1/n}(1+u_1+u_2)^{-n-2}du_1du_2}$$

$$= n(n+1)(n+2)\frac{\left[\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}(1+u_1+u_2+u_3)^{-n-3}du_1du_2du_3\right]^2}{\int_0^{1/n}\int_0^{1/n}(1+u_1+u_2)^{-n-2}du_1du_2}.$$

From the simple inequality

$$\frac{1}{(1+u_1+u_2+u_3)^{n+3}} = \left(1-\frac{u_1+u_2+u_3}{1+u_1+u_2+u_3}\right)^{n+3} \leq e^{-(n+3)\frac{u_1+u_2+u_3}{1+u_1+u_2+u_3}},$$

we get the next estimation

$$\begin{aligned}\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}\frac{du_1du_2du_3}{(1+u_1+u_2+u_3)^{n+3}} &\leq \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} e^{-(n+3)\frac{u_1+u_2+u_3}{1+u_1+u_2+u_3}}du_1du_2du_3\\ &\leq \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} e^{-(n+3)\frac{u_1+u_2+u_3}{1+\frac{3}{n}}}du_1du_2du_3\\ &= \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} e^{-n(u_1+u_2+u_3)}du_1du_2du_3\\ &= \frac{1}{n^3}\left(1-e^{-1}\right)^3.\end{aligned}$$

We have as well

$$\begin{aligned}\int_0^{1/n}\int_0^{1/n}\frac{du_1du_2}{(1+u_1+u_2)^{n+2}} &\geq \int_0^{1/n}\int_0^{1/n} e^{-(n+2)(u_1+u_2)}du_1du_2\\ &= \frac{\left(1-e^{-\frac{n+2}{n}}\right)^2}{(n+2)^2} \geq \frac{\left(1-e^{-1}\right)^2}{(n+2)^2}.\end{aligned}$$

Then

$$I_0 \leq n(n+1)(n+2)\frac{\left(n^{-3}\left(1-e^{-1}\right)^3\right)^2}{\left(1-e^{-1}\right)^2(n+2)^{-2}} = \frac{(n+1)(n+2)^3}{n^5}\left(1-e^{-1}\right)^4.$$

But

$$\frac{(n+1)(n+2)^3}{n^5} \le \frac{1}{n} + \frac{C}{n^2} \quad \text{and then} \quad I_0 \le \frac{1}{n}\left(1-e^{-1}\right)^4 + \frac{C}{n^2}.$$

From this and the above for $\sum_{k=1}^{\infty}$ it follows that

$$I \le \frac{1}{n}\left(1+\frac{C}{n}\right) \le \frac{1}{n+2}\left(1+\frac{C}{n}\right).$$

Case 2. $1 \le j$.

Again, by using two two times Cauchy's inequality we have

$$\frac{T_{3,k}^2\left(\frac{j}{n}\right)}{T_{2,k}\left(\frac{j}{n}\right)} \le \left(1+\frac{C}{n}\right) n^3 \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} \frac{V_{n+3,k}^2\left(\frac{j}{n}+u_1+u_2+u_3\right)}{V_{n+2,k}\left(\frac{j}{n}+u_1+u_2\right)} du_1du_2du_3$$

Let

$$A = \frac{\left(\frac{j}{n}+u_1+u_2+u_3\right)^2}{\frac{j}{n}+u_1+u_2}, \quad B = \frac{\left(1+\frac{j}{n}+u_1+u_2+u_3\right)^2}{1+\frac{j}{n}+u_1+u_2}.$$

From the above it follows that

$$I_k \le \frac{\left(1+\frac{C}{n}\right)n^3}{(k+1)(n+k+2)} \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} \frac{\binom{n+k+2}{k}^2 A^k B^{-(n+k+3)}}{\binom{n+k+1}{k}\left(1+\frac{j}{n}+u_1+u_2\right)} du_1du_2du_3$$

$$\le \left(1+\frac{C}{n}\right) n^2 \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} \frac{\binom{n+k+2}{k} A^k B^{-(n+k+3)}}{(k+1)\left(1+\frac{j}{n}+u_1+u_2\right)} du_1du_2du_3$$

and

$$I \le \left(1+\frac{C}{n}\right) n^2 \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} \frac{B^{-n-3}}{1+\frac{j}{n}+u_1+u_2} \sum_{k=0}^{\infty} \frac{\binom{n+k+2}{k}}{k+1}\left(\frac{A}{B}\right)^k du_1du_2du_3.$$

But

$$\sum_{k=0}^{\infty} \frac{\binom{n+k+2}{k}}{k+1}\left(\frac{A}{B}\right)^k = \frac{B^{n+3}(B-A)^{-n-2}}{(n+2)A} - \frac{B}{(n+2)A} \le \frac{B^{n+3}(B-A)^{-n-2}}{(n+2)A}.$$

Consequently

$$I \le \left(1+\frac{C}{n}\right) n \int_0^{1/n}\!\!\int_0^{1/n}\!\!\int_0^{1/n} A^{-1}\left(1+\frac{j}{n}+u_1+u_2\right)^{-1}(B-A)^{-n-2}du_1du_2du_3.$$

Now we will prove that

$$(B-A)^{-n-2} \le \left(1+\frac{C}{n}\right)e^{\frac{n^2u_3^2}{j+nu_1+nu_2}}. \tag{2.22}$$

Indeed,

$$B-A = 1-\frac{u_3^2}{\psi^2\left(\frac{j}{n}+u_1+u_2\right)} = 1-\frac{u_3^2}{\left(\frac{j}{n}+u_1+u_2\right)\left(1+\frac{j}{n}+u_1+u_2\right)}$$

$$= 1-\frac{u_3^2}{z(1+z)},$$

where $z=\dfrac{j}{n}+u_1+u_2$. Then

$$(B-A)^{-n-2} = \left[1+\frac{u_3^2}{z(1+z)-u_3^2}\right]^{n+2} \le e^{\frac{(n+2)u_3^2}{z(1+z)-u_3^2}}.$$

Now for $0\le u_3\le \dfrac{1}{n}$ we have

$$z(1+z)-u_3^2 \ge \left(\frac{j}{n}+u_1+u_2\right)\left(1+\frac{j}{n}+u_1+u_2\right)-\frac{1}{n^2}$$
$$=\frac{j}{n}+u_1+u_2+\left(\frac{j}{n}+u_1+u_2\right)^2-\frac{1}{n^2}\ge \frac{j}{n}+u_1+u_2,$$

and consequently

$$(B-A)^{-n-2} \le e^{\frac{n(n+2)u_3^2}{j+nu_1+nu_2}}.$$

But from

$$\frac{u_3^2}{j+nu_1+nu_2}\le \frac{1}{n^2},$$

it follows that

$$e^{\frac{2nu_3^2}{j+nu_1+nu_2}} \le e^{\frac{2}{n}} \le 1+\frac{C}{n} \quad \text{and} \quad e^{\frac{n(n+2)u_3^2}{j+nu_1+nu_2}} \le \left(1+\frac{C}{n}\right) e^{\frac{n^2u_3^2}{j+nu_1+nu_2}},$$

and (2.22) is proved.

Putting (2.22) in the above we get

$$I \le \left(1+\frac{C}{n}\right) n \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} e^{\frac{n^2u_3^2}{j+nu_1+nu_2}} \frac{\frac{j}{n}+u_1+u_2}{\left(\frac{j}{n}+u_1+u_2+u_3\right)^2} \frac{du_1du_2du_3}{\left(1+\frac{j}{n}+u_1+u_2\right)}$$
$$\le \frac{1+\frac{C}{n}}{n\left(1+\frac{j}{n}\right)} \int_0^1\int_0^1\int_0^1 \frac{j+t_1+t_2}{(j+t_1+t_2+t_3)^2} e^{\frac{t_3^2}{j+t_1+t_2}} dt_1dt_2dt_3,$$

and using the inequality (2.8) we have

$$I \le \frac{1+\frac{C}{n}}{(j+1)(n+j+2)}.$$

With this (2.21) is proved and Lemma 2.5 as well. □

3. Proof of Theorem 1.1. The method to prove Theorem 1.1 is based on the idea of Ivanov and Ditzian [2] by using $g = V_n^N(f)$ in the K-functional.

We have

$$K_\psi\left(f,\frac{1}{n}\right) = \inf\left\{\|f-g\| + \frac{1}{n}\|\psi^2D^2g\|\right\} \le \|f-V_nf\| + \frac{1}{n}\,\|\psi^2D^2(V_nf)\|$$

which means that it is sufficient to show that, for some constant C,

(3.1) $$\frac{1}{n}\,\|\psi^2D^2(V_nf)\| \le C\|f-V_nf\|.$$

We have

$$\frac{1}{n}\,\|\psi^2D^2\big(V_nf\big)\| = \frac{1}{n}\|\psi^2D^2\big(V_nf - V_n^{N+1}f + V_n^{N+1}f\big)\|$$
$$\le \frac{1}{n}\|\psi^2D^2V_n(f-V_n^Nf)\| + \frac{1}{n}\|\psi^2D^2\big(V_n^{N+1}f\big)\|.$$

Applying (2.6) and (2.1) we obtain

$$\frac{1}{n}\|\psi^2D^2V_n(f-V_n^Nf)\| \le C_3\|f-V_n^Nf\| \le C_3\sum_{i=0}^{N-1}\|V_n^if-V_n^{i+1}f\| \le C_3N\|f-V_nf\|.$$

For the second term, using Theorem 2.1, Theorem 2.2 and (2.1) we have

$$
\begin{aligned}
&\frac{1}{2n}\|\psi^2 D^2\big(V_n^{N+1}f\big)\| \\
&\qquad \le \|V_n^{N+2}f - V_n^{N+1}f - \frac{1}{2n}\psi^2 D^2\big(V_n^{N+1}f\big)\| + \|V_n^{N+2}f - V_n^{N+1}f\| \\
&\qquad \le C_4 n^{-3/2}\|\psi^3 D^3\big(V_n^{N+1}f\big)\| + \|f - V_n f\| \\
&\qquad \le C_4\frac{K(N)}{n}\|\psi^2 D^2(V_n f)\| + \|f - V_n f\|
\end{aligned}
$$

or

$$
\frac{1}{n}\|\psi^2 D^2(V_n f)\| \le (C_3 N + 2)\|f - V_n f\| + 2C_4\,\frac{K(N)}{n}\|\psi^2 D^2(V_n f)\|.
$$

Because of $\lim\limits_{N\to\infty} K(N) = 0$, we can choose N such that $2C_4K(N) \le \dfrac{1}{2}$ which is possible since C_4 is an absolute constant and then fixing this, say $\dfrac{L-2}{2}$, we have (3.1) for all $n \ge L$ with constant $C_3(L-2)+4$.

This completes the proof of Theorem 1.1.

4. Proof of Theorem 2.2. We will prove Theorem 2.2 following the method of Knoop and Zhou [5]. In order to prove it we will need some additional lemmas. For the first one we need a representation of the third derivative of $V_n^N f$. Let us first compute the second derivative. Using (2.5),

$$
\begin{aligned}
&\big(D^2(V_n^2 f)\big)(x) \\
&\quad = \big(D^2 V_n(V_n f)\big)(x) \\
&\quad = \sum_{k_2=0}^{\infty} n(n+1)\int_0^{1/n}\!\!\int_0^{1/n} (D^2 V_n f)\left(\frac{k_2}{n} + u_2 + v_2\right) du_2 dv_2 \cdot V_{n+2,k_2}(x) \\
&\quad = \sum_{k_2=0}^{\infty} n(n+1)\int_0^{1/n}\!\!\int_0^{1/n} \sum_{k_1=0}^{\infty} n(n+1)\int_0^{1/n}\!\!\int_0^{1/n} (D^2 f)\left(\frac{k_1}{n} + u_1 + v_1\right) du_1 dv_1 \\
&\qquad\qquad \times V_{n+2,k_1}\left(\frac{k_2}{n} + u_2 + v_2\right) du_2 dv_2 \cdot V_{n+2,k_2}(x),
\end{aligned}
$$

and inductively

$$\begin{aligned}
&\left(D^2(V_n^N f)\right)(x) \\
&= \sum_{k_1=0}^{\infty}\cdots\sum_{k_N=0}^{\infty} n(n+1)\int_0^{1/n}\!\!\int_0^{1/n}(D^2 f)\left(\frac{k_1}{n}+u_1+v_1\right)du_1dv_1\cdot V_{n+2,k_N}(x) \\
&\times\prod_{j=1}^{N-1} n(n+1)\int_0^{1/n}\!\!\int_0^{1/n} V_{n+2,k_j}\left(\frac{k_{j+1}}{n}+u_{j+1}+v_{j+1}\right)du_{j+1}dv_{j+1}.
\end{aligned}$$

Before continuing we will introduce some notations, which will be very useful later.

(4.1)
$$\sum_N = \sum_{k_1=0}^{\infty}\cdots\sum_{k_N=0}^{\infty},$$

(4.2)
$$P(k_1,\dots,k_N;n) = \prod_{j=1}^{N-1} T_{2,k_j}\left(\frac{k_{j+1}}{n}\right),$$

(4.3)
$$l_j = l_j(k_j,k_{j+1};n) = \frac{T_{3,k_j}\left(\frac{k_{j+1}}{n}\right)}{T_{2,k_j}\left(\frac{k_{j+1}}{n}\right)},$$

(4.4)
$$l_j^* = l_j^*(k_j,k_{j+1};n) = \frac{(n+2)\int_0^{1/n}(DT_{2,k_j})\left(\frac{k_{j+1}}{n}+t_{j+1}\right)dt_{j+1}}{T_{2,k_j}\left(\frac{k_{j+1}}{n}\right)}, \qquad 1\le j\le N-2,$$

(4.5)
$$Q_j = Q_j\left(k_j,\dots,k_N;n\right) = l_j^* l_{j+1}\cdots l_{N-1}, \quad 1\le j\le N-2,$$

(4.6)
$$\begin{aligned}
l_{N-1}^* &= l_{N-1}^*(k_{N-1},k_N;n) = Q_{N-1} \\
&= \frac{(n+2)\int_0^{1/n}(DT_{2,k_{N-1}})\left(\frac{k_N}{n}+t_N\right)dt_N}{T_{2,k_{N-1}}\left(\frac{k_N}{n}\right)},
\end{aligned}$$

and

(4.7)
$$Q = \sum_{i=1}^{N-1} Q_i.$$

We note that all of the quantities above are well defined. In this notations the second derivative can be written as

$$(4.8)\quad \big(D^2(V_n^N f)\big)(x) = \sum_N n(n+1)\int_0^{1/n}\int_0^{1/n}(D^2 f)\Big(\frac{k_1}{n}+u_1+v_1\Big)du_1dv_1 \times V_{n+2,k_N}(x)P(k_1,\dots,k_N;n)\,.$$

For the third derivative we prove an analogous expression in the next lemma.

Lemma 4.1. *In notations* (4.1), (4.2), (4.5) *the next identity is true*

$$(4.9)\quad \big(D^3(V_n^N f)\big)(x) = \sum_N n(n+1)\int_0^{1/n}\int_0^{1/n}(D^2 f)\left(\frac{k_1}{n}+u_1+v_1\right)du_1dv_1 \times V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\,Q_j$$

for all $1 \le j \le N-1$.

P r o o f. We will prove this identity by induction.
Differentiating (4.8) we obtain for the third derivative

$$(4.10)\quad \big(D^3(V_n^N f)\big)(x) = \sum_N n(n+1)\int_0^{1/n}\int_0^{1/n}(D^2 f)\Big(\frac{k_1}{n}+u_1+v_1\Big)du_1dv_1 \times (DV_{n+2,k_N})(x)P(k_1,\dots,k_N;n)\,.$$

Let us first prove (4.9) for $j = N-1$. Using (2.3) we have

$$\begin{aligned}
&\sum_{k_N=0}^{\infty}(DV_{n+2,k_N})(x)P(k_1,\dots,k_N;n)\\
&\quad=\sum_{k_N=0}^{\infty}(n+2)\left[V_{n+3,k_N-1}(x)-V_{n+3,k_N}(x)\right]T_{2,k_{N-1}}\left(\frac{k_N}{n}\right)P(k_1,\dots,k_{N-1};n)\\
&\quad=(n+2)P(k_1,\dots,k_{N-1};n)\\
&\qquad\times\left[\sum_{k_N=0}^{\infty}V_{n+3,k_N-1}(x)T_{2,k_{N-1}}\left(\frac{k_N}{n}\right)-\sum_{k_N=0}^{\infty}V_{n+3,k_N}(x)T_{2,k_{N-1}}\left(\frac{k_N}{n}\right)\right]
\end{aligned}$$

$$= (n+2)P(k_1,\dots,k_{N-1};n) \times \sum_{k_N=0}^{\infty}\left[V_{n+3,k_N}(x)T_{2,k_{N-1}}\left(\frac{k_N+1}{n}\right) - V_{n+3,k_N}(x)T_{2,k_{N-1}}(\frac{k_N}{n})\right]$$

$$= (n+2)P(k_1,\dots,k_{N-1};n) \times \sum_{k_N=0}^{\infty}\left[T_{2,k_{N-1}}\left(\frac{k_N+1}{n}\right) - T_{2,k_{N-1}}\left(\frac{k_N}{n}\right)\right]V_{n+3,k_N}(x)$$

$$= (n+2)P(k_1,\dots,k_{N-1};n)\sum_{k_N=0}^{\infty}\int_0^{1/n}(DT_{2,k_{N-1}})\left(\frac{k_N}{n}+t_N\right)\,dt_N\cdot V_{n+3,k_N}(x).$$

Putting this in (4.10) we get

$$\begin{aligned}\big(D^3(V_n^N f)\big)(x) &= \sum_N n(n+1)(n+2)\int_0^{1/n}\int_0^{1/n}(D^2f)\left(\frac{k_1}{n}+u_1+v_1\right)du_1dv_1 \\ &\quad\times\frac{P(k_1,\dots,k_N;n)\int_0^{1/n}(DT_{2,k_{N-1}})\left(\frac{k_N}{n}+t_N\right)dt_N}{T_{2,k_{N-1}}\left(\frac{k_N}{n}\right)}V_{n+3,k_N}(x)\\ &= \sum_N n(n+1)\int_0^{1/n}\int_0^{1/n}(D^2f)\left(\frac{k_1}{n}+u_1+v_1\right)du_1dv_1\\ &\qquad\times V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)Q_{N-1}.\end{aligned}$$

This proves (4.9) for $j = N-1$.

Now, let us suppose (4.9) is true for $j = m$. Then we have

$$\text{(4.11)}\quad \sum_{k=0}^{\infty} g\left(\frac{k}{n}\right)(n+2)\int_0^{1/n}(DT_{2,k})(x+t)\,dt = \sum_{k=0}^{\infty}(n+2)\int_0^{1/n}(Dg)\left(\frac{k}{n}+t\right)dt\cdot T_{3,k}(x).$$

Indeed,

$$\sum_{k=0}^{\infty}(n+2)\int_0^{1/n}(Dg)\left(\frac{k}{n}+t\right)dt\cdot T_{3,k}(x)$$

$$= n(n+1)(n+2)^2 \sum_{k=0}^{\infty} \int_0^{1/n} (Dg)\left(\frac{k}{n}+t\right) dt$$
$$\times \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} V_{n+3,k}(x+t_1+t_2+t_3)\, dt_1 dt_2 dt_3$$
$$= n(n+1)(n+2)^2 \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} \sum_{k=0}^{\infty} \left[g\left(\frac{k+1}{n}\right) - g\left(\frac{k}{n}\right)\right]$$
$$\times V_{n+3,k}(x+t_1+t_2+t_3) dt_1 dt_2 dt_3$$
$$= n(n+1)(n+2)^2 \int_0^{1/n}\int_0^{1/n}\int_0^{1/n} \sum_{k=0}^{\infty} g\left(\frac{k}{n}\right)$$
$$\times \left(V_{n+3,k-1} - V_{n+3,k}\right)(x+t_1+t_2+t_3)\, dt_1 dt_2 dt_3$$
$$= \sum_{k=0}^{\infty} g\left(\frac{k}{n}\right)(n+2)\int_0^{1/n} n(n+1)\int_0^{1/n}\int_0^{1/n} (DV_{n+2,k})(x+t_1+t_2+t_3)\, dt_1 dt_2 dt_3$$
$$= \sum_{k=0}^{\infty} g\left(\frac{k}{n}\right)(n+2)\int_0^{1/n} (DT_{2,k})(x+t)\, dt.$$

Obviously, we can replace in the left side of (4.11)

$$g\left(\frac{k}{n}\right) \quad \text{with} \quad (n+2)\int_0^{1/n} (Dg)\left(\frac{k}{n}+t\right) dt$$

and

$$(n+2)\int_0^{1/n} (DT_{2,k})(x+t) dt \quad \text{with} \quad T_{3,k}(x)$$

and consequently (4.11) is proved.

Therefore using (4.11) and summing with respect to indeces k_m we can replace

$$T_{2,k_{m-1}}\left(\frac{k_m}{n}\right) \quad \text{with} \quad (n+2)\int_0^{1/n} (DT_{2,k_{m-1}})\left(\frac{k_m}{n}+t\right) dt,$$

$$(n+2)\int_0^{1/n} (DT_{2,k_m})\left(\frac{k_{m+1}}{n}+t\right) dt \quad \text{with} \quad T_{3,k_m}\left(\frac{k_{m+1}}{n}\right),$$

$$T_{2,k_{m-1}}\left(\frac{k_m}{n}\right) T_{2,k_m}\left(\frac{k_{m+1}}{n}\right) l_m^* \quad \text{with} \quad T_{2,k_{m-1}}\left(\frac{k_m}{n}\right) T_{2,k_m}\left(\frac{k_{m+1}}{n}\right) l_{m-1}^* l_m.$$

That means $P(k_1,\dots,k_N;n)\,Q_m$ changes to $P(k_1,\dots,k_N;n)\,Q_{m-1}$ and (4.9) is true for $j=m-1$ and, consequently for all $1\le j\le N-1$. $\square$

The next two lemmas are proved in sections 5 and 6.

Lemma 4.2. *For* $V_{n,k}$ *defined by* (1.2), $P(k_1,\dots,k_N;n)$ *by* (4.2), Q *by* (4.7), $n\ge 10$

$$\sum_N V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\,Q^2\le CnN\psi^{-2}(x)$$

is true for $2\le N\le n$.

Lemma 4.3. *For* $V_{n,k}$ *defined by* (1.2), $P(k_1,\dots,k_N;n)$ *by* (4.2), $n\ge 10$

$$\sum_N \psi^{-4}\left(\frac{k_1+1}{n}\right)V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\le CN^{3/4}\ln N\psi^{-4}(x)$$

is true for $2\le N\le \dfrac{n-2}{2}$.

P r o o f o f T h e o r e m 2.2. Summing up (4.9) with respect to j, $1\le j\le N-1$, we obtain

$$\left(D^3(V_n^N f)\right)(x)=\frac{1}{N-1}\sum_N n(n+1)\int_0^{1/n}\!\!\int_0^{1/n}(D^2f)\left(\frac{k_1}{n}+u_1+v_1\right)du_1dv_1 \times V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\,Q.$$

By Cauchy's inequality we have

$$\left|\left(D^3(V_n^N f)\right)(x)\right|\le\frac{n(n+1)}{N-1}\sqrt{\sum_N V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\,Q^2} \times\sqrt{\sum_N\left[\int_0^{1/n}\!\!\int_0^{1/n}(D^2f)\Bigl(\frac{k_1}{n}+u_1+v_1\Bigr)du_1dv_1\right]^2V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)}. \tag{4.12}$$

But

$$\Bigl|\int_0^{1/n}\!\!\int_0^{1/n}(D^2f)\left(\frac{k_1}{n}+u_1+v_1\right)du_1dv_1\Bigr| \le\|\psi^2D^2f\|\int_0^{1/n}\!\!\int_0^{1/n}\psi^{-2}\left(\frac{k_1}{n}+u_1+v_1\right)du_1dv_1$$

and

$$\int_0^{1/n}\int_0^{1/n}\psi^{-2}\left(\frac{k_1}{n}+u_1+v_1\right)du_1dv_1 \le \frac{1}{1+\frac{k_1}{n}}\int_0^{1/n}\frac{du_1}{\sqrt{\frac{k_1}{n}+u_1}}\int_0^{1/n}\frac{dv_1}{\sqrt{\frac{k_1}{n}+v_1}}$$
$$\le \frac{8}{1+\frac{k_1+1}{n}}\left(\sqrt{\frac{k_1+1}{n}}-\sqrt{\frac{k_1}{n}}\right)^2.$$

Because of

$$\sqrt{\frac{k_1+1}{n}}-\sqrt{\frac{k_1}{n}}=\frac{1}{n\left(\sqrt{\frac{k_1+1}{n}}+\sqrt{\frac{k_1}{n}}\right)}\le\frac{1}{n\sqrt{\frac{k_1+1}{n}}},$$

we obtain

$$\int_0^{1/n}\int_0^{1/n}\psi^{-2}\left(\frac{k_1}{n}+u_1+v_1\right)du_1dv_1 \le \frac{8}{n^2\psi^2\left(\frac{k_1+1}{n}\right)},$$

which means

$$\sum_N\left[\int_0^{1/n}\int_0^{1/n}(D^2f)\left(\frac{k_1}{n}+u_1+v_1\right)du_1dv_1\right]^2 V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)$$
$$\le\frac{64}{n^4}\|\psi^2D^2f\|^2\sum_N\psi^{-4}\left(\frac{k_1+1}{n}\right)V_{n+3,k_N}(x)P(k_1,\dots,k_N;n) \tag{4.13}$$

Observe that from (4.12), (4.13), Lemma 4.2 and Lemma 4.3 we obtain

$$\left|\left(D^3(V_n^Nf)\right)(x)\right|\le\frac{n(n+1)}{N-1}\sqrt{CnN\psi^{-2}(x)}\,\frac{8}{n^2}\|\psi^2D^2f\|\sqrt{\psi^{-4}(x)CN^{3/4}\ln N}$$
$$\le C\sqrt{n}\,\|\psi^2D^2f\|\psi^{-3}(x)\sqrt{N^{-1/4}\ln N}$$

and Theorem 2.2 follows immediately if we set $C\sqrt{N^{-1/4}\ln N}=K(N)$. □

5. Proof of Lemma 4.2. First, we will prove an identity for the numbers Q_i, which will allow us to replace Q^2 by $\sum_i Q_i^2$ in some sence.

Lemma 5.1. *For* $V_{n,k}$ *defined by* (1.2), $P(k_1,\dots,k_N;n)$ *by* (4.2), Q_i *by* (4.5) *and* (4.6), *and* $x\in(0,\infty)$,

$$\sum_N V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\,Q_iQ_j=0,\qquad i\ne j.$$

P r o o f. We have

$$\sum_N V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\,Q_iQ_j$$

$$=\sum_{k_1=0}^{\infty}T_{2,k_1}\left(\frac{k_2}{n}\right)\sum_{k_2=0}^{\infty}\cdots\sum_{k_N=0}^{\infty}V_{n+3,k_N}(x)Q_iQ_j\prod_{\nu=2}^{N-1}T_{2,k_\nu}\left(\frac{k_{\nu+1}}{n}\right).$$

Let $i<j$. Observe that Q_i does not depend on k_m, $m=1,2,\dots,i-1$, so we may sum with respect to all k_m, $m=1,2,\dots,i-1$. And using the equality

$$\sum_{k_m=0}^{\infty}T_{2,k_m}\left(\frac{k_{m+1}}{n}\right)=n(n+1)\int_0^{1/n}\int_0^{1/n}\sum_{k_m=0}^{\infty}V_{n+2,k_m}\left(\frac{k_{m+1}}{n}+t_1+t_2\right)dt_1dt_2$$

$$=\frac{n+1}{n}$$

we obtain

$$\sum_N V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\,Q_iQ_j$$

$$=\sum_{k_1=0}^{\infty}T_{2,k_1}\left(\frac{k_2}{n}\right)\cdots\sum_{k_{i-1}=0}^{\infty}T_{2,k_{i-1}}\left(\frac{k_i}{n}\right)$$

$$\times\sum_{k_i=0}^{\infty}\cdots\sum_{k_N=0}^{\infty}V_{n+3,k_N}(x)Q_iQ_j\prod_{\nu=i}^{N-1}T_{2,k_\nu}\left(\frac{k_{\nu+1}}{n}\right)$$

$$=\left(\frac{n+1}{n}\right)^{i-1}\sum_{k_i=0}^{\infty}\cdots\sum_{k_N=0}^{\infty}V_{n+3,k_N}(x)Q_iQ_j\prod_{\nu=i}^{N-1}T_{2,k_\nu}\left(\frac{k_{\nu+1}}{n}\right)$$

$$=\left(\frac{n+1}{n}\right)^{i-1}\sum_{k_N=0}^{\infty}\cdots\sum_{k_{i+1}=0}^{\infty}H(k_{i+1},\dots,k_N)\sum_{k_i=0}^{\infty}Q_iT_{2,k_i}\left(\frac{k_{i+1}}{n}\right)$$

where $H(k_{i+1},\dots,k_N)$ does not depend on k_i.

Now by using the notations (4.3) and (4.4) we get Lemma 5.1 from

$$\sum_{k_i=0}^{\infty}Q_iT_{2,k_i}\left(\frac{k_{i+1}}{n}\right)=\sum_{k_i=0}^{\infty}l_i^*l_{i+1}\cdots l_{N-1}T_{2,k_i}\left(\frac{k_{i+1}}{n}\right)$$

$$=\left(\sum_{k_i=0}^{\infty}(n+2)\int_0^{1/n}(DT_{2,k_i})\left(\frac{k_{i+1}}{n}+t_{i+1}\right)dt_{i+1}\right)l_{i+1}\cdots l_{N-1}$$

$$= (n+2)l_{i+1}\cdots l_{N-1}\int_0^{1/n}\sum_{k_i=0}^{\infty}(DT_{2,k_i})\left(\frac{k_{i+1}}{n}+t_{i+1}\right)dt_{i+1} = 0$$

because

$$\sum_{k_i=0}^{\infty}(DT_{2,k_i})\left(\frac{k_{i+1}}{n}+t_{i+1}\right) = 0. \qquad \square$$

P r o o f o f L e m m a 4.2. By Lemma 5.1 we have

$$\begin{aligned}
&\sum_N V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\,Q^2 \\
&\qquad = \sum_{i=1}^{N-1}\sum_N V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\,Q_i^2 \\
&\qquad\qquad + \sum_{i\neq j}\sum_N V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\,Q_iQ_j \\
&\qquad = \sum_{i=1}^{N-1}\sum_N V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\,Q_i^2.
\end{aligned}$$

Consequently, it is enough to show

$$\sum_N V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\,Q_i^2 \le Cn\psi^{-2}(x), \qquad i=1,2,\dots,n-1.$$

The left side is

$$\begin{aligned}
&\sum_N V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)\,Q_m^2 \\
&\quad = \sum_N V_{n+3,k_N}(x)\prod_{j=1}^{N-1}T_{2,k_j}\left(\frac{k_{j+1}}{n}\right)\left(l_m^* l_{m+1}\cdots l_{N-1}\right)^2 \\
&\quad = \sum_{k_N=0}^{\infty}V_{n+3,k_N}(x)\sum_{k_{N-1}=0}^{\infty}T_{2,k_{N-1}}\left(\frac{k_N}{n}\right)l_{N-1}^2\cdots\sum_{k_{m+1}=0}^{\infty}T_{2,k_{m+1}}\left(\frac{k_{m+2}}{n}\right)l_{m+1}^2 \\
&\qquad \times\sum_{k_m=0}^{\infty}T_{2,k_m}\left(\frac{k_{m+1}}{n}\right)(l_m^*)^2\sum_{k_{m-1}=0}^{\infty}T_{2,k_{m-1}}\left(\frac{k_m}{n}\right)\cdots\sum_{k_1=0}^{\infty}T_{2,k_1}\left(\frac{k_2}{n}\right)
\end{aligned}$$

$$= \left(\frac{n+1}{n}\right)^{m-1} \sum_{k_N=0}^{\infty} V_{n+3,k_N}(x) \sum_{k_{N-1}=0}^{\infty} T_{2,k_{N-1}}\left(\frac{k_N}{n}\right) l_{N-1}^2$$
$$\cdots \sum_{k_{m+1}=0}^{\infty} T_{2,k_{m+1}}\left(\frac{k_{m+2}}{n}\right) l_{m+1}^2 \sum_{k_m=0}^{\infty} T_{2,k_m}\left(\frac{k_{m+1}}{n}\right) (l_m^*)^2.$$

Now

$$\sum_{k_m=0}^{\infty} T_{2,k_m}\left(\frac{k_{m+1}}{n}\right)(l_m^*)^2$$
$$= \sum_{k_m=0}^{\infty} T_{2,k_m}\left(\frac{k_{m+1}}{n}\right)\left[\frac{(n+2)\int_0^{1/n}(DT_{2,k_m})\left(\frac{k_{m+1}}{n}+t_{m+1}\right)dt_{m+1}}{T_{2,k_m}\left(\frac{k_{m+1}}{n}\right)}\right]^2$$
$$= n(n+1)(n+2)^2$$
$$\times \sum_{k_m=0}^{\infty} \frac{\left[\int_0^{1/n}\int_0^{1/n}\int_0^{1/n}(DV_{n+2,k})\left(\frac{k_{m+1}}{n}+t_1+t_2+t_3\right)dt_1dt_2dt_3\right]^2}{\int_0^{1/n}\int_0^{1/n}V_{n+2,k}\left(\frac{k_{m+1}}{n}+t_1+t_2\right)dt_1dt_2}.$$

And by Lemma 2.2

$$\sum_{k_m=0}^{\infty} T_{2,k_m}\left(\frac{k_{m+1}}{n}\right)(l_m^*)^2 \le n(n+1)(n+2)^2\frac{C_5}{n^3}\psi^{-2}\left(\frac{k_{m+1}+1}{n+1}\right)$$
$$\le \frac{C_8 n^3}{(k_{m+1}+1)(n+k_{m+1}+2)}.$$

Then

$$\sum_{k_{m+1}=0}^{\infty} T_{2,k_{m+1}}\left(\frac{k_{m+2}}{n}\right) l_{m+1}^2 \sum_{k_m=0}^{\infty} T_{2,k_m}\left(\frac{k_{m+1}}{n}\right)(l_m^*)^2$$
$$\le \sum_{k_{m+1}=0}^{\infty} \frac{T_{3,k_{m+1}}^2\left(\frac{k_{m+2}}{n}\right)}{T_{2,k_{m+1}}\left(\frac{k_{m+2}}{n}\right)} \frac{C_8 n^3}{(k_{m+1}+1)(n+k_{m+1}+2)},$$

and by Lemma 2.5

$$\sum_{k_{m+1}=0}^{\infty} T_{2,k_{m+1}}\left(\frac{k_{m+2}}{n}\right) l_{m+1}^2 \sum_{k_m=0}^{\infty} T_{2,k_m}\left(\frac{k_{m+1}}{n}\right)(l_m^*)^2$$
$$\leq \frac{C_8\left(1+\frac{C_7}{n}\right)n^3}{(k_{m+2}+1)(n+k_{m+2}+2)}.$$

Recurrently, for $N \leq n$, we obtain

$$\sum_N V_{n+3,k_N}(x)P(k_1,\ldots,k_N;n)\,Q_m^2 \leq \left(\frac{n+1}{n}\right)^{m-1}\frac{C_8 n^3\left(1+\frac{C_7}{n}\right)^{N-m}}{n^2x(1+x)} \leq \frac{Cn}{\psi^2(x)}.$$

The proof of Lemma 4.2 is complete. □

6. Proof of Lemma 4.3. The equivalence

$$\frac{1}{2}\psi^{-4}(x) \leq \frac{1}{x^2(1+x)^4}+\frac{1}{(1+x)^4} \leq \psi^{-4}(x) \tag{6.1}$$

implies

$$\sum_{k_1=0}^{\infty}\psi^{-4}\left(\frac{k_1+1}{n}\right)T_{2,k_1}\left(\frac{k_2}{n}\right)$$
$$\leq 2\sum_{k_1=0}^{\infty}\frac{T_{2,k_1}\left(\frac{k_2}{n}\right)}{\left(\frac{k_1+1}{n}\right)^2\left(1+\frac{k_1+1}{n}\right)^4}+2\sum_{k_1=0}^{\infty}\frac{T_{2,k_1}\left(\frac{k_2}{n}\right)}{\left(1+\frac{k_1+1}{n}\right)^4}.$$

Using the simple equality

$$\frac{1}{(k+1)^2}=\int_0^1\int_0^1 (w_1w_2)^k dw_1dw_2,$$

we obtain for the first sum on the right side

$$\sum_{k_1=0}^{\infty}\frac{T_{2,k_1}\left(\frac{k_2}{n}\right)}{\left(\frac{k_1+1}{n}\right)^2\left(1+\frac{k_1+1}{n}\right)^4} = n^2\int_0^1\int_0^1\sum_{k_1=0}^{\infty}\frac{(w_1w_2)^{k_1}T_{2,k_1}\left(\frac{k_2}{n}\right)dw_1dw_2}{\left(1+\frac{k_1+1}{n}\right)^4}$$
$$\leq n^2\int_0^1\int_0^1\sum_{k_1=0}^{\infty}\frac{(w_1w_2)^{k_1}T_{2,k_1}\left(\frac{k_2}{n}\right)dw_1dw_2}{\left(1+\frac{k_1}{n}\right)^4}.$$

Now, by (2.19) we get

$$\sum_{k_1=0}^{\infty}\frac{T_{2,k_1}\left(\frac{k_2}{n}\right)}{\left(\frac{k_1+1}{n}\right)^2\left(1+\frac{k_1+1}{n}\right)^4}\le\frac{\left(1+\frac{4}{n}\right)n^2}{\left(1+\frac{k_2}{n}\right)^4}\left(\frac{n}{b(n-2)}\right)^2$$
$$\times\int_0^1\!\!\int_0^1\!\!\left(\frac{1-e^{-\frac{(n-2)}{n}b(1-w_1w_2)}}{1-w_1w_2}\right)^2 e^{-\frac{(n-2)}{n}b(1-w_1w_2)k_2}dw_1dw_2+\frac{C_6\left(1+\frac{4}{n}\right)}{n^{1/4}(1+\frac{k_2}{n})^4},$$

and consequently by (6.1) and (2.20),

$$\begin{aligned}\sum_{k_1=0}^{\infty}&\psi^{-4}\left(\frac{k_1+1}{n}\right)T_{2,k_1}\left(\frac{k_2}{n}\right)\\ &\le 2\sum_{k_1=0}^{\infty}\frac{T_{2,k_1}\left(\frac{k_2}{n}\right)}{\left(\frac{k_1+1}{n}\right)^2\left(1+\frac{k_1+1}{n}\right)^4}+2\sum_{k_1=0}^{\infty}\frac{T_{2,k_1}\left(\frac{k_2}{n}\right)}{\left(1+\frac{k_1+1}{n}\right)^4}\\ &\le\frac{2\left(1+\frac{4}{n}\right)n^2}{\left(1+\frac{k_2}{n}\right)^4}\left(\frac{n}{b(n-2)}\right)^2\\ &\qquad\times\int_0^1\int_0^1\left(\frac{1-e^{-\frac{(n-2)}{n}b(1-w_1w_2)}}{1-w_1w_2}\right)^2 e^{-\frac{(n-2)}{n}b(1-w_1w_2)k_2}dw_1dw_2\\ &\quad+\frac{2\left(1+\frac{4}{n}\right)\left(1+C_6n^{-1/4}\right)}{\left(1+\frac{k_2}{n}\right)^4}.\end{aligned}\tag{6.2}$$

Now we define two sequences of functions, which we will need later.

For $0\le t\le 1$, $n=3,4,\dots$, $N=1,2,\dots$ and b given by (2.15), we define $H_j(t)$ and $h_j(t)$ in the following way.

$$H_0(t)=1-t,\quad H_{j+1}(t)=1-e^{-\frac{n-2}{n}bH_j(t)},\quad j=0,1,\dots,N-2\tag{6.3}$$

and

$$h_j(t)=H_j(1-t),\quad j=0,1,\dots,N-2.\tag{6.4}$$

It is clear that

$$h_0(t)=t,\quad h_{j+1}(t)=1-e^{-\frac{n-2}{n}bh_j(t)},\quad j=0,1,\dots,N-2.\tag{6.5}$$

Using the sequence $H_j(t)$, (6.2) can be rewritten as

$$\sum_{k_1=0}^{\infty} \psi^{-4}\left(\frac{k_1+1}{n}\right) T_{2,k_1}\left(\frac{k_2}{n}\right)$$
$$\leq \frac{2\left(1+\frac{4}{n}\right)n^2}{b^2\left(1+\frac{k_2}{n}\right)^4}\left(\frac{n}{n-2}\right)^2 \int_0^1\int_0^1 \frac{H_1^2(w_1w_2)}{H_0^2(w_1w_2)} e^{-\frac{n-2}{n}bH_0(w_1w_2)k_2} dw_1dw_2$$
$$+\frac{2\left(1+\frac{4}{n}\right)\left(1+C_6n^{-1/4}\right)}{\left(1+\frac{k_2}{n}\right)^4}.$$

Multiplying by $T_{2,k_2}\left(\frac{k_3}{n}\right)$ and summing with respect to k_2, using (2.19) and (2.20) with $a=\exp\left(-\frac{n-2}{n}bH_0\right)$

$$\sum_{k_2=0}^{\infty}\sum_{k_1=0}^{\infty} \psi^{-4}\left(\frac{k_1+1}{n}\right) T_{2,k_1}\left(\frac{k_2}{n}\right) T_{2,k_2}\left(\frac{k_3}{n}\right)$$
$$\leq \frac{2\left(1+\frac{4}{n}\right)^2 n^2}{b^4\left(1+\frac{k_3}{n}\right)^4}\left(\frac{n}{n-2}\right)^4 \int_0^1\int_0^1 \frac{H_2^2(w_1w_2)}{H_0^2(w_1w_2)} e^{-\frac{n-2}{n}bH_1(w_1w_2)k_3} dw_1dw_2$$
$$+\frac{2\left(1+\frac{4}{n}\right)^2\left(1+2C_6n^{-1/4}\right)}{\left(1+\frac{k_3}{n}\right)^4},$$

because

$$H_1(w_1w_2)=1-e^{-\frac{n-2}{n}bH_0(w_1w_2)} \leq \frac{n-2}{n}bH_0(w_1w_2)$$

or

$$\frac{H_1^2(w_1w_2)}{H_0^2(w_1w_2)} \leq \left(\frac{n-2}{n}b\right)^2.$$

Inductively we have

$$H_j(w_1w_2) \leq \left(\frac{n-2}{n}b\right)^j H_0(w_1w_2) \quad \text{and} \quad \frac{H_j^2(w_1w_2)}{H_0^2(w_1w_2)} \leq \left(\frac{n-2}{n}b\right)^{2j}.$$

So, after N times and using (2.11), (2.20), (2.12) we obtain

$$\text{(6.6)}\quad \sum_N \psi^{-4}\left(\frac{k_1+1}{n}\right) V_{n+3,k_N}(x)P(k_1,\dots,k_N;n)$$
$$\le \frac{2\left(1+\frac{4}{n}\right)^N n^2}{b^{2N}(1+x)^4}\Big(\frac{n}{n-2}\Big)^{2N}\int_0^1\int_0^1 \frac{H_{N-1}^2(w_1w_2)}{H_0^2(w_1w_2)}e^{-(n-1)bH_{N-1}(w_1w_2)x}dw_1dw_2$$
$$+\frac{2\left(1+\frac{4}{n}\right)^N\!\left(1+NC_6n^{-1/4}\right)}{(1+x)^4}.$$

We need to estimate the double integral in (6.6). We will do it in Lemma 6.2 and will use the sequence $h_j(t)$, defined by (6.4).

Lemma 6.1. *For the sequence* $h_j(t)$*, defined by* (6.4) *or* (6.5)

$$\text{(6.7)}\qquad \left(\frac{n-2}{n}b\right)^N\left(t-\frac{Nt^2}{2}\right) \le h_{N-1}(t) \le \left(\frac{n-2}{n}b\right)^N t.$$

P r o o f. From Taylor's formula

$$h_{N-1}(t) = h_{N-1}(0) + h'_{N-1}(0)t + \frac{1}{2}h''_{N-1}(0)t^2 + \frac{1}{6}h'''_{N-1}(\theta t),\quad 0<\theta<1.$$

$$h'_{j+1}(t) = \frac{n-2}{n}bh'_j(t)e^{-\frac{n-2}{n}bh_j(t)} = \dots = \Big(\frac{n-2}{n}b\Big)^{j+1}e^{-\frac{n-2}{n}b\sum_{i=0}^{j}h_i(t)},$$

and consequently

$$h'_{N-1}(t) = \Big(\frac{n-2}{n}b\Big)^{N-1}e^{-\frac{n-2}{n}b\sum_{i=0}^{N-2}h_i(t)}.$$

From

$$h_0(0)=0 \quad\text{we get}\quad h_1(0) = 1-e^{-\frac{n-2}{n}bh_0(0)} = 0,$$

and inductively

$$h_j(0)=0 \quad\text{and}\quad h'_{N-1}(0) = \left(\frac{n-2}{n}b\right)^{N-1}.$$

Also

$$h''_j(t) = -\left(\frac{n-2}{n}b\right)^{j+1}e^{-\frac{n-2}{n}b\sum_{i=0}^{j-1}h_i(t)}\sum_{i=0}^{j-1}h'_i(t).$$

From

$$h'_j(0) = \Big(\frac{n-2}{n}b\Big)^j \le 1$$

we have

$$\sum_{i=0}^{N-2} h_i'(0) \le N-1 < N \quad \text{and} \quad |h_{N-1}''(0)| \le \Big(\frac{n-2}{n}b\Big)^N N.$$

$$h_{N-1}'''(t) = \Big(\frac{n-2}{n}b\Big)^{N+1} e^{-\frac{n-2}{n}b\sum_{i=0}^{N-2} h_i(t)} \Big(\sum_{i=0}^{N-2} h_i'(t)\Big)^2 - \Big(\frac{n-2}{n}b\Big)^N e^{-\frac{n-2}{n}b\sum_{i=0}^{N-2} h_i(t)} \sum_{i=0}^{N-2} h_i''(t).$$

Since $h_i'(t) \ge 0$ we get $h_j''(t) \le 0$ and consequently $h_{N-1}'''(t) \ge 0$. Then the left inequality in (6.7) follows immediately.

$$\begin{aligned} h_{N-1}(t) &\ge \Big(\frac{n-2}{n}b\Big)^{N-1} t - \Big(\frac{n-2}{n}b\Big)^N \frac{Nt^2}{2} \\ &\ge \Big(\frac{n-2}{n}b\Big)^N t - \Big(\frac{n-2}{n}b\Big)^N \frac{Nt^2}{2} = \Big(\frac{n-2}{n}b\Big)^N \left(t - \frac{Nt^2}{2}\right). \end{aligned}$$

At the same time

$$h_{j+1}(t) = 1 - e^{-\frac{n-2}{n}bh_j(t)} \le \frac{n-2}{n}bh_j(t), \quad h_0(t) = t$$

and inductively

$$h_j(t) \le \Big(\frac{n-2}{n}b\Big)^j t.$$

Combining we have

$$\left(\frac{n-2}{n}b\right)^N \left(t - \frac{Nt^2}{2}\right) \le h_{N-1}(t) \le \left(\frac{n-2}{n}b\right)^N t. \qquad \square$$

Lemma 6.2. *For the sequence $H_j(t)$, defined by (6.3) there exists an absolute constant C_9 such that*

$$\int_0^1 \int_0^1 \frac{H_{N-1}^2(w_1w_2)}{H_0^2(w_1w_2)} e^{-(n-1)bH_{N-1}(w_1w_2)x} dw_1 dw_2 \le \frac{C_9 \ln N}{(nx)^2}.$$

P r o o f. First,

$$H_{N-1}^2(t)e^{-(n-1)bH_{N-1}(t)x} \le \frac{C}{(nx)^2}$$

because of the fact that for

$$a \ge 0, \quad y^2e^{-ay} \le 2a^{-2}$$

and consequently

$$H_{N-1}^2(t)e^{-(n-1)bH_{N-1}(t)x} \le \frac{2}{((n-1)\,bx)^2} \le \frac{C}{(nx)^2}.$$

Now,

$$H_0(t) = 1 - t \ge \frac{1}{2} \quad \text{if} \quad t \le \frac{1}{2} \quad \text{and} \quad H_0(w_1w_2) \ge \frac{1}{2} \quad \text{if} \quad w_1 \le \frac{1}{2} \quad \text{or} \quad w_2 \le \frac{1}{2},$$

so

$$\int_0^1\int_0^1 \frac{H_{N-1}^2(w_1w_2)}{H_0^2(w_1w_2)}e^{-(n-1)bH_{N-1}(w_1w_2)x}dw_1dw_2$$

$$\le \int_{1/2}^1\int_{1/2}^1 \frac{H_{N-1}^2(w_1w_2)}{H_0^2(w_1w_2)}e^{-(n-1)bH_{N-1}(w_1w_2)x}dw_1dw_2 + \frac{C}{(nx)^2}.$$

If we make the substitute $w_1w_2 = u, w_1 = s$ in the integral, we obtain

$$\int_{1/2}^1\int_{1/2}^1 \frac{H_{N-1}^2(w_1w_2)}{H_0^2(w_1w_2)}e^{-(n-1)bH_{N-1}(w_1w_2)x}dw_1dw_2$$

$$= \int_{1/2}^1 \frac{1}{s}\int_{s/2}^s \frac{H_{N-1}^2(u)}{H_0^2(u)}e^{-(n-1)bH_{N-1}(u)x}duds$$

$$\le 2\int_{1/2}^1\int_{s/2}^s \frac{H_{N-1}^2(u)}{H_0^2(u)}e^{-(n-1)bH_{N-1}(u)x}duds$$

$$\le 2\int_{1/2}^1\int_{1/2}^s \frac{H_{N-1}^2(u)}{H_0^2(u)}e^{-(n-1)bH_{N-1}(u)x}duds + \frac{C}{(nx)^2}$$

$$= 2\int_{1/2}^1\int_u^1 \frac{H_{N-1}^2(u)}{H_0^2(u)}e^{-(n-1)bH_{N-1}(u)x}dsdu + \frac{C}{(nx)^2}$$

$$= 2\int_{1/2}^1 \frac{H_{N-1}^2(u)}{1-u}e^{-(n-1)bH_{N-1}(u)x}du + \frac{C}{(nx)^2}.$$

Obviously, it is sufficient to show that

$$\int_{1/2}^{1} \frac{H_{N-1}^2(u)}{1-u} e^{-(n-1)bH_{N-1}(u)x} du \le \frac{C \ln N}{(nx)^2}. \tag{6.8}$$

Let $1-u=t$. Then

$$\begin{aligned}\int_{1/2}^{1} \frac{H_{N-1}^2(u)}{1-u} e^{-(n-1)bH_{N-1}(u)x} du &= \int_{0}^{1/2} \frac{h_{N-1}^2(t)}{t} e^{-(n-1)bh_{N-1}(t)x} dt \\ &= \int_0^{2/(3N)} + \int_{2/(3N)}^{1/2}.\end{aligned}$$

For the first one, using Lemma 6.1, we have

$$\begin{aligned}&\int_0^{2/(3N)} \frac{h_{N-1}^2(t)}{t} e^{-(n-1)bh_{N-1}(t)x} dt \\ &\qquad \le \int_0^{2/(3N)} \Big(\frac{n-2}{n}b\Big)^{2N} t e^{-(n-1)\left(\frac{n-2}{n}b\right)^N \left(t-\frac{Nt^2}{2}\right)bx} dt \\ &\qquad \le \Big(\frac{n-2}{n}b\Big)^{2N} \int_0^{2/(3N)} t e^{-\frac{2}{3}(n-1)t\left(\frac{n-2}{n}b\right)^N bx} dt.\end{aligned}$$

Computing

$$\int_0^{2/(3N)} t e^{-at} dt = -\frac{2}{3aN} e^{-\frac{2}{3N}a} - \frac{1}{a^2} e^{-\frac{2}{3N}a} + \frac{1}{a^2} \le \frac{1}{a^2},$$

we obtain

$$\begin{aligned}&\Big(\frac{n-2}{n}b\Big)^{2N} \int_0^{2/(3N)} t e^{-\frac{2}{3}(n-1)t\left(\frac{n-2}{n}b\right)^N bx} dt \\ &\qquad \le \Big(\frac{n-2}{n}b\Big)^{2N} \left[\frac{2}{3}(n-1)\left(\frac{n-2}{n}b\right)^N bx\right]^{-2} \\ &\qquad = \frac{9}{4(n-1)^2(bx)^2} \le \frac{C}{(nx)^2}.\end{aligned}$$

For the second one we use again

$$h_{N-1}^2(t) e^{-(n-1)bh_{N-1}(t)x} \le \frac{C}{\big((n-1)bx\big)^2} \le \frac{C}{(nx)^2}.$$

Then

$$\int_{2/(3N)}^{1/2} \frac{h_{N-1}^2(t)}{t} e^{-(n-1)bh_{N-1}(t)x} dt \le \frac{C}{(nx)^2} \int_{2/(3N)}^{1/2} \frac{dt}{t} \le \frac{C \ln N}{(nx)^2}.$$

This completes the proof of (6.8) and Lemma 6.2 as well. □

So, by using Lemma 6.2, we obtain from (6.6)

$$\begin{aligned}
&\sum_N \psi^{-4}\left(\frac{k_1+1}{n}\right) V_{n+3,k_N}(x) P(k_1,\dots,k_N;n) \\
&\quad \le \frac{\left(1+\frac{4}{n}\right)^N n^2}{b^{2N}(1+x)^4} \left(\frac{n}{n-2}\right)^{2N} \frac{C_9 \ln N}{(nx)^2} + \frac{2\left(1+\frac{4}{n}\right)^N \left(1+NC_6 n^{-1/4}\right)}{(1+x)^4} \\
&\quad \le \frac{C \ln N}{b^{2N} x^2 (1+x)^2} + \frac{CNn^{-1/4}}{x^2(1+x)^2} = C\left(\frac{\ln N}{b^{2N}} + Nn^{-1/4}\right)\psi^{-4}(x).
\end{aligned}$$

But

$$b^{-2N} = \left(1 + \frac{3}{4}\frac{\ln(n-2)}{n-2}\right)^{2N} \le \left(1 + \frac{3}{4}\frac{\ln(2N)}{2N}\right)^{2N} \le (2N)^{3/4},$$

because

$$\frac{\ln(n-2)}{n-2} \le \frac{\ln(2N)}{2N} \quad \text{for} \quad N \le \frac{n-2}{2}.$$

Then

$$\begin{aligned}
\sum_N \psi^{-4}\left(\frac{k_1+1}{n}\right) V_{n+3,k_N}(x) P(k_1,\dots,k_N;n) &\le C\left(N^{3/4}\ln N + N^{3/4}\right)\psi^{-4}(x) \\
&\le CN^{3/4} \ln N \psi^{-4}(x),
\end{aligned}$$

which proves Lemma 4.3. □

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REFERENCES

[1] V. A. Baskakov. An instance of a sequence of linear positive operators in the space of continuous functions. *Docl. Akad. Nauk SSSR* **113** (1957), 249–251 (in Russian).

[2] Z. Ditzian, K. G. Ivanov. Strong converse inequalities. *J. Anal. Math.* **61** (1993), 61–111.

[3] Z. Ditzian, V. Totik. Moduli of Smoothness. Springer Series in Computational Mathematics vol. **9**. New York, Springer-Verlag, 1987.

[4] Sh. Guo, Q. Qi. Strong converse inequalities for Baskakov operators. *J. Approx. Theory* **124**, *2* (2003), 219–231.

[5] H.-B. Knoop, X. L. Zhou. The lower estimate for linear positive operators II. *Results Math.* **25**, *3–4* (1994), 315–330.

[6] G. G. Lorentz. Bernstein Polynomials. Mathematical Expositions No 8. Toronto, University of Toronto Press, 1953.

[7] V. Totik. Approximation by Bernstein polynomials. *Amer. J. Math.* **116**, *4* (1994), 995–1018.

[8] V. Totik. Strong converse inequalities. *J. Approx. Theory* **76**, *3* (1994), 369–375.

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