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NOTES ON LATTICE ORDERED ALGEBRAS

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Abstract. In this paper we introduce a new class of lattice ordered algebra, which will be called a *pseudo-almost f-algebra*; namely, a lattice ordered algebra A in which $a \wedge b = 0$ in A implies $ab \wedge ba = 0$. We present some fundamental properties of pseudo-almost f-algebras and consider their relationships with various type of lattice ordered algebras; mainely, f-algebras, almost f-algebras and d-algebras.

1. Introduction. The importance of the certain lattice ordered algebras in the theory of vector lattices has steadily grown and much attention has been paid to the classes of f-algebras, almost f-algebras and d-algebras. Very little attention has been paid to lattice ordered algebras which differ from any of these certain lattice ordered algebras. Henriksen [7] recently indicated his idea to see workers in lattice ordered algebras to start paying much more attention to lattice ordered algebras rather than f-algebras. In order to contribute to this idea, there have recently been some researches in this direction. Papers [5] and [6] introduce

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new classes of lattice ordered algebras so-called a *pseudo f-algebra* (a lattice ordered algebra A having the property that $ab = 0$ if $a \wedge b$ is a nilpotent element of A) and *generalized almost f-algebra* (a lattice ordered algebra A such that ab is an annihilator of A if $a \wedge b = 0$), respectively. As a consequence of results given in [5] and [6], we see that the implications

$$f\text{-algebra} \Rightarrow \text{pseudo } f\text{-algebra} \Rightarrow \text{almost } f\text{-algebra} \Rightarrow \text{generalized almost } f\text{-algebra}$$

hold, but not conversely. So various other classes of lattice ordered algebras have recently been getting attention, which offers an interesting side road. In this aspect the present paper introduces a "new" class of lattice ordered algebra which is a wider class than each of an almost f-algebra and a d-algebra, but is in general independent from a generalized almost f-algebra by examples below. A lattice ordered algebra A in which $a \wedge b = 0$ in A implies $ab \wedge ba = 0$ is called a *pseudo-almost f-algebra*, which need not be an almost f-algebra or a d-algebra.

In this work we mainly focus on properties of pseudo-almost f-algebras and their relations with the other classes of lattice ordered algebras.

For the elementary theory of ℓ-spaces and terminology not explained here we refer to [1, 9, 11].

2. Definitions and properties. A lattice ordered space is said to be *Archimedean* if for each non zero $a \in A$ the set $\{na : n = \pm 1, \pm 2, \ldots\}$ has no upper bound in A. *All ℓ-spaces and ℓ-algebras under consideration are supposed to be Archimedean.* A (real) vector lattice (briefly, an ℓ-space) or a Riesz space A is said to be a *lattice ordered algebra* (briefly, an ℓ-algebra) or *a Riesz algebra* if it is a linear algebra (associative, but not necessarily commutative or unital) such that if $a, b \in A^+$, then $ab \in A^+$. An ℓ-algebra A is said to be a *f-algebra* if $a \wedge b = 0$ in A implies $ac \wedge b = ca \wedge b = 0$ for all $c \in A^+$. An ℓ-algebra A for which $a \wedge b = 0$ in A implies $ab = 0$ is called an *almost f-algebra.* We call an ℓ-algebra A a *d-algebra* if $a \wedge b = 0$ in A implies $ac \wedge bc = ca \wedge cb = 0$ for all $c \in A^+$.

The class of f-algebras, as given here, first appeared in a paper by Birkhoff and Pierce [3] in 1956, to be followed a decade later by the class of almost f-algebras introduced by Birkhoff [4]. Prior to that, in 1962 Kudláček [8] introduced the notion of d-algebras. In this paper we present their connections with pseudo-almost f-algebras. Although these various classes of algebras are distinct, there are relations between them; for example, from the definitions it is obvious that every f-algebra is both an almost f-algebra and a d-algebra, but not conversely. We next discuss some properties of these classes of ℓ-algebras. Every Archimedean

f-algebra is automatically associative and every almost f-algebra (and so an f-algebra) is commutative, but not necessarily associative. Archimedean almost f-algebras have positive squares (i.e., $a^2 \geq 0$ for all a), whereas d-algebras do not have these properties. Almost f-algebras and d-algebras share the property that $|a^2| = |a|^2$ for all a. An almost f-algebra and a d-algebra are in general independent. However, if some conditions are imposed, then we have the following relationships between the two classes. Any d-algebra which is commutative or has positive squares is an almost f-algebra. Moreover, under certain conditions, these three classes of ℓ-algebras turn out to be the same; that is, in any semi-prime (associative) ℓ-algebra the classes of f-algebras, almost f-algebras and d-algebras are equivalent. In particular, this holds for an (associative) ℓ-algebra with unit element $e > 0$. More details about these classes of ℓ-algebras can be found in [2]. In general, Archimedean pseudo-almost f-algebras need not be commutative, associative and have positive squares, as will be illustrated in the examples given below. Other relationships are investigated in the next section.

In this paper we also deal with quotient ℓ-algebras. A two-sided algebra ideal in an ℓ-algebra which is also a solid linear subspace is called an *lattice ideal* (briefly, *ℓ-ideal*). If A is an Archimedean ℓ-space and I is a solid subspace in A then the *quotient ℓ-space* A/I of all equivalence classes modulo I is an ℓ-space with respect to the partial ordering, $[a] \leq [b]$ whenever there exist elements $a_1 \in [a]$ and $b_1 \in [b]$ such that $a_1 \leq b_1$ (see, for example, [9, Theorem 18.9]). Moreover, $[x] \wedge [y] = [x \wedge y]$ and $[x] \vee [y] = [x \vee y]$ for all $x, y \in A$. In general, A/I is not Archimedean even if A is Archimedean ([9, Example 60.1]). The necessary and sufficient conditions for A/I to be Archimedean is that I is uniformly closed ([9, Example 60.2]). If A is an algebra and I is an algebra ideal in A, then $[x] \cdot [y] = [xy]$ for all $x, y \in A$. Hence, in the case that $I = N(A)$, the nilpotent elements of A, the space $A/N(A)$ is semi-prime. (An element a in an algebra A is said to be *nilpotent* if $a^k = 0$ for some positive integer k, and an algebra A is called *semi-prime* if $N(A) = \{0\}$.)

We are now in a position to introduce a new class of ℓ-algebra and establish some relation with the other main classes of ℓ-algebras.

Definition 2.1. An ℓ-algebra A is said to be a *pseudo-almost f-algebra* if

$$a \wedge b = 0 \textit{ in } A \textit{ implies } ab \wedge ba = 0.$$

Theorem 2.2. **(1)** *Every almost f-algebra (and so pseudo f-algebra) is a pseudo-almost f-algebra.*

(2) *Every d-algebra is a pseudo-almost f-algebra.*

Proof. **(1)** Let A be an almost f-algebra and $a \wedge b = 0$ in A. It follows from $a, b \in A^+$ that $ab, ba \in A^+$, and so $0 \leq ab \wedge ba \leq ab = 0$; that is, $ab \wedge ba = 0$.

(2) Let A be an d-algebra and $a \wedge b = 0$ in A. Now $a^2, b^2 \in A^+$, and so

$$\begin{aligned} 0 \leq ab \wedge ba &\leq (a^2 + ab) \wedge (ba + b^2) = a(a+b) \wedge b(a+b) \\ &\leq (a \wedge b)(a+b) = 0. \end{aligned}$$

Thus $ab \wedge ba = 0$, as required.

We now give two examples to show that the classes of almost f-algebras, d-algebras and pseudo-almost f-algebras are all distinct. Here although ℓ-algebras are supposed to be associative, we also discuss the case for non-associativity.

Example 2.3. Let $A = \mathbb{R}^4$ be with the usual operations of addition, scalar multiplication and partial ordering. We define multiplication by

$$(a_1, a_2, a_3, a_4)(b_1, b_2, b_3, b_4) = (a_2 b_2 + a_3 b_3, a_2 b_2, a_3 b_3, a_3 b_4).$$

It can easily be verified that A is an Archimedean associative pseudo-almost f-algebra. However, it is neither an almost f-algebra nor a d-algebra; for, if $a = (1,0,1,0)$, $b = (0,1,0,1)$ and $c = (1,1,1,1)$, then $a \wedge b = 0$ but $ab = (0,0,0,1)$ and $ca \wedge cb = (1,0,0,0)$. A is not semi-prime since $(1,0,0,0)^2 = 0$ but $(1,0,0,0) \neq 0$. A is not positive squares; for, $(0,0,1,-1)^2 = (0,0,1,-1)(0,0,1,-1) = (1,0,1,-1)$. We observe that A does not have a unit element, either. It can easily be checked that the nilpotent elements of A are given by

$$N(A) = \{(\alpha, 0, 0, \beta) \in A : \alpha, \beta \in \mathbb{R}\}.$$

Example 2.4. Let $A = \mathbb{R}^2$ be as Example 2.3 and define multiplication by

$$(a_1, a_2)(b_1, b_2) = (a_1 b_2 + a_2 b_2, a_2 b_2).$$

It is routine to show that A is an Archimedean non-associative pseudo-almost f-algebra. However, it is neither an almost f-algebra nor a d-algebra; for, if $a = (1,0)$, $b = (0,1)$ and $c = (1,1)$, then $a \wedge b = 0$ but $ac \wedge bc = (1,0)$ and $ab = (1,0)$. The Archimedean pseudo-almost f-algebra A is not commutative, either. A straightforward calculation shows that

$$N(A) = \{(\alpha, 0) \in A : \alpha \in \mathbb{R}\}.$$

It follows that an Archimedean pseudo-almost f-algebra is not in general semiprime. We also see that the algebra under consideration does not posses the property of being positive squares; for, if $a = (-2, 1)$, then $a^2 = (-1, 1)$.

Remark 2.5. As observed earlier, almost f-algebras and d-algebras share the property that $|a^2| = |a|^2$ for all a. However, in general, this property does not hold for pseudo-almost f-algebras; for, if $a = (-2, 1)$ in the pseudo-almost f-algebra A in Example 2.4, then $|a^2| = (1, 1)$ whereas $|a|^2 = (3, 1)$. A generalized almost f-algebra A has a remarkable property that $a^4 = (a^+)^4 + (a^-)^4 \geq 0$ (hence $|a^4| = |a|^4$) for all $a \in A$ [6], which does not hold for pseudo-almost f-algebras. This also shows that a pseudo-almost f-algebra is not necessarily a generalized almost f-algebra. By the following example which appears in [6], we see that these two classes are in general distinct.

Example 2.6. Let $A = \left\{ \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$ be with the usual operations of addition, scalar multiplication, partial ordering and multiplication $*$ defined by

$$\begin{pmatrix} 0 & a & b & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix} * \begin{pmatrix} 0 & a' & b' & c' \\ 0 & 0 & a' & b' \\ 0 & 0 & 0 & a' \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & aa' & ab' & a'b \\ 0 & 0 & 0 & aa' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then A is an Archimedean commutative generalized almost f-algebra which is not a pseudo-almost f-algebra.

We now give some characterizations of pseudo-almost f-algebras.

Theorem 2.7. *In an ℓ-algebra A, the following are equivalent.*
(1) *A is a pseudo-almost f-algebra.*
(2) *$a \perp b$ implies $ab \perp ba = 0$ for all $a, b \in A$.*
(3) *$(a - a \wedge b)(b - a \wedge b) \wedge (b - a \wedge b)(a - a \wedge b) = 0$ for all $a, b \in A$.*
(4) *$a^+a^- \wedge a^-a^+ = 0$ for all $a \in A$.*

Proof. The implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are clear. (We note that, for (3), $(a - a \wedge b) \wedge (b - a \wedge b) = (a \wedge b) - (a \wedge b) = 0$, which holds for any ℓ-algebra.)

(**3**) $\Rightarrow$ (**4**). Letting $b = 0$ in (3), we have (4).

(**4**) $\Rightarrow$ (**1**). Suppose that $a \wedge b = 0$ and let $c = a - b$. Then $c^+ = a$ and $c^- = b$ by the decomposition property of ℓ-spaces (see, for example, [1, Theorem 1.3]). Hence, by (4), $ab \wedge ba = 0$. This completes the proof. $\square$

Theorem 2.8. *For any pseudo-almost f-algebra A, we have*
(1) $aa^+ = (a^+)^2 - (a^-a^+ - a^-a^+)^+$ *for all* $a \in A$.
(2) $a^+a = ((a^+)^2 - a^+a^- + a^-a^+) \wedge (a^+)^2$ *for all* $a \in A$.
(3) $(a \wedge b)^2 = (a(a \wedge b) - (a-b)^+b) \vee (b(a \wedge b) - (b-a)^+a)$ *for all* $a, b \in A$.
(4) *It follows from* $a \wedge b = 0$ *in* A *that* $ab^2a \wedge ba^2b = 0$.

Proof. **(1)** If A is a pseudo-almost f-algebra and $a \in A$, then

$$\begin{aligned} aa^+ &= (a^+)^2 - a^-a^+ = (a^+)^2 - a^-a^+ + a^-a^+ \wedge a^+a^- \\ &= (a^+)^2 + (a^-a^+ - a^-a^+) \wedge (a^+a^- - a^-a^+) \\ &= (a^+)^2 - (a^-a^+ - a^-a^+)^+. \end{aligned}$$

(2) Is proved similarly.

(3) If A is a pseudo-almost f-algebra and $a, b \in A$, it follows from $(a - a \wedge b) \wedge (b - a \wedge b) = 0$ that $(a - a \wedge b)(b - a \wedge b) \wedge (b - a \wedge b)(a - a \wedge b) = 0$. Hence

$$(ab - a(a \wedge b) - (a \wedge b)b + (a \wedge b)^2) \wedge (ba - b(a \wedge b) - (a \wedge b)a + (a \wedge b)^2) = 0,$$

and so $((ab - a(a \wedge b) - (a \wedge b)b) \wedge (ba - b(a \wedge b) - (a \wedge b)a)) + (a \wedge b)^2 = 0$. This implies that

$$\begin{aligned} (a \wedge b)^2 &= (a(a \wedge b) + (a \wedge b)b - ab) \vee (b(a \wedge b) + (a \wedge b)a - ba) \\ &= (a(a \wedge b) + (a \wedge b - a)b) \vee (b(a \wedge b) + (a \wedge b - b)a) \\ &= (a(a \wedge b) - (a-b)^+b) \vee (b(a \wedge b) - (b-a)^+a). \end{aligned}$$

(4) Suppose that $a \wedge b = 0$ in a pseudo-almost f-algebra A. It follows that $ab \wedge ba = 0$, and so $0 = (ab)(ba) \wedge (ba)(ab) = ab^2a \wedge ba^2b$. □

In the end of this section we discuss quotients of ℓ-algebras. It is well-known that, if A is an f-algebra, then A/I is an f-algebra whenever I is an ℓ-ideal in A. Moreover, I has the property that $ca \wedge b, ac \wedge b \in I$ for all $c \in A^+$ whenever $a \wedge b \in I$ and $a, b \in A$. Likewise, $a \vee b \in I$ implies that $ca \vee b, ac \vee b \in I$ for all $c \in A^+$. A similar problem was considered by Bernau and Huijsmans in [2] and they shown that, if I is an ℓ-ideal in an almost f-algebra A, then A/I is an almost f-algebra, and furthermore, I has the property that $a, b \in A$ and $a \wedge b \in I$ or $a \vee b \in I$ imply $ab \in I$ [2, Proposition 3.6]. Analogously we prove the following result for pseudo-almost f-algebras.

Theorem 2.9. *If A is a pseudo-almost f-algebra and I is an ℓ-ideal of A, then the following hold.*

(1) *A/I is a pseudo-almost f-algebra.*

(2) *$ab \wedge ba \in I$ whenever either $a \wedge b \in I$ or $a \vee b \in I$.*

Proof. **(1)** Clearly the quotient A/I is an ℓ-algebra. In order to show that it is a pseudo-almost f-algebra, suppose that $[a] \wedge [b] = [0]$ in A/I. Since $a \wedge b \in I$, $[a] = [a - a \wedge b]$ and $[b] = [b - a \wedge b]$. It follows from $(a - a \wedge b)(b - a \wedge b) \wedge (b - a \wedge b)(a - a \wedge b) = 0$ that $[a] \cdot [b] \wedge [b] \cdot [a] = [0]$.

(2) Suppose that $a \wedge b \in I$. Then $I = (a \wedge b) + I = a + I \wedge b + I$, i.e., $[a] \wedge [b] = [a \wedge b] = [0]$ holds in A/I. By (1), since A/I is a pseudo-almost f-algebra, $[a] \cdot [b] \wedge [b] \cdot [a] = [0]$, which implies that $ab \wedge ba \in I$.

If $a \vee b \in I$, then $(-a) \wedge (-b) = -(a \vee b) \in I$. Hence $(-a)(-b) \wedge (-b)(-a) \in I$, i.e., $ab \wedge ba \in I$. $\square$

3. The relationships. As observed in preceding section, the classes of ℓ-algebras considered in this paper are in general distinct. For instant, a pseudo-almost f-algebra need be neither an almost f-algebra nor a d-algebra. If some extra condition is imposed however, then the situation improves. It was shown by Bernau and Huijsmans in [2, Theorem 4.3] that every commutative d-algebra is an almost f-algebra. A similar connection between almost f-algebras and pseudo-almost f-algebras holds as given in the following theorem.

Theorem 3.1. *A pseudo-almost f-algebra in which positive disjoint elements commute is an almost f-algebra. In particular, every commutative pseudo-almost f-algebra is an almost f-algebra.*

Proof. This follows immediately from the definition of a pseudo-almost f-algebra; for, $ab = ab \wedge ba = 0$ whenever $a \wedge b = 0$ in A. $\square$

Theorem 3.2. *Let A be an ℓ-algebra with unit element $e > 0$, then the following statements are equivalent.*

(1) *A is a pseudo-almost f-algebra.*

(2) *e is a weak order unit (i.e., $|a| \wedge e = 0$ implies $a = 0$ for all $a \in A$).*

Proof. $(\mathbf{1}) \Rightarrow (\mathbf{2})$. Let a be an element of A such that $|a| \wedge e = 0$. Then $|a|e \wedge e|a| = 0$ since A is a pseudo-almost f-algebra. Thus $|a| = 0$, and so $a = 0$.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$. If A is an ℓ-algebra with unit element $e > 0$ which is also a weak order unit, then A is an f-algebra [2, Corollary 1.10], and so it is a pseudo-almost f-algebra. $\square$

The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) in the next theorem are a combination of the results given in [2] for semi-prime ℓ-algebras. In particular, this is true for ℓ-algebras with unit element $e > 0$. We here give a direct and short proof.

Theorem 3.3. *Let A be a semi-prime ℓ-algebra, which is associative. Then the following statements are equivalent.*

(1) *A is an f-algebra.*

(2) *A is a d-algebra.*

(3) *A is an almost f-algebra.*

(4) *A is a generalized almost f-algebra.*

In particular, every semi-prime generalized almost f-algebra is a pseudo-almost f-algebra.

Proof. As already observed, (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) hold for an arbitrary ℓ-algebra.

(**2**) $\Rightarrow$ (**3**). Let A be a d-algebra and $a \wedge b = 0$ in A^+. Then $(ab)^2 = 0$ by Theorem 5.1 of [2], which implies that $ab = 0$ since A is semi-prime. Hence A is an almost f-algebra.

(**3**) $\Rightarrow$ (**1**). Suppose that $a \wedge b = 0$ in A. For all $c \in A^+$,

$$0 \leq (ca \wedge b)^2 = (ca \wedge b)(ca \wedge b) \leq (ca)b = c(ab) = 0;$$

that is, $(ca \wedge b)^2 = 0$, which implies that $ca \wedge b = 0$ since A is semi-prime. Similarly $ac \wedge b = 0$ for all $c \in A^+$.

(**4**) $\Rightarrow$ (**1**). If $a \wedge b = 0$ in A and $c \in A^+$, then $c(ab) = 0$ and $(ab)c = 0$ since A is a generalized almost f-algebra. It follows from $0 \leq (ca \wedge b)^2 \leq c(ab) = 0$ and $0 \leq (ca \wedge b)^2 \leq (ab)c = 0$ that $ca \wedge b = ac \wedge b = 0$, as required. □

As in the proof of Theorem 3.2, it can be seen that if A is an ℓ-algebra with unit element $e > 0$, then A is an almost f-algebra if and only if e is a weak order unit [2, Theorem 1.9]. On the other hand, since every f-algebra A with a unit element is semi-prime (see, for example, [10, Theorem 142.5]), in an ℓ-algebra A with unit element $e > 0$ the classes of f-algebras, almost f-algebras and d-algebras are equivalent, by Theorem 3.3. Thus summarizing our results gives the following relations amount the various classes of ℓ-algebras.

Corollary 3.4. *In an (associative) ℓ-algebra A with unit element $e > 0$, the following statements are equivalent.*

(1) *A is an f-algebra.*

(2) *A is an almost f-algebra.*

(3) *A is a d-algebra.*

(4) *A is a generalized almost f-algebra.*

(5) *A is a pseudo-almost f-algebra.*

(6) *e is a weak order unit.*

Remark 3.5. Recall that, by definition, an ℓ-algebra is supposed to be associative, however in some literatures it is not necessarily assumed to be associative in advance. In the preceding theorem we therefore emphasize the associativity of the ℓ-algebra A which plays an important role. Indeed, the ℓ-algebra A in the example given in [2, Remark 1.12] satisfies all the properties of an Archimedean semi-prime almost f-algebra apart from the associativity. Nevertheless, A is neither an f-algebra nor a d-algebra.

REFERENCES

[1] C. D. Aliprantis, O. Burkinshaw. Positive Oparators. Pure and Applied Mathematics vol. **119**, Orlando, FL, Academic Press, Inc., 1985.

[2] S. J. Bernau, C. B. Huijsmans. Almost f-algebras and d-algebras. *Math. Proc. Cambridge Philos. Soc.* **107**, *2* (1990), 287–308.

[3] G. Birkhoff, R. S. Pierce. Lattice-ordered rings. *An. Acad. Brasil. Ciénc.* **28** (1956), 41–49.

[4] G. Birkhoff. Lattice Theory. Third edition. American Mathematical Society Colloquium Publications vol. **XXV**, Providence, RI, American Mathematical Society, 1967.

[5] K. Boulabiar, F. Hadded. A class of Archimedean lattice ordered algebras. *Algebra Universalis* **50**, *3–4* (2003), 305–323.

[6] E. Chil. Generalized almost f-algebras. *Bull. Belg. Math. Soc. Simon Stevin* **16**, *2* (2009), 223–234.

[7] M. Henriksen. Old and new unsolved problems in lattice-ordered rings. In: Ordered Algebraic Structures. Proceedings of the conference on lattice-ordered groups and f-rings held at the University of Florida, Gainesville, FL, USA, February 28March 3, 2001 (Ed. J. Martinez). Dordrecht, Kluwer Academic Publishers. Developments in Mathematics **7** (2002), 11–17.

[8] V. Kudláček. On some types of ℓ-rings. *Sb. Vysoké. Učeni Tech. Brno 1–2* (1962), 179–181.

[9] W. A. J. Luxemburg, A. C. Zaanen. Riesz Spaces I. North-Holland Mathematical Library, vol. **XI**. Amsterdam-London, North-Holland Publishing Company, 1971.

[10] A. C. Zaanen. Riesz Spaces II. North-Holland Mathematical Library, Vol. **30**. Amsterdam-New York-Oxford, North-Holland Publishing Company, 1983.

[11] A. C. Zaanen. Introduction to Operator Theory in Riesz Spaces. Berlin, Springer, 1997.

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