Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **41** (2015), 49–82

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

OPERATOR ALGEBRAS: AN INTRODUCTION

Aristides Katavolos

Communicated by I. G. Todorov

Abstract. The first part of these notes contains a sketch of the elementary parts of C*-algebra theory, culminating in the two Gelfand–Naimark theorems. The final section is a presentation of the basic facts of the theory of weak-* closed (possibly non-selfadjoint) unital algebras containing maximal abelian selfadjoint algebras (masas), or more generally bimodules over masas.

The following is a brief and sketchy introduction to the rudiments of the theory of operator algebras, particularly C*-algebras. The text consists of rough lecture notes given by the author in the summer school in Operator Theory held in July 2011 at the University of the Aegean in Chios.

The notion of a C*-algebra is a fascinating common abstraction of the structure of two seemingly very different objects: on the one hand, the algebra of continuous functions on a (locally) compact space; and on the other, an algebra of bounded operators on Hilbert space closed in the norm and under adjoints.

2010 *Mathematics Subject Classification:* Primary 46L05; Secondary 47L55.
Key words: C*-algebras, Gelfand Theory; masa bimodules.

The aim of the first part of these notes (sections one to five) is to describe as briefly and as simply as possible this process of abstraction, culminating in the two Gelfand–Naimark theorems (3.9, 5.3). In the final section we give a brief sketch of what one can do when the 'adjoint' operation is not available, at least in the case where there exists a 'parametrization' in terms of '(possibly continuous) coordinates'.

Many proofs are only sketched and many others are omitted altogether.

1. C*-algebras: basics.

1.1. $\mathcal{B}(\mathcal{H})$.

In these notes, the action takes place via *bounded operators on Hilbert space* – either directly, or indirectly through representations by operators of more abstract structures (e.g. C*-algebras).

The space of all bounded linear operators $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is denoted $\mathcal{B}(\mathcal{H})$. It is complete under the norm

$$\|T\| = \sup\{\|Tx\| : x \in \mathrm{b}_1(\mathcal{H})\}$$

(here $\mathrm{b}_1(\mathcal{X})$ denotes the closed unit ball of a normed space $\mathcal{X}$) and is an *algebra* under composition. Moreover, because it acts on a Hilbert space, it has additional structure: an *involution* $T \to T^*$ defined via

$$\langle T^*x, y\rangle = \langle x, Ty\rangle \quad \text{for all } x, y \in \mathcal{H}.$$

This satisfies

$$\|T^*T\| = \|T\|^2 \qquad \text{the } C^* \text{ property.}$$

These fundamental properties of $\mathcal{B}(\mathcal{H})$ (norm-completeness, involution, C^* property) motivate the definition of an abstract *C*-algebra*.

1.2. C*-algebras.

Definition 1. (a) *A* **Banach algebra** $\mathcal{A}$ *is a complex algebra equipped with a complete norm which is sub-multiplicative:*

$$\|ab\| \leqslant \|a\|\,\|b\| \qquad \textit{for all } a, b \in \mathcal{A}.$$

(b) *A* **C*-algebra** $\mathcal{A}$ *is a Banach algebra equipped with an involution*[1] $a \to a^*$ *satisfying the* **C*-condition**

$$\|a^*a\| = \|a\|^2 \qquad \textit{for all } a \in \mathcal{A}.$$

[1]that is, a map on $\mathcal{A}$ such that $(a + \lambda b)^* = a^* + \bar{\lambda} b^*$, $(ab)^* = b^*a^*$, $a^{**} = a$ for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$

If $\mathcal{A}$ has a unit $\mathbf{1}$ then necessarily $\mathbf{1}^* = \mathbf{1}$ and $\|\mathbf{1}\| = 1$. If not, adjoin a unit:

Definition 2. *If $\mathcal{A}$ is a C^*-algebra let*

$$\mathcal{A}^\sim =: \mathcal{A} \oplus \mathbb{C}$$

$$\textit{with} \quad (a,z)(b,w) =: (ab + wa + zb, zw) \quad (a,z)^* =: (a^*, \bar{z})$$

$$\|(a,z)\| =: \sup\{\|ab + zb\| : b \in \mathrm{b}_1 \mathcal{A}\}$$

Thus the norm of $\mathcal{A}^\sim$ is defined by identifying each $(a,z) \in \mathcal{A}^\sim$ with the operator $L_{(a,z)} : \mathcal{A} \to \mathcal{A} : b \to ab + zb$ acting on the Banach space $\mathcal{A}$.

Definition 3. *A* **morphism** *$\phi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is a linear map that preserves products and the involution.* We will see later that morphisms are automatically contractive (hence continuous), and that 1-1 morphisms are isometric (this is one instance where the algebraic structure 'forces' the topological behaviour).

Basic Examples:

- $\mathbb{C}$, the set of complex numbers.
- $C(K)$, the set of all continuous functions $f : K \to \mathbb{C}$, where K is a compact Hausdorff space. With pointwise operations, $f^*(t) = \overline{f(t)}$ and the sup norm, $C(K)$ is an abelian, unital algebra.
- $C_0(X)$, where X is a locally compact Hausdorff space. This consists of all functions $f : X \to \mathbb{C}$ which are continuous and 'vanish at infinity', meaning that given $\varepsilon > 0$ there is a compact $K_{f,\varepsilon} \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \notin K_{f,\varepsilon}$. With the same operations and norm as above, this is an abelian C*-algebra, which is nonunital if and only if X is non-compact.

  We will see later (*section* 3.2) *that all abelian C^*-algebras can be represented as $C_0(X)$ for suitable X.*

- $M_n(\mathbb{C})$, the set of all $n \times n$ matrices with complex entries. With matrix operations, A^* = conjugate transpose, and $\|A\| = \sup\{\|Ax\|_2 : x \in \ell^2(n), \|x\|_2 = 1\}$, this is a non-abelian (when $n > 1$), unital algebra.
- $\mathcal{B}(\mathcal{H})$ is a non-abelian, unital C*-algebra; it is infinite dimensional when $\mathcal{H}$ is infinite-dimensional.

  We will see later (*section* 5.2) *that all C^*-algebras can be represented as closed selfadjoint subalgebras of $\mathcal{B}(\mathcal{H})$ for suitable $\mathcal{H}$.*

- $\mathcal{K}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : \overline{A(\mathrm{b}_1(\mathcal{H}))}$ compact in $\mathcal{H}\}$: the compact operators. This is a closed selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$, hence a C*-algebra. It is nonabelian and non-unital when $\mathcal{H}$ is infinite-dimensional.

Non-examples:

- $A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}}$ holomorphic$\}$: the disc algebra (here $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$).

  This is a closed subalgebra of the C-algebra $C(\overline{\mathbb{D}})$ but not a *-subalgebra, because if $f \in A(\mathbb{D})$ then $\bar{f}$ is not holomorphic unless it is constant; thus the* diagonal $A(\mathbb{D}) \cap A(\mathbb{D})^* = \mathbb{C}\mathbf{1}$ *is trivial: $A(\mathbb{D})$ is an* antisymmetric *algebra.*

- $T_n = \{(a_{ij}) \in M_n(\mathbb{C}) : a_{ij} = 0$ for $i > j\}$: the upper-triangular matrices.

  A closed subalgebra of the C-algebra $M_n(\mathbb{C})$ but not a *-subalgebra. Here the diagonal $T_n \cap T_n^*$ is D_n, the diagonal matrices: a maximal abelian selfadjoint algebra (masa) in M_n.*

- $M_{oo}(\mathbb{C})$: infinite matrices with finite support.

  To define a norm (and operations), consider its elements as operators acting on $\ell^2(\mathbb{N})$ with its usual basis. This is a selfadjoint algebra and its norm satisfies the C-condition, but it is not complete.*

  Its completion is $\mathcal{K}$, the set of compact operators on ℓ^2: a non-unital, non-abelian C-algebra.*

2. Examples and constructions.

• If X is an index set and $\mathcal{A}$ is a C*-algebra, the Banach space $\ell^\infty(X, \mathcal{A})$ of all bounded functions $a : X \to \mathcal{A}$ (with norm $\|a\|_\infty = \sup\{\|a(x)\|_{\mathcal{A}} : x \in X\}$) becomes a C*-algebra with pointwise product and involution.
Its subspace $c_0(X, \mathcal{A})$ consisting of all $a : X \to \mathcal{A}$ with[2] $\lim\limits_{x\to\infty} \|a(x)\|_{\mathcal{A}} = 0$ is a C*-algebra.
The subset $c_{00}(X, \mathcal{A})$ consisting of all functions of finite support is a dense *-subalgebra, which is proper when X is infinite.

• If X is a locally compact Hausdorff space then $C_b(X, \mathcal{A})$ is the *-subalgebra of $\ell^\infty(X, \mathcal{A})$ consisting of continuous bounded functions. It is closed, hence a C*-algebra. (This is denoted $C(X, \mathcal{A})$ when X is compact.)

[2] i.e. such that for each $\varepsilon > 0$ there is a *finite* subset $X_\varepsilon \subseteq X$ s.t. $x \notin X_\varepsilon \Rightarrow \|a(x)\|_{\mathcal{A}} < \varepsilon$

• The C*-algebra $C_0(X, \mathcal{A})$ consists of those $f \in C_b(X, \mathcal{A})$ which 'vanish at infinity', i.e. such that the function $t \to \|f(t)\|_{\mathcal{A}}$ is in $C_0(X)$ (see *Basic Examples*).

More generally, consider subsets of the Cartesian product $\prod \mathcal{A}_i$ of a family of C*-algebras:

Definition 4. (*i*) *The direct sum* $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ *of C*-algebras is a C*-algebra under pointwise operations and involution and the norm*

$$\|(a_1, \ldots, a_n)\| = \max\{\|a_1\|, \ldots, \|a_n\|\}.$$

(*ii*) *Let* $\{\mathcal{A}_i\}$ *be a family of C*-algebras. Their* **direct product** *or* ℓ^∞**-direct sum** $\bigoplus_{\ell^\infty} \mathcal{A}_i$ *is the subset of the Cartesian product* $\prod \mathcal{A}_i$ *consisting of all* $(a_i) \in \prod \mathcal{A}_i$ *such that* $i \to \|a_i\|_{\mathcal{A}_i}$ *is bounded. It is a C*-algebra under pointwise operations and involution and the norm*

$$\|(a_i)\| = \sup\{\|a_i\|_{\mathcal{A}_i} : i \in I\}.$$

(*iii*) *The* **direct sum** *or* c_0**-direct sum** $\bigoplus_{c_0} \mathcal{A}_i$ *of a family* $\{\mathcal{A}_i\}$ *of C*-algebras is the closed selfadjoint subalgebra of their direct product consisting of all* $(a_i) \in \prod \mathcal{A}_i$ *such that* $i \to \|a_i\|_{\mathcal{A}_i}$ *vanishes at infinity.*

In case $\mathcal{A}_i = \mathcal{A}$ for all i, the direct product is just $\ell^\infty(I, \mathcal{A})$ and the direct sum is $c_0(X, \mathcal{A})$.

• If $\mathcal{A}$ is a C*-algebra and $n \in \mathbb{N}$, the space $M_n(\mathcal{A})$ of all matrices $[a_{ij}]$ with entries $a_{ij} \in \mathcal{A}$ becomes a *-algebra with product $[a_{ij}][b_{ij}] = [c_{ij}]$ where $c_{ij} = \sum_k a_{ik} b_{kj}$ and involution $[a_{ij}]^* = [d_{ij}]$ where $d_{ij} = a_{ji}^*$. It is of course non-commutative when $n > 1$.

But how does one define a norm on $M_n(\mathcal{A})$ *satisfying the C*-condition*?

Consider two *special cases*:

• Suppose $\mathcal{A}$ is $C_0(X)$. Then we may identify $M_n(C_0(X))$ (as a *-algebra) with $C_0(X, M_n)$, i.e. M_n-valued continuous functions on X vanishing at infinity: each matrix $[f_{ij}] \in M_n(C_0(X))$ defines naturally a function $F : X \to M_n : x \to [f_{ij}(x)]$ which is continuous with respect to the norm on M_n.[3]

[3] Conversely, of course, if $F : X \to M_n$ is continuous, then its entries f_{ij} given by $f_{ij}(x) = \langle F(x)e_j, e_i \rangle$ form an $n \times n$ matrix of continuous functions.

Thus we may define

$$\|[f_{ij}]\| = \|F\|_\infty = \sup\{\|[f_{ij}(x)]\|_{M_n} : x \in X\}$$

and it is easy to verify that this satisfies the C*-condition, because the norm on M_n satisfies the C*-condition.

• Suppose $\mathcal{A}$ is $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then we may identify $M_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\mathcal{H}^n)$: Given a matrix $[a_{ij}]$ of bounded operators a_{ij} on $\mathcal{H}$, we define an operator A on $\mathcal{H}^n$ by

$$A \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} \sum_j a_{1j}\xi_j \\ \vdots \\ \sum_j a_{nj}\xi_j \end{bmatrix}$$

(this identification preserves the algebraic operations, including the involution).[4] Hence one *defines the norm* $\|[a_{ij}]\|$ of $[a_{ij}] \in M_n(\mathcal{B}(\mathcal{H}))$ to be the norm $\|A\|$ of the corresponding operator on $\mathcal{H}^n$.

• In order to define a C*-algebra structure on $M_n(\mathcal{A})$ for a *general C*-algebra* $\mathcal{A}$, one uses the Gelfand–Naimark Theorem (see section 5.2).

3. Spectral theory.

3.1. The spectrum.

Definition 5. *If $\mathcal{A}$ is a unital C*-algebra and $GL(\mathcal{A})$ denotes the group of invertible elements of $\mathcal{A}$, the* **spectrum** *of an element $a \in \mathcal{A}$ is*

$$\sigma(a) = \sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin GL(\mathcal{A})\}.$$

If $\mathcal{A}$ is non-unital, the spectrum of $a \in \mathcal{A}$ is defined by

$$\sigma(a) = \sigma_{\mathcal{A}^\sim}(a).$$

In this case, necessarily $0 \in \sigma(a)$.

Proposition 3.1. *The spectrum $\sigma(a)$ is a compact nonempty subset of $\mathbb{C}$.*

[4] Conversely any $A \in \mathcal{B}(\mathcal{H}^n)$ defines an $n \times n$ matrix of operators a_{ij} on $\mathcal{H}$ by $\langle a_{ij}\xi, \eta\rangle_{\mathcal{H}} = \langle A\xi_j, \eta_i\rangle_{\mathcal{H}^n}$, where $\xi_j \in \mathcal{H}^n$ is the column vector having ξ at the j-th entry and zeroes elsewhere (and η_i is defined analogously).

Sketch of proof. *(i) $\sigma(a)$ is bounded:* In a unital C*-algebra, if $\|x\| < 1$ then $\sum_{n\geqslant 0} x^n$ converges to an element y such that $(\mathbf{1}-x)y = y(\mathbf{1}-x) = \mathbf{1}$, hence $(\mathbf{1} - x) \in GL(\mathcal{A})$. The proof is the same[5] as the case $\mathcal{A} = \mathbb{C}$. Hence if $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ satisfies $|\lambda| > \|a\|$ then $\left\|\frac{a}{\lambda}\right\| < 1$ so $\lambda \notin \sigma(a)$: the spectrum is bounded by $\|a\|$.

(ii) $\sigma(a)$ is closed: To prove this, we prove that $GL(\mathcal{A})$ is open; see Lemma 3.2 below.

(iii) $\sigma(a)$ is nonempty: This is proved by contradiction: one shows that for each ϕ in the Banach space dual of $\mathcal{A}$, the function $f : \lambda \to \phi((\lambda\mathbf{1} - a)^{-1})$ is analytic on its domain $\mathbb{C}\backslash\sigma(a)$ and $\lim_{|\lambda|\to\infty} f(\lambda) = 0$; so if $\sigma(a)$ were empty, this function would be analytic on $\mathbb{C}$ and vanishing at infinity, hence would be zero by Liouville's theorem; hence $\phi(a^{-1}) = f(0) = 0$ for all ϕ, which is absurd by Hahn-Banach. □

Lemma 3.2. *The set $GL(\mathcal{A})$ is open in $\mathcal{A}$ and the map $x \to x^{-1}$ is continuous (hence a homeomorphism) on $GL(\mathcal{A})$.*

Proof. We have seen that if $\|\mathbf{1} - x\| < 1$ then $x \in GL(\mathcal{A})$. Thus $\mathbf{1}$ is an interior point of $GL(\mathcal{A})$. To show that every $a \in GL(\mathcal{A})$ is an interior point of $GL(\mathcal{A})$, just notice that the map $x \to ax$ is a homeomorphism of $GL(\mathcal{A})$ (with inverse $y \to a^{-1}y$) and it maps $\mathbf{1}$ to a.[6]
To show that inversion is continuous, let $a, b \in GL(\mathcal{A})$. Then

$$\begin{aligned}\left\|a^{-1} - b^{-1}\right\| &= \left\|b^{-1}(b-a)a^{-1}\right\| = \left\|(b^{-1} - a^{-1})(b-a)a^{-1} + a^{-1}(b-a)a^{-1}\right\| \\ &\leqslant \left\|b^{-1} - a^{-1}\right\| \|b-a\| \left\|a^{-1}\right\| + \left\|a^{-1}\right\|^2 \|b-a\|\end{aligned}$$

hence

$$\left\|a^{-1} - b^{-1}\right\| (1 - \|b-a\| \left\|a^{-1}\right\|) \leqslant \left\|a^{-1}\right\|^2 \|b-a\| .$$

It follows that

$$\lim_{b\to a} \left\|b^{-1} - a^{-1}\right\| = 0.$$

□

The **spectral radius** of $a \in \mathcal{A}$ is defined to be

$$\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

[5] since $\sum \|x^n\| \leqslant \sum \|x\|^n$, the series $\sum x^n$ converges absolutely, hence (completeness!) it converges in $\mathcal{A}$

[6] In fact, if $y \in GL(\mathcal{A})$, the ball $\left\{x \in \mathcal{A} : \|x - y\| < \frac{1}{\|y^{-1}\|}\right\}$ is in $GL(\mathcal{A})$.

It satisfies $\rho(a) \leqslant \|a\|$, but equality may fail.[7] In fact, it can be shown that

$$\rho(a) = \lim_n \|a^n\|^{1/n}. \tag{1}$$

This is the **Gelfand–Beurling** formula.

Exercise 3.3. *Any morphism $\phi : \mathcal{A} \to \mathcal{B}$ between (non-unital) C^*-algebras extends uniquely to a unital morphism $\phi^\sim : \mathcal{A}^\sim \to \mathcal{B}^\sim$ by $\phi^\sim(a, \lambda) = (\phi(a), \lambda)$.*

If $\phi : \mathcal{A} \to \mathcal{B}$ is a morphism, then $\sigma(\phi(a)) \subseteq \sigma(a) \cup \{0\}$ for all $a \in \mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are unital and $\phi(\mathbf{1}) = \mathbf{1}$ then $\sigma(\phi(a)) \subseteq \sigma(a)$ for all $a \in \mathcal{A}$.

An element $a \in \mathcal{A}$ is said to be *normal* if $a^*a = aa^*$, *selfadjoint* if $a = a^*$ and *unitary* if ($\mathcal{A}$ is unital and) $u^*u = \mathbf{1} = uu^*$.

Proposition 3.4.
(i) $a = a^* \Longrightarrow \sigma(a) \subseteq \mathbb{R}$
(ii) $a = b^*b \Longrightarrow \sigma(a) \subseteq \mathbb{R}^+$
(iii) $u^*u = \mathbf{1} = uu^* \Longrightarrow \sigma(u) \subseteq \mathbb{T}$.

Proof of *(iii)*. We have $\rho(u) \leqslant \|u\| = 1$ so $\sigma(u) \subseteq \mathbb{D}$. It remains to show that if $\lambda \in \sigma(u)$ then $|\lambda| \geqslant 1$. Now $\lambda \neq 0$ since u is invertible; and if $|\lambda| < 1$, then since $\sigma(u^{-1}) \subseteq \mathbb{D}$ (because $\rho(u^{-1}) \leqslant \|u^{-1}\| = 1$) the element $x = (\lambda^{-1} - u^{-1})$ is invertible. But then $(\lambda - u)u^{-1} = \lambda(u^{-1} - \lambda^{-1})$ is invertible and hence so is $\lambda - u$, contradiction. Hence $|\lambda| \geqslant 1$.

Proof of *(i)*. Let $u(t) = \exp(ita)$ ($t \in \mathbb{R}$) (defined by the power series which converges absolutely). Note that $u(t)^* = \exp(-ita)$ because $a = a^*$. As in the case $a \in \mathbb{R}$, one shows that the function $t \to u(t)$ is norm-differentiable and $u'(t) = au(t) = u(t)a$. It follows that if $f(t) = u(t)u(-t)$ then $f'(t) = 0$ for all $t \in \mathbb{R}$ so $f(t) = f(0) = \mathbf{1}$ hence $u(t)u(t)^* = u(t)^*u(t) = \mathbf{1}$. Thus by (iii) we have $\sigma(\exp ita) \subseteq \mathbb{T}$.

Let $\lambda \in \sigma(a)$. Then[8]

$$\begin{aligned}\exp(ia) - \exp(i\lambda)\mathbf{1} &= e^{i\lambda}(\exp i(a - \lambda) - \mathbf{1}) = e^{i\lambda} \sum_{n=1}^{\infty} \frac{i^n}{n!}(a - \lambda)^n \\ &= e^{i\lambda}(a - \lambda)b\end{aligned}$$

where $b \in \mathcal{A}$ commutes with $a - \lambda$. Thus $\exp(ia) - \exp(i\lambda)\mathbf{1}$ cannot be invertible. Therefore $e^{i\lambda} \in \sigma(\exp(ia)) \subseteq \mathbb{T}$ and so $\lambda \in \mathbb{R}$.

---

[7] Consider for instance any $a \neq 0$ with $a^2 = 0$.

[8] One can show that $e^{-i\lambda}\exp(ia) = \exp i(a - \lambda)$ because a and $\lambda\mathbf{1}$ commute.

Second proof of (i). Let $a = a^*$. Suppose that $\lambda + i\mu \in \sigma(a)$ for some $\lambda, \mu \in \mathbb{R}$; we show that $\mu = 0$. If $\mu \neq 0$, then the element $a - (\lambda + i\mu)\mathbf{1} = \mu\left(\frac{a - \lambda\mathbf{1}}{\mu} - i\mathbf{1}\right)$ would not be invertible. But then $i \in \sigma(b)$ where $b = \frac{a - \lambda\mathbf{1}}{\mu}$ is selfadjoint. Let $n \in \mathbb{N}$. Then $n + 1 \in \sigma(n\mathbf{1} - ib)$ because $(n\mathbf{1} - ib) - (n+1)\mathbf{1} = -i(b - i\mathbf{1})$ is not invertible. Therefore $|n+1| \leqslant \|n\mathbf{1} - ib\|$ and hence

$$(n+1)^2 \leqslant \|n\mathbf{1} - ib\|^2 \overset{(C^*)}{=} \|(n\mathbf{1} - ib)^*(n\mathbf{1} - ib)\| \overset{(b=b^*)}{=} \left\|n^2\mathbf{1} + b^2\right\| \leqslant n^2 + \left\|b^2\right\|.$$

Thus $2n + 1 \leqslant \left\|b^2\right\|$ for all n, a contradiction.

The proof of (ii) is non-trivial: see Theorem 4.7.

Lemma 3.5. *If $aa^* = a^*a$ then $\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} = \|a\|$.*

Proof. Since $a^*a = aa^*$, we have

$$\|a\|^4 = \|a^*a\|^2 = \|(a^*a)^*(a^*a)\| = \left\|(a^2)^*a^2\right\| = \|a^2\|^2$$

hence $\|a\|^2 = \|a^2\|$ and inductively $\|a\|^{2^n} = \|a^{2^n}\|$ for all n. Thus, by the Gelfand–Beurling formula (1), $\rho(a) = \lim \left\|a^{2^n}\right\|^{2^{-n}} = \|a\|$. □

A fundamental consequence of the C^*-property combined with completeness is the following:

Proposition 3.6. *The norm of a C^*-algebra is determined by its algebraic structure. Thus if $\mathcal{A}$ is a *-algebra, there is at most one norm $\|\cdot\|$ on $\mathcal{A}$ such that $(\mathcal{A}, \|\cdot\|)$ is a C^*-algebra.*

Proof. $\|a\|^2 = \|a^*a\| = \rho(a^*a)$. □

Corollary 3.7. *Every morphism $\rho : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is automatically contractive.*

Using Gelfand Theory (see the next section) one can show that an injective morphism is in fact an isometry.

Dependence of the spectrum on the algebra. If $\mathcal{A}$ is a unital C*-algebra and $\mathcal{B}$ is a closed subalgebra of $\mathcal{A}$ containing the identity of $\mathcal{A}$, then every $b \in \mathcal{B}$ satisfies $\sigma_{\mathcal{A}}(b) \subseteq \sigma_{\mathcal{B}}(b)$. Indeed if $\lambda \notin \sigma_{\mathcal{B}}(b)$ then $\lambda\mathbf{1} - b$ has an inverse in $\mathcal{B}$ hence also in $\mathcal{A}$. But equality need not hold:

For example suppose $\mathcal{A} = C(\mathbb{T})$, the continuous functions on the unit circle. Let $\mathcal{B}$ be the subalgebra consisting of all $f \in \mathcal{A}$ having a continuous extension to $\overline{\mathbb{D}}$ which is holomorphic in $\mathbb{D}$.[9] Let $b \in \mathcal{B}$ be the function $b(z) = z$. The function b^{-1} given by $b^{-1}(z) = \dfrac{1}{z}$ is continuous on $\mathbb{T}$, but is not in $\mathcal{B}$.

It is remarkable that if $\mathcal{B}$ is a C*-subalgebra this cannot happen:

Proposition 3.8 (Permanence of spectrum). *If $\mathcal{A}$ is a unital C*-algebra and $\mathcal{B}$ is a C*-subalgebra of $\mathcal{A}$ containing the identity of $\mathcal{A}$ (i.e. $\mathbf{1}_{\mathcal{A}} \in \mathcal{B} \subseteq \mathcal{A}$), then every $b \in \mathcal{B}$ satisfies*

$$\sigma_{\mathcal{A}}(b) = \sigma_{\mathcal{B}}(b).$$

Proof. It is enough to show that if $b \in \mathcal{B}$ has an inverse in $\mathcal{A}$, then this inverse is contained in $\mathcal{B}$.

Suppose first that $b = b^*$. Since $\sigma_{\mathcal{B}}(b) \subseteq \mathbb{R}$, for each $n \in \mathbb{N}$ we have $\dfrac{i}{n} \notin \sigma_{\mathcal{B}}(b)$. Thus the elements $x_n = b - \dfrac{i}{n}\mathbf{1}$ are invertible in $\mathcal{B}$: each x_n^{-1} belongs to $\mathcal{B}$. But since $x_n \to b$ and inversion is continuous *on the space* $GL(\mathcal{A})$, $x_n^{-1} \to b^{-1}$. Since $x_n^{-1} \in \mathcal{B}$ and $\mathcal{B}$ is closed, it follows that $b^{-1} \in \mathcal{B}$ as required.

For the general case, if $b \in \mathcal{B}$ is invertible in $\mathcal{A}$, so is b^* (verify) and hence so is $x = b^*b$. But x is selfadjoint, so by the previous paragraph $x \in GL(\mathcal{B})$: if $y = x^{-1}$, then $y \in GL(\mathcal{B})$. We have $yb^*b = yx = \mathbf{1}$ and so

$$b^{-1} = (yb^*b)b^{-1} = (yb^*)(bb^{-1}) = yb^*$$

hence $b^{-1} \in \mathcal{B}$, which completes the proof. □

3.2. Gelfand theory for commutative C*-algebras.

Theorem 3.9 (Gelfand–Naimark 1). *Every commutative C*-algebra $\mathcal{A}$ is isometrically *-isomorphic to $C_0(\hat{\mathcal{A}})$ where $\hat{\mathcal{A}}$ is the set of* nonzero *morphisms $\phi : \mathcal{A} \to \mathbb{C}$ which, equipped with the topology of pointwise convergence, is a locally compact Hausdorff space. For each $a \in \mathcal{A}$ the function $\hat{a} : \hat{\mathcal{A}} \to \mathbb{C} : \phi \to \phi(a)$ is in $C_0(\hat{\mathcal{A}})$. The* Gelfand transform*:*

$$\mathcal{A} \to C_0(\hat{\mathcal{A}}) : \ a \to \hat{a}$$

*is an isometric *-isomorphism. The algebra $\mathcal{A}$ is unital if and only if $\hat{\mathcal{A}}$ is compact.*

In more detail: $\hat{\mathcal{A}}$ is the set of all *nonzero* multiplicative linear forms (*characters*) $\phi : \mathcal{A} \to \mathbb{C}$. Each $\phi \in \hat{\mathcal{A}}$ necessarily satisfies $\|\phi\| \leqslant 1$ and, when $\mathcal{A}$ is unital, $\|\phi\| = \phi(\mathbf{1}) = 1$. The topology on $\hat{\mathcal{A}}$ is[10] pointwise convergence: $\phi_i \to \phi$ iff $\phi_i(a) \to \phi(a)$ for all $a \in \mathcal{A}$.

---

[9] This is isomorphic to the disc algebra.

[10] In fact, since $\hat{\mathcal{A}}$ is contained in the unit ball of the (Banach space) dual $\mathcal{A}^*$ of $\mathcal{A}$, this topology is just the restriction of the w*-topology of $\mathcal{A}^*$ to $\hat{\mathcal{A}}$.

When $\mathcal{A}$ is non-abelian there may be no characters (consider $M_2(\mathbb{C})$ or $\mathcal{B}(\mathcal{H})$, for example).

When $\mathcal{A}$ is abelian there are 'many' characters: for each $a \in \mathcal{A}$ there exists $\phi \in \hat{\mathcal{A}}$ such that $\|a\| = |\phi(a)|$.

When $\mathcal{A}$ is also unital, $\hat{\mathcal{A}}$ is compact and $\mathcal{A}$ is isometrically *-isomorphic to $C(\hat{\mathcal{A}})$.

When $\mathcal{A}$ is abelian but non-unital every $\phi \in \hat{\mathcal{A}}$ extends uniquely to a character $\phi^{\sim} \in \widehat{\mathcal{A}^{\sim}}$ by $\phi^{\sim}(\mathbf{1}) = 1$, and there is exactly one $\phi_\infty \in \widehat{\mathcal{A}^{\sim}}$ that vanishes on $\mathcal{A}$. Thus $\mathcal{A}$ is *-isomorphic the algebra of those continuous functions on the 'one-point compactification' $\hat{\mathcal{A}} \cup \{\phi_\infty\}$ of $\hat{\mathcal{A}}$ which vanish at ϕ_∞; this algebra is in fact isomorphic to $C_0(\hat{\mathcal{A}})$.

S k e t c h o f p r o o f i n t h e u n i t a l c a s e. We assume that $\mathcal{A}$ is abelian and unital.

(a) The compact space $\hat{\mathcal{A}}$. Let $\phi : \mathcal{A} \to \mathbb{C}$ be a character. Then $\ker\phi$ is an ideal, so $\phi(\mathbf{1})^2 = \phi(\mathbf{1}) \neq 0$ (for if $\phi(\mathbf{1}) = 0$ then $\phi(a) = \phi(a\mathbf{1}) = 0$ for all a, a contradiction), hence $\phi(\mathbf{1}) = 1$. Also for all $a \in \mathcal{A}$ we have $\phi(a) \in \sigma(a)$ because $\phi(a)\mathbf{1} - a$ is in $\ker\phi$ which cannot contain invertible elements, being a proper ideal. Thus $|\phi(a)| \leqslant \rho(a) \leqslant \|a\|$.

In fact the equality

$$(\dagger) \qquad \sigma(a) = \{\phi(a) : \phi \in \hat{\mathcal{A}}\}$$

holds; we prove this in the Appendix.

Note also that each character ϕ is *selfadjoint*:

$$\phi(a^*) = \overline{\phi(a)} \quad \text{for all} \quad a \in \mathcal{A}.$$

Indeed, it suffices to prove that if $a = a^*$ then $\phi(a) \in \mathbb{R}$; but this is clear since $\phi(a) \in \sigma(a)$ and $\sigma(a) \subseteq \mathbb{R}$.

The inequality $|\phi(a)| \leqslant \|a\|$ shows that $\hat{\mathcal{A}}$ is contained in the space $\Pi_{a\in\mathcal{A}}\mathbb{D}_a$, the Cartesian product of the compact spaces $\mathbb{D}_a = \{z \in \mathbb{C} : |z| \leqslant \|a\|\}$; and the product topology is just the topology of pointwise convergence. But in fact $\hat{\mathcal{A}}$ is closed in this product: if $\phi_i \to \psi$ pointwise, then it is clear that ψ is linear and multiplicative, because each ϕ_i is linear and multiplicative, and $\psi \neq 0$ because $\psi(\mathbf{1}) = \lim_i \phi_i(\mathbf{1}) = 1$; thus $\psi \in \hat{\mathcal{A}}$.

(b) The Gelfand map $\mathcal{G} : a \to \hat{a}$. For each $a \in \mathcal{A}$ the function

$$\hat{a} : \hat{\mathcal{A}} \to \mathbb{C} \quad \text{where} \quad \hat{a}(\phi) = \phi(a), \ (\phi \in \hat{\mathcal{A}})$$

is continuous by the very definition of the topology on $\hat{\mathcal{A}}$. This gives a well defined map

$$\mathcal{G} : \mathcal{A} \to C(\hat{\mathcal{A}}) : a \to \hat{a}\,.$$

If $a, b \in \mathcal{A}$, since each $\phi \in \hat{\mathcal{A}}$ is linear, multiplicative and *-preserving, we have

$$\begin{aligned}
\widehat{(a+b)}(\phi) &= \phi(a+b) = \phi(a) + \phi(b) = \hat{a}(\phi) + \hat{b}(\phi) \\
\widehat{(ab)}(\phi) &= \phi(ab) = \phi(a)\phi(b) = \hat{a}(\phi)\hat{b}(\phi) \\
\widehat{(a^*)}(\phi) &= \phi(a^*) = \overline{\phi(a)} = \overline{\hat{a}(\phi)}
\end{aligned}$$

therefore

$$\mathcal{G}(a+b) = \mathcal{G}(a) + \mathcal{G}(b), \quad \mathcal{G}(ab) = \mathcal{G}(a)\mathcal{G}(b) \quad \text{and} \quad \mathcal{G}(a^*) = (\mathcal{G}(a))^*$$

that is, the map $\mathcal{G}$ is a morphism of *-algebras. Hence it is automatically contractive; but in fact it can be seen directly to be isometric:

$$\begin{aligned}
\|\hat{a}\|_\infty &= \sup\{|\hat{a}(\phi)| : \phi \in \hat{\mathcal{A}}\} \\
&= \sup\{|\phi(a)| : \phi \in \hat{\mathcal{A}}\} = \sup\{|\lambda| : \lambda \in \sigma(a)\} \qquad \text{(by †)} \\
&= \|a\|
\end{aligned}$$

by Lemma 3.5, because a is normal since $\mathcal{A}$ is abelian.

(c) The Gelfand map is onto $C(\hat{\mathcal{A}})$. Consider the range $\mathcal{G}(\mathcal{A})$: it is a *-subalgebra of $C(\hat{\mathcal{A}})$, because $\mathcal{G}$ is a *-homomorphism. It contains the constants, because $\mathcal{G}(\mathbf{1}) = \mathbf{1}$ (:the constant function 1). It separates the points of $\hat{\mathcal{A}}$, because if $\phi, \psi \in \hat{\mathcal{A}}$ are different, they must differ at some $a \in \mathcal{A}$, so

$$\mathcal{G}(a)(\phi) = \phi(a) \neq \psi(a) = \mathcal{G}(a)(\psi).$$

By the Stone – Weierstrass Theorem, $\mathcal{G}(\mathcal{A})$ must be dense in $C(\hat{\mathcal{A}})$. But it is closed, since $\mathcal{A}$ is complete and $\mathcal{G}$ is isometric. Hence $\mathcal{G}(\mathcal{A}) = C(\hat{\mathcal{A}})$. □

Appendix: A note on characters. Let $\mathcal{A}$ be an *abelian unital Banach algebra*, and let $\widehat{\mathcal{A}}$ be the set of all nonzero morphisms $\phi : \mathcal{A} \to \mathbb{C}$.

In Section 3.2, we saw that

$$\{\phi(a) : \phi \in \widehat{\mathcal{A}}\} \subseteq \sigma(a).$$

We wish to show that equality in fact holds.
So fix a $\lambda_0 \in \sigma(a)$ and let $\mathcal{J}_0 = \{x(a - \lambda_0\mathbf{1}) : x \in \mathcal{A}\}$. One easily sees that $\mathcal{J}_0$ is

an ideal of $\mathcal{A}$, and it is proper since $a - \lambda_0 \mathbf{1}$ is not invertible. It is enough to find $\phi \in \widehat{\mathcal{A}}$ such that the ideal $\ker \phi$ contains $\mathcal{J}_0$.

We will show that $\mathcal{J}_0$ is contained in a maximal proper ideal of $\mathcal{A}$.

Remark 3.10. If $\mathcal{J}$ is a proper ideal of $\mathcal{A}$, then $\|\mathbf{1} - x\| \geqslant 1 \quad$ for all $\; x \in \mathcal{J}$. In particular, the closure of a proper ideal is a proper ideal.

Indeed, if $\|\mathbf{1} - x\| < 1$ then, as we know, $x \in GL(\mathcal{A})$, so x cannot belong to a proper ideal.

Remark 3.11. $\mathcal{J}_0$ is contained in a maximal proper ideal $\mathcal{M}$ of $\mathcal{A}$, which is therefore closed.

Proof. Let F be the family of all ideals $\mathcal{J}$ of $\mathcal{A}$ which contain $\mathcal{J}_0$ but do not contain $\mathbf{1}$; order F by inclusion. If $G \subseteq F$ is a totally ordered subset of F, let $\mathcal{J}_G$ be the union of all elements of G. Of course $\mathcal{J}_G$ contains $\mathcal{J}_0$ and does not contain $\mathbf{1}$; it is easy to verify that $\mathcal{J}_G$ is an ideal, hence it is an upper bound for G.

Zorn's lemma shows that there exists $\mathcal{M} \in F$ which is maximal in the partially ordered set $(F, \subseteq)$. Thus $\mathcal{M}$ is an ideal containing $\mathcal{J}_0$ and it is proper because $\mathbf{1} \notin \mathcal{M}$. In fact it is a maximal proper ideal; for if $\mathcal{N}$ is a proper ideal of $\mathcal{A}$ containing $\mathcal{M}$, then it contains $\mathcal{J}_0$ and, since it is proper, cannot contain $\mathbf{1}$; thus $\mathcal{N} \in F$, hence $\mathcal{N} = \mathcal{M}$ because $\mathcal{M}$ is a maximal member of F.

In particular $\mathcal{M}$ is closed, because its closure $\overline{\mathcal{M}}$ is an ideal and does not contain $\mathbf{1}$ by Remark 3.10, hence $\overline{\mathcal{M}} = \mathcal{M}$ by maximality. $\square$

Note the essential use of $\mathbf{1}$ in the above argument: in fact the conclusion may fail in non-unital algebras: If for example $\mathcal{A} = c_0$, the Banach algebra of null sequences, then it can be shown that the ideal $\mathcal{J} = c_{00}$ (the set of sequences of finite support) is contained in no maximal ideal.

Now let $\mathcal{B} = \mathcal{A}/\mathcal{M}$. It is well known that (since $\mathcal{M}$ is a closed subspace) $\mathcal{B}$ is a Banach space with respect to the quotient norm

$$\|a + \mathcal{M}\| = \inf\{\|a + x\| : x \in \mathcal{M}\} = \operatorname{dist}(a, \mathcal{M}).$$

Remark 3.12. $\mathcal{A}/\mathcal{M}$ is a Banach algebra.

Proof. Of course $\mathcal{A}/\mathcal{M}$ is an algebra. We have to prove that

$$\|ab + \mathcal{M}\| \leqslant \|a + \mathcal{M}\| \, \|b + \mathcal{M}\|, \quad a, b \in \mathcal{A}.$$

If $x, y \in \mathcal{M}$ then

$$\|a + x\| \, \|b + y\| \geqslant \|(a + x)(b + y)\| = \|ab + xb + ay + xy\|.$$

But $xb + ay + xy \in \mathcal{M}$, so $\|ab + xb + ay + xy\| \geqslant \|ab + \mathcal{M}\|$. Thus

$$\|a + x\| \, \|b + y\| \geqslant \|ab + \mathcal{M}\|$$

and the required inequality follows by taking the inf over x and y in $\mathcal{M}$. $\square$

Remark 3.13. $\mathcal{B} = \mathcal{A}/\mathcal{M}$ is a division algebra with identity $\mathbf{1} + \mathcal{M}$: that is, if $a + \mathcal{M}$ is not the zero element $0 + \mathcal{M}$ of $\mathcal{B}$, then $a + \mathcal{M}$ is invertible.

Proof. We need to find $b \in \mathcal{A}$ so that $(a + \mathcal{M})(b + \mathcal{M}) = \mathbf{1} + \mathcal{M}$, equivalently $ab + \mathcal{M} = \mathbf{1} + \mathcal{M}$, i.e. $ab - \mathbf{1} \in \mathcal{M}$. Set

$$\mathcal{J} = a\mathcal{A} + \mathcal{M} = \{ab + x : b \in \mathcal{A}, x \in \mathcal{M}\}.$$

This is easily seen to be an ideal of $\mathcal{A}$ and it clearly contains $\mathcal{M}$. But it also contains a which is not in $\mathcal{M}$; hence, by maximality of $\mathcal{M}$, we must have $\mathcal{J} = \mathcal{A}$. Thus there exists $b \in \mathcal{A}$ and $x \in \mathcal{M}$ so that $ab + x = \mathbf{1}$, in other words $ab - \mathbf{1} = -x \in \mathcal{M}$. $\square$

Remark 3.14. If $\mathcal{B}$ is a division Banach algebra, there is an isomorphism $a \to \lambda(a) : \mathcal{B} \to \mathbb{C}$.

Proof. The spectrum $\sigma(a)$ of each $a \in \mathcal{B}$ is nonempty. Thus there exists $\lambda(a) \in \mathbb{C}$ such that $a - \lambda(a)\mathbf{1}$ is not invertible. By the last remark, $a - \lambda(a)\mathbf{1} = 0$, i.e. $a = \lambda(a)\mathbf{1}$. Now if $\mu \in \sigma(a)$ then $a - \mu\mathbf{1}$ is not invertible, hence $a = \mu\mathbf{1}$ and so $\mu = \lambda(a)$.

Thus $\sigma(a) = \{\lambda(a)\}$ is a singleton. Therefore we have a well defined map

$$a \to \lambda(a) : \mathcal{B} \to \mathbb{C}.$$

It is easy to verify that this is an injective algebra morphism: for example, $a = \lambda(a)\mathbf{1}$ and $b = \lambda(b)\mathbf{1}$ gives $ab = \lambda(a)\lambda(b)\mathbf{1}$, but then $\lambda(a)\lambda(b) \in \sigma(ab) = \{\lambda(ab)\}$ and so $\lambda(a)\lambda(b) = \lambda(ab)$. $\square$

Conclusion of the proof. To show that $\{\phi(a) : \phi \in \widehat{\mathcal{A}}\} = \sigma(a)$, we need a character ϕ of $\mathcal{A}$ such that $\phi(a) = \lambda_0$. Consider a maximal ideal $\mathcal{M}$ of $\mathcal{A}$ containing $\mathcal{J}_0$ and define $\phi : \mathcal{A} \to \mathbb{C}$ as follows:

$$\begin{array}{rcccccc} \phi : & \mathcal{A} & \to & \mathcal{B} & \to & \mathbb{C} \\ & x & \to & x + \mathcal{M} & \to & \lambda(x + \mathcal{M}) \end{array}$$

where $\lambda : \mathcal{B} \to \mathbb{C}$ is the isomorphism of the last Remark. This is a composition of morphisms, hence a morphism. Its kernel is precisely $\mathcal{M}$, so $\phi \neq 0$ and, since $a - \lambda_0\mathbf{1} \in \mathcal{J}_0 \subseteq \mathcal{M}$, we have $\phi(a - \lambda_0\mathbf{1}) = 0$ i.e. $\phi(a) = \lambda_0$. $\square$

3.3. Functional calculus and spectral theorem.

3.3.1. The continuous functional calculus for selfadjoint operators. Let A be a selfadjoint[11] element of the unital C*-algebra $\mathcal{B}(\mathcal{H})$.
For any (complex) polynomial $p(\lambda) = \sum_{k=0}^{n} c_k \lambda^k$ we have a (normal) element $p(A) = \sum_{k=0}^{n} c_k A^k$ of $\mathcal{B}(\mathcal{H})$. We wish to extend the map

$$\Phi_0 : p \to p(A)$$

to a map $f \to f(A)$ defined on all continuous functions $f : \sigma(A) \to \mathbb{C}$. Since the polynomials are dense in $C(\sigma(A))$, it is enough to prove that Φ_0 is continuous in the norm of $C(\sigma(A))$.

Theorem 3.15. *If $A \in \mathcal{B}(\mathcal{H})$ is selfadjoint and p is a polynomial,*

$$\|p(A)\| = \sup\{|p(\lambda)| : \lambda \in \sigma(A)\} \equiv \|p\|_{\sigma(A)}.$$

In particular $\Phi_0(p)$ only depends on the values of p on $\sigma(A)$; thus Φ_0 is well defined on the subspace of $C(\sigma(A))$ consisting of polynomial functions.

The proof of Theorem 3.15 is an immediate consequence of the fact that the spectral radius of a normal element ($p(A)$ is normal) equals its norm, together with the following entirely algebraic fact:

Lemma 3.16 (Spectral mapping lemma)**.** *If $A \in \mathcal{B}(\mathcal{H})$ is selfadjoint and p is a polynomial,*

$$\sigma(p(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}.$$

Definition 6. *Let $A = A^* \in \mathcal{B}(\mathcal{H})$. The* ***continuous functional calculus*** *for A is the unique continuous extension*

$$\Phi_c : (C(\sigma(A)), \|.\|_{\sigma(A)}) \to (\mathcal{B}(\mathcal{H}), \|.\|) : f \to f(A)$$

of the map $\Phi_o : p \to p(A)$. Thus if f is continuous on $\sigma(A)$, the operator $f(A) \in \mathcal{B}(\mathcal{H})$ is defined by the limit

$$f(A) = \lim p_n(A)$$

[11] The functional calculus can be defined for normal operators as well. We restrict to the selfadjoint case for simplicity.

where (p_n) *is any sequence of polynomials such that* $\|p_n - f\|_{\sigma(A)} \to 0$.

It is easily verified that Φ_c is an isometric *-homomorphism, which is uniquely determined by the conditions $\Phi_c(\mathbf{1}) = I$ and $\Phi_c(\mathrm{id}) = A$ (where $\mathrm{id}(\lambda) = \lambda$ is the identity function on $\sigma(A)$).

3.3.2. Connection with Gelfand theory. Keeping the notations of the last section, let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be the C*-algebra generated by A and the identity. It is a unital, abelian C*-algebra, the norm closure of $\{p(A) : p \text{ a polynomial}\}$. But this closure is precisely the set

$$\{f(A) : f \in C(\sigma(A))\}.$$

We determine $\widehat{\mathcal{A}}$:

Given any $\lambda \in \sigma(A)$, the map $\phi_\lambda : \mathcal{A} \to \mathbb{C}$ given by

$$\phi_\lambda(f(A)) = f(\lambda)$$

is obviously a nonzero multiplicative linear functional.

Conversely, let $\phi \in \widehat{\mathcal{A}}$. Then the number $\lambda = \phi(A)$ is in $\sigma_{\mathcal{A}}(A) = \sigma(A)$ (Proposition 3.8). For any polynomial $p(t) = \sum_{k=0}^{n} c_k t^k$, we have, since ϕ is linear and multiplicative,

$$\phi(p(A)) = \sum_{k=0}^{n} c_k \phi(A)^k = p(\lambda) = \phi_\lambda(p(A)).$$

But ϕ and ϕ_λ are continuous on $\mathcal{A}$ and the set $\{p(A) : p \text{ a polynomial}\}$ is dense in $\mathcal{A}$; therefore $\phi = \phi_\lambda$.

Thus we have a bijection

$$\lambda \to \phi_\lambda : \sigma(A) \to \widehat{\mathcal{A}}.$$

In fact this bijection is continuous and hence, since $\sigma(A)$ is compact, a homeomorphism. For this we have to show that if $\lambda_n \to \lambda$ then $\phi_{\lambda_n}(B) \to \phi_\lambda(B)$ for all $B \in \mathcal{A}$. Indeed, each $B \in \mathcal{A}$ is of the form $B = f(A)$ for some $f \in C(\sigma(A))$; and the definition of ϕ_λ gives

$$\phi_{\lambda_n}(B) = \phi_{\lambda_n}(f(A)) = f(\lambda_n) \to f(\lambda) = \phi_\lambda(f(A)) = \phi_\lambda(B)$$

since f is a continuous function. We summarize

Theorem 3.17. *If A is a selfadjoint operator and $\mathcal{A} = \{f(A) : f \in C(\sigma(A))\}$ is the unital C*-algebra generated by A, then the map*

$$\lambda \to \phi_\lambda : \sigma(A) \to \widehat{\mathcal{A}},$$

where $\phi_\lambda(f(A)) = f(\lambda)$, is a homeomorphism. If $\widehat{\mathcal{A}}$ is identified with $\sigma(A)$ via this homeomorphism, then the functional calculus

$$f \to f(A) : C(\sigma(A)) \to \mathcal{A}$$

is the inverse *of the Gelfand transform.*

To prove the last sentence, take any $B = f(A) \in \mathcal{A}$ and, for any $\phi = \phi_\lambda \in \widehat{\mathcal{A}}$, consider

$$\hat{B}(\phi_\lambda) = \phi_\lambda(B) = \phi_\lambda(f(A)) = f(\lambda).$$

So, if we identify each λ with ϕ_λ, then $\hat{B}$ is identified with f.

3.3.3. The spectral theorem. If $A \in \mathcal{B}(\mathcal{H})$ is selfadjoint and $K = \sigma(A)$, the continuous functional calculus $\Phi_c : C(K) \to \mathcal{B}(\mathcal{H})$ is a *representation* of the (abelian) C*-algebra $C(K)$ on $\mathcal{H}$.

We will construct a 'measure' $E(\cdot)$ whose values are not numbers, but projections on $\mathcal{H}$, satisfying

$$\Phi_c(f) = \int_K f(\lambda) dE_\lambda$$

for each $f \in C(K)$ and in particular

$$A = \Phi_c(\mathrm{id}) = \int_K \lambda dE_\lambda.$$

In fact, this construction works for any (automatically contractive) *-representation $\pi : C(K) \to \mathcal{B}(\mathcal{H})$:

Sketch of the construction. Fix $x, y \in \mathcal{H}$ and consider the map

$$C(K) \longrightarrow \mathbb{C} : \ f \longrightarrow \langle \pi(f)x, y \rangle.$$

This is a linear functional, bounded by $\|x\|.\|y\|$, because

$$|\langle \pi(f)x, y\rangle| \leqslant \|\pi(f)\|.\|x\|.\|y\| \leqslant \|f\|_\infty.\|x\|.\|y\|.$$

By the Riesz representation theorem, there is a unique complex regular Borel measure $\mu_{x,y}$ on K so that

$$\int_K f d\mu_{x,y} = \langle \pi(f)x, y \rangle \qquad \text{for each } f \in C(K) \tag{2}$$

satisfying[12]

$$\|\mu_{x,y}\| \leqslant \|x\|.\|y\|.$$

Now fix a Borel set $\Omega \subseteq K$ and consider the map

$$\mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C} : \ (x, y) \longrightarrow \mu_{x,y}(\Omega).$$

One shows that this is sesquilinear and bounded by 1, that is

$$|\mu_{x,y}(\Omega)| \leqslant \|\mu_{x,y}\| \leqslant \|x\|.\|y\|.$$

Therefore there is a unique bounded operator $E(\Omega) \in \mathcal{B}(\mathcal{H})$ satisfying

$$\begin{aligned} \langle E(\Omega)x, y\rangle &= \mu_{x,y}(\Omega) \qquad \text{for all } x, y \in \mathcal{B}(\mathcal{H}) \\ \text{and} \quad \|E(\Omega)\| &\leqslant 1 \qquad \text{for all Borel } \Omega \subseteq K. \end{aligned}$$

One shows that $E(\cdot)$ is a 'spectral measure', that is:

1. $E(\Omega)^* = E(\Omega)$
2. $E(\Omega_1 \cap \Omega_2) = E(\Omega_1).E(\Omega_2)$
3. $E(\varnothing) = 0$ and $E(K) = I$
4. for $x, y \in H$, the map $\mu_{xy} : \Omega \to \langle E(\Omega)x, y\rangle$ is a σ-additive complex-valued set function on the Borel σ-algebra of K.

We now define integration with respect to the 'measure' $E(\cdot)$: If

$$f = \sum_i \lambda_i \chi_{\Omega_i}$$

is a simple Borel function (with $\lambda_i \in \mathbb{C}$ and $\Omega_i \subseteq K$ pairwise disjoint Borel sets such that $\cup\Omega_i = K$), define

$$\int_K f(\lambda)dE_\lambda = \sum_i \lambda_i E(\Omega_i) \in \mathcal{B}(\mathcal{H}).$$

Observe that

$$\left\langle \left(\int_K f(\lambda)dE_\lambda\right) x, y \right\rangle = \int_K f d\mu_{x,y}$$

[12] $\|\mu_{x,y}\|$ is the *total variation* of the measure $\mu_{x,y}$; it equals the norm of the corresponding functional on $C(K)$.

for all $x, y \in H$.

One shows that the mapping $f \to \int f dE$ is linear, and also a *-homomorphism, that is

$$\int \bar{f} dE = \left(\int f dE\right)^* \qquad \text{and} \qquad \int fg dE = \left(\int f dE\right)\left(\int g dE\right)$$

for all simple Borel functions f, g.

One shows that

$$\left\| \int f dE \right\| \leqslant \sup\{|f(\lambda)| : \lambda \in K\}.$$

Hence the map $f \to \int f dE$ extends uniquely to a contractive linear mapping $\mathcal{L}^\infty(K) \to \mathcal{B}(\mathcal{H})$, where $\mathcal{L}^\infty(K)$ is the C*-algebra of all bounded Borel *functions* on K. This extension is also a *-homomorphism. Finally, if $f : K \to \mathbb{C}$ is continuous, then

$$\begin{aligned} \left\langle \left(\int_K f(\lambda) dE_\lambda\right) x, y \right\rangle &= \int_K f d\mu_{x,y} \\ &= \langle \pi(f)x, y\rangle \qquad \text{for all } \ x, y \in \mathcal{H}, \end{aligned}$$

$$\text{and so} \qquad \int_K f(\lambda) dE_\lambda = \pi(f).$$

This concludes the (sketch of the) construction of the spectral measure corresponding to the representation π. Notice that $E(\cdot)$ is 'regular' in the sense that $\mu_{x,x}$ is (by construction) a regular Borel (positive) measure for each $x \in \mathcal{H}$. Uniqueness of $E(\cdot)$ follows by the uniqueness part of the Riesz representation theorem.

We summarize:

Theorem 3.18. *Every representation π of $C(K)$ on a Hilbert space $\mathcal{H}$ determines a* unique *regular Borel spectral measure $E(\cdot)$ on K so that*

$$\int_K f dE = \pi(f) \qquad (f \in C(K)).$$

Applying this to the representation given by the continuous functional calculus
$\Phi_c : C(\sigma(A)) \to \mathcal{B}(\mathcal{H})$, we obtain

Theorem 3.19. *If $A \in \mathcal{B}(\mathcal{H})$ is a selfadjoint operator, there exists a* unique *regular Borel spectral measure $E(.)$ on $\sigma(A)$ so that*

$$\int_{\sigma(A)} f dE = f(A) \qquad (f \in C(K)) \quad \textit{and in particular} \quad A = \int_{\sigma(A)} \lambda dE_\lambda .$$

Notice that in the course of the construction leading to Theorem 3.18 we have defined the operator-valued integral $\int f dE$ for every bounded Borel function. This leads to an extension of the functional calculus:

Proposition 3.20 (Borel Functional calculus)**.** *The map $\Phi_c : C(\sigma(A) \to \mathcal{B}(\mathcal{H})$ extends uniquely to a contractive *-representation $f \to f(A) := \int_{\sigma(A)} f dE$ of the C^*-algebra $\mathcal{L}^\infty(\sigma(A))$ of all bounded Borel functions on $\sigma(A)$. In particular, if $\Omega \subseteq \sigma(A)$ is a Borel set, $\chi_\Omega(A) = E(\Omega)$.*

Remark 3.21. The spectral Theorem and the Borel functional calculus are also valid for a *normal* operator $A \in \mathcal{B}(\mathcal{H})$. The proof is the same as the selfadjoint case, provided one extends the continuous functional calculus to normal operators.

4. Positivity.

Definition 7. *An element $a \in \mathcal{A}$ is* ***positive*** *(written $a \geqslant 0$) if $a = a^*$ and $\sigma(a) \subseteq \mathbb{R}_+$. We write $\mathcal{A}_+ = \{a \in \mathcal{A} : a \geqslant 0\}$.*

If a, b are selfadjoint, we define $a \leqslant b$ by $b - a \in \mathcal{A}_+$.

Examples 4.1. In $C(X)$: $f \geqslant 0$ iff $f(t) \in \mathbb{R}_+$ for all $t \in X$ because $\sigma(f) = f(X)$.

In $\mathcal{B}(\mathcal{H})$: $T \geqslant 0$ iff $\langle T\xi, \xi\rangle \geqslant 0$ for all $\xi \in H$.

Remark 4.2. Any morphism $\pi : \mathcal{A} \to \mathcal{B}$ between C*-algebras preserves order:

$$a \geqslant 0 \quad \Rightarrow \quad \pi(a) \geqslant 0.$$

Proof. If $a = a^*$ and $\sigma(a) \subseteq [0, +\infty)$ then $\pi(a)^* = \pi(a^*)$ and

$$\sigma(\pi(a)) \subseteq \sigma(a) \cup \{0\} \subseteq [0, +\infty)$$

so $\pi(a) \geqslant 0$. $\square$

Remark 4.3. In a unital C*-algebra, if $a = a^*$ then $-\|a\|\,\mathbf{1} \leqslant a \leqslant \|a\|\,\mathbf{1}$.

Proof. Observe that $\|a\|\,\mathbf{1} - a$ is selfadjoint and

$$\sigma(\|a\|\,\mathbf{1} - a) = \{\|a\| - \lambda : \lambda \in \sigma(a)\} \subseteq \mathbb{R}_+$$

because $\lambda \in \mathbb{R}$ and $\lambda \leqslant \|a\|$ for $\lambda \in \sigma(a)$. Hence $\|a\|\,\mathbf{1} - a \geqslant 0$; the other inequality is proved similarly. □

Proposition 4.4. *Every positive element of a C*-algebra has a unique positive square root. In fact*

$$a \in \mathcal{A}_+ \quad \textit{if and only if there exists } b \in \mathcal{A}_+ \textit{ such that } a = b^2.$$

Proof. If $a = b^2$ where $b \in \mathcal{A}_+$, then $a = a^*$ and $\sigma(a) = \{\lambda^2 : \lambda \in \sigma(b)\}$ by the Spectral mapping Lemma 3.16; thus $\sigma(a) \subseteq \mathbb{R}_+$ since $b \geqslant 0$ and therefore $a \geqslant 0$.

Conversely, suppose $a \geqslant 0$ and consider the C*-subalgebra $\mathcal{C} = C^*(a)$ of $\mathcal{A}$ generated by a; it is *-isomorphic to the algebra $C_o(X)$ for some space X via the Gelfand transform $x \to \hat{x}$. Note that $a \in \mathcal{C}_+$ since $\sigma_{\mathcal{C}}(a) = \sigma_{\mathcal{A}}(a)$. The Gelfand transform and its inverse preserve order. Since $a \geqslant 0$, we have $\hat{a} \geqslant 0$. Look at the function $\sqrt{\hat{a}} \in C_o(X)$. This is the image of some $b \in \mathcal{A}$, which must be positive because $\sqrt{\hat{a}} \geqslant 0$; also $(\hat{b})^2 = \hat{a}$, so $b^2 = a$.

Uniqueness: Let $b \in \mathcal{A}_+$ be as in the last paragraph and suppose there exists $c \in \mathcal{A}_+$ which also satisfies $c^2 = a$. Observe that $ac = ca$. Since b is in $C^*(a)$, it is a limit of polynomials in a, so it follows that $bc = cb$. Now consider the C*-algebra $C^*(b, c)$: it is abelian and contains a, so we may view b, c, a as continuous functions on *the same space* and then it is clear that $b = c$. □

Proposition 4.5. *For any C*-algebra the set $\mathcal{A}_+$ is a* cone*:*

$$a, b \in \mathcal{A}_+,\ \lambda \geqslant 0 \quad \Longrightarrow \quad \lambda a \in \mathcal{A}_+, a + b \in \mathcal{A}_+.$$

Proof. The first assertion is immediate from the definition of positivity. Hence, for the second one, passing to the unitisation if necessary, it is enough to assume that $0 \leqslant a \leqslant \mathbf{1}$ and $0 \leqslant b \leqslant \mathbf{1}$ and prove that $\dfrac{a+b}{2} \geqslant 0$.

But we have the following characterization:

Lemma 4.6. *In a unital C*-algebra, if $x = x^*$ and $\|x\| \leqslant 1$, then*

$$x \geqslant 0 \quad \Longleftrightarrow \quad \|\mathbf{1} - x\| \leqslant 1.$$

Thus if a and b are positive contractions then $\dfrac{a+b}{2}$ is a selfadjoint contraction and $\left\|\mathbf{1}-\dfrac{a+b}{2}\right\| = \dfrac{1}{2}\|(\mathbf{1}-a)+(\mathbf{1}-b)\| \leqslant \dfrac{1}{2}(\|\mathbf{1}-a\|+\|\mathbf{1}-b\|) \leqslant 1$ so that $\dfrac{a+b}{2} \geqslant 0$, completing the proof of the Proposition.

P r o o f o f t h e L e m m a. Considering the C*-algebra generated by x and $\mathbf{1}$, there is no loss in assuming that x is a continuous function on a compact set. Then the Lemma is just an application of the triangle inequality: The assumption is that $-1 \leqslant x(t) \leqslant 1$ for all t and we need to conclude that

$$x(t) \geqslant 0 \quad \iff \quad |1-x(t)| \leqslant 1.$$

But this is obvious!

We now have the machinery to complete the proof of Proposition 3.4:

Theorem 4.7. *In any C*-algebra, any element of the form a^*a is positive.*

P r o o f. Of course a^*a is selfadjoint.[13] So it can be written

$$a^*a = b - c \quad \text{where } b, c \geqslant 0,\ bc = 0$$

(to see this, consider a^*a as a function and let b and c be its positive and negative parts).
We will show that $c = 0$.

Let $x = ca^*$. Observe that

$$xx^* = ca^*ac = c(b-c)c = -c^3$$

and so, since $c \geqslant 0$,

$$-xx^* \in \mathcal{A}_+.$$

On the other hand, if we write $x = u + iv$ with u, v selfadjoint, we find

$$xx^* + x^*x = 2u^2 + 2v^2 \in \mathcal{A}_+$$

since $\mathcal{A}_+$ is a cone. Again using the fact that $\mathcal{A}_+$ is a cone, we conclude that

$$x^*x = -xx^* + (xx^* + x^*x) \in \mathcal{A}_+.$$

[13] If a were normal, we could consider it as a function $\hat{a}$ on a locally compact space, and then we could conclude that a^*a corresponds to the function $\hat{a}^*\hat{a} = |\hat{a}|^2$ which is nonnegative; the difficulty is that a need not lie in an abelian C*-algebra.

Thus we have

$$\sigma(x^*x) \subseteq \mathbb{R}_+ \quad \text{and} \quad \sigma(xx^*) \subseteq \mathbb{R}_-.$$

But in any unital algebra we have $\sigma(kh) \subseteq \sigma(hk) \cup \{0\}$.[14]

It follows that $\sigma(xx^*) = \{0\}$. Thus $\|xx^*\| = 0$ (xx^* is selfadjoint) showing that $-c^3 = xx^* = 0$ and so $c = 0$. $\square$

5. The Gelfand–Naimark theorem.

5.1. The GNS construction.

Definition 8. *A* **state** *on a C^*-algebra $\mathcal{A}$ is a positive linear map $\phi : \mathcal{A} \to \mathbb{C}$ of norm 1, i.e. such that $\phi(a^*a) \geqslant 0$ for all $a \in \mathcal{A}$ and $\|\phi\| = 1$. A state is called* **faithful** *if $\phi(a^*a) > 0$ whenever $a \neq 0$.*

Note. When $\mathcal{A}$ is unital and ϕ is positive, $\|\phi\| = \phi(\mathbf{1})$.

Examples 5.1. On $\mathcal{B}(\mathcal{H})$,

- the map $\phi(T) = \langle T\xi, \xi\rangle$ (where $\xi \in \mathcal{H}$ is a unit vector)
- the map $\psi(T) = \sum_i \langle T\xi_i, \xi_i\rangle$ where $\sum \|\xi_i\|^2 = 1$ (called a 'density matrix' in physics).

On $C(K)$,

- the map $\phi(f) = f(t)$ for $t \in K$
- the map $\psi(f) = \int f d\mu$ for a probability measure μ.

For a C*-algebra $\mathcal{A}$, if $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation and $\xi \in \mathcal{H}$ a unit vector, the map $\phi(a) = \langle \pi(a)\xi, \xi\rangle$.

In fact, every state on a C*-algebra arises as in the last example.

Theorem 5.2 (Gelfand, Naimark, Segal)**.** *For every state ϕ on a C^*-algebra $\mathcal{A}$ there is a triple $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ where π_ϕ is a representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\phi$ and $\xi_\phi \in \mathcal{H}_\phi$ a* cyclic[15] *unit vector such that*

$$\phi(a) = \langle \pi_\phi(a)\xi_\phi, \xi_\phi\rangle \quad \textit{for all } a \in \mathcal{A}.$$

The GNS triple $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$ is uniquely determined by this relation up to unitary equivalence.

---

[14] Indeed if $\lambda \notin \sigma(hk)$ is nonzero then the element $y = \lambda^{-1}\mathbf{1} + \lambda^{-1}k(\lambda\mathbf{1} - hk)^{-1}h$ satisfies $y(\lambda\mathbf{1} - kh) = (\lambda\mathbf{1} - kh)y = \mathbf{1}$ and so $\lambda \notin \sigma(kh)$.

[15] i.e. such that $\pi_\phi(\mathcal{A})\xi_\phi$ is dense in $\mathcal{H}_\phi$.

Motivation: the abelian case. Consider a state ϕ on $\mathcal{A} = C(K)$. By the *Riesz representation theorem*, there is a unique positive Borel measure μ on K so that

$$\phi(f) = \int_K f d\mu \qquad \text{for all } f \in C(K).$$

Define the seminorm

$$|||f||| = \left(\int_K |f|^2 d\mu\right)^{1/2} = \phi(|f|^2)^{1/2}, \quad f \in C(K).$$

The set $N_\phi = \left\{ f \in C(K) : \int |f|^2 d\mu = 0 \right\}$ is a subspace of $C(K)$ (it consists of all $f \in C(K)$ that vanish μ-a.e.) and the seminorm $|||\cdot|||$ induces a norm, $\|\cdot\|_2$ on $H_0 := C(K)/N_\phi$. The completion is of course just the Hilbert space $L^2(K, \mu)$.

We may represent the C*-algebra $\mathcal{A}$ on this Hilbert space by observing that for each $f \in \mathcal{A}$ the map $C(K) \to C(K) : g \to fg$ preserves N_ϕ (it is a (left) ideal of $\mathcal{A}$) and hence induces a map

$$\pi_0(f) : H_0 \to H_0 : [g] \to [fg]$$

(here $[g]$ denotes the coset $g + N_\phi$). But this map is bounded in the norm $\|\cdot\|_2$:

$$\|\pi_0(f)[g]\|_2^2 = \|[fg]\|_2^2 = \int |fg|^2 d\mu \leqslant \sup |f|^2 \int |g|^2 d\mu = \|f\|_\infty^2 \|[g]\|_2^2$$

and hence extends to a bounded operator $\pi(f)$ on $L^2(K, \mu)$ (: the operator of multiplication by f). It is now easy to check that $\pi : \mathcal{A} \to \mathcal{B}(L^2(K, \mu))$ is a representation. Note finally that the vector $\xi_\phi := [\mathbf{1}]$ is cyclic for π (indeed $\pi(\mathcal{A})\xi_\phi = \{[f\mathbf{1}] : f \in \mathcal{A}\} = H_0$) and satisfies

$$\langle \pi(f)\xi_f, \xi_f \rangle = \int (f\mathbf{1})\overline{\mathbf{1}} d\mu = \int f d\mu = \phi(f)$$

for all $f \in \mathcal{A}$.

P r o o f o f t h e T h e o r e m (s k e t c h). Assume for simplicity that $\mathcal{A}$ is unital. Define the sesquilinear form

$$\langle a, b \rangle_\phi = \phi(b^* a), \quad a, b \in \mathcal{A}.$$

The set

$$N_\phi = \{a \in \mathcal{A} : \phi(a^* a) = 0\}$$

is a left ideal of $\mathcal{A}$. This follows from the Cauchy–Schwarz inequality

$$|\phi(b^*a)|^2 \leqslant \phi(a^*a)\phi(b^*b), \quad a, b \in \mathcal{A}.$$

In particular

(a) N_ϕ is a linear subspace of $\mathcal{A}$ and the quotient $H_0 = \mathcal{A}/N_\phi$ acquires the scalar product

$$\langle [a], [b] \rangle = \phi(b^*a), \quad a, b \in \mathcal{A}.$$

(b) For each $a \in \mathcal{A}$ the map $b \to ab$ leaves N_ϕ invariant, so it induces a linear map

$$\pi_0(a) : H_0 \to H_0 : [b] \to [ab].$$

Now observe that the map $\pi_0(a)$ is bounded on $(H_0, \|\cdot\|)$ (where $\|[b]\|^2 = \langle [b], [b] \rangle = \phi(b^*b)$). Indeed, if $[b], [c]$ are in H_0,

$$\begin{aligned}
|\langle \pi_0(a)[b], [c] \rangle|^2 &= |\langle [ab], [c] \rangle|^2 = |\phi(c^*ab)|^2 \\
&\leqslant \phi(c^*c)\phi((ab)^*ab) = \phi(c^*c)\phi(b^*a^*ab) \\
&= \phi(c^*c)\phi_b(a^*a) \qquad \text{where } \ \phi_b(x) = \phi(b^*xb) \\
&\leqslant \phi(c^*c)\,\|\phi_b\|\,\|a^*a\| = \phi(c^*c)\phi_b(\mathbf{1})\,\|a\|^2 \\
&= \phi(c^*c)\phi(b^*b)\,\|a\|^2 = \|[c]\|^2\,\|[b]\|^2\,\|a\|^2
\end{aligned}$$

(where we have used the fact that ϕ_b is a positive linear form and its norm is $\phi_b(\mathbf{1})$).

So $\pi_0(a)$ extends to a bounded operator $\pi(a)$ on the completion $\mathcal{H}_\phi$ of H_0. It is easy to see that the map

$$\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\phi) : a \to \pi(a)$$

is a *-representation (it suffices to verify that π_0 is a *-homomorphism). Finally, setting $\xi_\phi = [\mathbf{1}] \in \mathcal{H}_\phi$ (a unit vector), we have $\pi(\mathcal{A})\xi_\phi = \{\pi(a)[\mathbf{1}] : a \in \mathcal{A}\} = \{[a] : a \in \mathcal{A}\} = H_0$, which is dense in $\mathcal{H}_\phi$ and

$$\langle \pi(a)\xi_\phi, \xi_\phi \rangle = \langle [a], [\mathbf{1}] \rangle = \phi(\mathbf{1}^*a) = \phi(a). \qquad \square$$

5.2. The universal representation.

Theorem 5.3 (Gelfand, Naimark). *For every C^*-algebra $\mathcal{A}$ there exists a representation $(\pi, \mathcal{H})$ which is one to one (called* faithful*).*

T h e i d e a o f t h e p r o o f. We may adjoin an identity, if necessary; so we may assume $\mathcal{A}$ unital. Let $\mathcal{S}(\mathcal{A})$ be the set of all states. For each $\phi \in \mathcal{S}(\mathcal{A})$

consider the triple $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$; 'adding up' all these representations, we obtain a representation $(\pi, \mathcal{H})$.

More precisely:

The space $\mathcal{H}$ consists of all families $(x_\phi)_{\phi\in\mathcal{S}(\mathcal{A})}$ of vectors $x_\phi \in \mathcal{H}_\phi$ such that $\sum_\phi \|x_\phi\|^2_{\mathcal{H}_\phi} < \infty$.[16] Given a family $(A_\phi)_{\phi\in\mathcal{S}(\mathcal{A})}$ of operators where $A_\phi \in \mathcal{B}(\mathcal{H}_\phi)$ such that $\sup_\phi \|A_\phi\| < \infty$, the map $A =: \oplus A_\phi$ given by $A((x_\phi)) = (A_\phi x_\phi)$ is a well defined bounded operator on $\mathcal{H}$. Thus for each $a \in \mathcal{A}$, since $\sup_\phi \|\pi_\phi(a)\| \leqslant \|a\|$, we may define the operator $\pi(a) := \sum_\phi \pi_\phi(a) \in \mathcal{B}(\mathcal{H})$; one can readily verify that the map $\mathcal{A} \to \mathcal{B}(\mathcal{H}) : a \to \pi(a)$ is a *-representation of $\mathcal{A}$.

It remains to prove that it is faithful. This follows from the fact (see the following lemma) that for each nonzero $a \in \mathcal{A}$ there exists $\psi \in \mathcal{S}(\mathcal{A})$ such that $\psi(a^*a) > 0$. Denoting by $\mathbf{x}_\psi \in \mathcal{H}$ the family $(x_\phi)_\phi$ with $x_\psi = \xi_\psi$ and $x_\phi = 0$ for all $\phi \neq \psi$ we have

$$\|\pi(a)\mathbf{x}_\phi\|^2_{\mathcal{H}} = \|\pi_\psi(a)\xi_\psi\|^2 = \langle \pi_\psi(a^*a)\xi_\psi, \xi_\psi\rangle = \psi_a(a^*a) > 0$$

which proves that $\pi(a) \neq 0$, as required.

It remains to prove the following

Lemma 5.4. *For each nonzero $a \in \mathcal{A}$ there exists $\psi \in \mathcal{S}(\mathcal{A})$ such that $\psi(a^*a) > 0$.*

Proof. Consider the real Banach space $\mathcal{A}_h$ of all selfadjoint elements of $\mathcal{A}$. The set $\mathcal{A}_+$ is a closed convex cone in $\mathcal{A}_h$ and the element $b := -a^*a \in \mathcal{A}_h$ is not in $\mathcal{A}_+$. By the Hahn - Banach separation theorem, there is a (real-linear) functional $\omega : \mathcal{A}_h \to \mathbb{R}$ and a $c \in \mathbb{R}$ such that $\omega(b) < c$ and $\omega(x) \geqslant c$ for all $x \in \mathcal{A}_+$. Note that $c \leqslant 0$ because $0 = \omega(0) \geqslant c$ since $0 \in \mathcal{A}_+$.

We claim that $\omega(\mathcal{A}_+) \subseteq \mathbb{R}_+$. Indeed, if $\omega(y) < 0$ for some $y \in \mathcal{A}_+$ then $\omega(ny) = n\omega(y) < c$ for large enough $n \in \mathbb{N}$, contradicting the fact that $\omega(x) \geqslant c$ for all $x \in \mathcal{A}_+$.

We extend ω to a complex linear map $\omega_c : \mathcal{A} \to \mathbb{C}$ by setting

$$\omega_c(x + iy) = \omega(x) + i\omega(y), \quad x, y \in \mathcal{A}_h.$$

[16] that is, such that $\sup_{\mathcal{F}} \sum_{\phi\in\mathcal{F}} \|x_\phi\|^2_{\mathcal{H}_\phi} < \infty$, where the supremum ranges over all finite subsets $\mathcal{F}$ of $\mathcal{S}(\mathcal{A})$

Then $\omega_c|_{\mathcal{A}_h} = \omega$, hence ω_c is positive and so $\psi := \frac{\omega_c}{\|\omega_c\|}$ is a state; finally $\|\omega_c\| \, \psi(b) = \omega(b) < c$ and so $\psi(a^*a) = -\psi(b) > 0$ since $c \leqslant 0$. $\square$

6. Bimodules over masas.

6.1. Von Neumann algebras. Apart from the norm, $\mathcal{B}(\mathcal{H})$ is equipped with other natural topologies.

We will concentrate on the weak* topology that $\mathcal{B}(\mathcal{H})$ has as a dual Banach space:

For $\xi, \eta \in \mathcal{H}$, we denote by $\omega_{\xi,\eta}$ the linear form on $\mathcal{B}(\mathcal{H})$ given by

$$\omega_{\xi,\eta}(T) = \langle T\xi, \eta \rangle, \quad T \in \mathcal{B}(\mathcal{H}).$$

This is clearly bounded (by $\|\xi\| \, \|\eta\|$). We denote by $\mathcal{B}(\mathcal{H})_\sim$ the linear space spanned by these linear forms, and by $\mathcal{B}(\mathcal{H})_*$ its closure in the dual Banach space of $\mathcal{B}(\mathcal{H})$.

Each $T \in \mathcal{B}(\mathcal{H})$ defines a bounded linear form ϕ_T on $\mathcal{B}(\mathcal{H})_\sim$ by evaluation: $\phi_T(\omega) = \omega(T)$, and in particular, $\phi_T(\omega_{\xi,\eta}) = \langle T\xi, \eta \rangle$. Conversely, each bounded linear form ϕ on $\mathcal{B}(\mathcal{H})_\sim$ defines an operator $T_\phi \in \mathcal{B}(\mathcal{H})$ such that $\langle T_\phi \xi, \eta \rangle = \phi(\omega_{\xi,\eta})$ for all $\xi, \eta \in \mathcal{H}$.[17]

Proposition 6.1. *The map $T \to \phi_T$ is an isometric isomorphism from $\mathcal{B}(\mathcal{H})$ onto the Banach space dual of $\mathcal{B}(\mathcal{H})_\sim$ (and hence of its closure $\mathcal{B}(\mathcal{H})_*$) with inverse $\phi \to T_\phi$.*

Thus $\mathcal{B}(\mathcal{H})$ acquires a weak* topology, as the dual of the Banach space $\mathcal{B}(\mathcal{H})_*$: a net T_i converges to 0 in this topology if and only if $\omega(T_i) \to 0$ for all $\omega \in \mathcal{B}(\mathcal{H})_*$. For *norm bounded* nets (in particular, for sequences), this is equivalent to the requirement that $\langle T_i\xi, \eta \rangle \to 0$ for all $\xi, \eta \in \mathcal{H}$.

A **von Neumann algebra** $\mathcal{M}$ is a selfadjoint unital subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the weak* topology.

Theorem 6.2 (von Neumann's bicommutant theorem). *If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a selfadjoint unital algebra and $T \in \mathcal{B}(\mathcal{H})$, the following are equivalent:*

(*a*) *$T \in \mathcal{A}''$.*

(*b*) *For each $\xi \in \mathcal{H}$, the operator T is in the closed linear span of $\{Ax : A \in \mathcal{A}\}$.*

(*c*) *T is in the weak*-closure of $\mathcal{A}$.*

[17] because the map $(\xi, \eta) \to \phi(\omega_{\xi,\eta})$ is a bounded sesquilinear form on $\mathcal{H} \times \mathcal{H}$

For later use, note that the equivalence of (b) and (c) says that a selfadjoint unital algebra is weak*-closed if and only if it is equal to the annihilator of a set of *vector (or rank one) functionals*, i.e. functionals of the form $\omega_{\xi,\eta}$.

Of course every von Neumann algebra is a C*-algebra but not conversely. For example the algebra of compact operators on an infinite dimensional Hilbert space is a C*-algebra, but is not weak*-closed in $\mathcal{B}(\mathcal{H})$, hence it is not a von Neumann algebra.

Similarly, the set of all multiplication operators $\{M_f : f \in C([0,1])\}$ is a C*-algebra $\mathcal{C}$ of operators on $L^2([0,1])$, but is not weak*-closed. It is not hard to see that the bicommutant $\mathcal{C}''$ is the set $\mathcal{M} = \{M_f : f \in L^\infty([0,1])\}$, and this is a von Neumann algebra.[18]

Abelian von Neumann algebras. It can be shown that any abelian von Neumann algebra $\mathcal{M}$ is *-isomorphic (isometrically, of course) to the algebra $L^\infty(X,\mu)$ for a suitable measure space (X,μ), where X may be taken locally compact Hausdorff and μ a regular Borel measure. In fact the *-isomorphism is bicontinuous for the weak* topology on $\mathcal{M}$ and the weak* topology on $L^\infty(X,\mu)$ as the dual of $L^1(X,\mu)$. For this reason, the theory of von Neumann algebras is sometimes described as "non-commutative measure theory", while the theory of C*-algebras is thought of as "non-commutative topology".

A *maximal abelian selfadjoint algebra (masa* for short) $\mathcal{M}$ is an abelian selfadjoint subalgebra of some $\mathcal{B}(\mathcal{H})$ which is maximal among abelian selfadjoint subalgebras of $\mathcal{B}(\mathcal{H})$. It is not hard to see that maximality is equivalent to the requirement that $\mathcal{M} = \mathcal{M}'$; hence a masa is automatically a von Neumann algebra.

A masa $\mathcal{M}$ is not only *-isomorphic, it is *spatially isomorphic* (that is, unitarily equivalent) to a *multiplication algebra*

$$\mathcal{M}_\mu := \{M_f : f \in L^\infty(X,\mu)\} \subseteq \mathcal{B}(L^2(\mu)).$$

In fact when $\mathcal{M}$ acts on a separable space, then it is spatially isomorphic to one of the following: $L^\infty([0,1])$ (with Lebesgue measure), $\ell^\infty(n)$, or $L^\infty([0,1]) \oplus \ell^\infty(n)$, for some $n \in \mathbb{N}$ or $n = \aleph_0$.

The first case arises when $\mathcal{M}$ has no minimal projections, the second when each projection in $\mathcal{M}$ dominates a minimal projection in $\mathcal{M}$, and the third when there are n minimal projections whose sum is not the identity operator.

In this last case $\mathcal{M}$ is unitarily equivalent to the von Neumann algebra $\mathcal{M}_\mu \oplus \mathcal{D}_n$ acting on $L^2([0,1],\mu) \oplus \ell^2(n)$ (here μ denotes Lebesgue measure and

[18] In fact $\mathcal{M} = \mathcal{C}'$.

$\mathcal{D}_n$ denotes the set of all bounded operators on $\ell^2(n)$ which are diagonal with respect to the usual orthonormal basis of $\ell^2(n)$).

6.2. The support of an operator. *In the sequel we shall assume that all Hilbert spaces are separable. In particular the predual of $\mathcal{B}(\mathcal{H})$, and of every von Neumann algebra, will be separable. The material that follows is based on* [1, 7, 22] *and* [23].

An operator $T \in \mathcal{B}(\ell^2)$ is said to *vanish* on a rectangle $A \times B \subseteq \mathbb{N} \times \mathbb{N}$ if $P(B)TP(A) = 0$, where $P(A)$ is the projection onto the space spanned by the basis elements $\{e_j : j \in A\}$. Notice that these projections belong to the masa $\mathcal{D} \subseteq \mathcal{B}(\ell^2)$ of all diagonal operators. Thus $\mathcal{D}$ codifies the 'coordinate system' induced by the usual basis of ℓ^2. More generally, every masa $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ can be said to introduce a 'coordinate system': After a unitary equivalence, we may identify $\mathcal{H}$ with $L^2(X,\mu)$ and $\mathcal{M}$ with the multiplication masa $\mathcal{M}_\mu$ of $L^\infty(X,\mu)$; the 'coordinate system' is indexed by X. In this representation, we say that an operator $T \in \mathcal{B}(\mathcal{H})$ *vanishes on a Borel rectangle* $A \times B \subseteq X \times X$ if $P(B)TP(A) = 0$, where $P(A)$ is the projection onto the space of all $f \in L^2(X,\mu)$ that vanish almost everywhere on A^c; thus $P(A)$ is an element of $\mathcal{M}$, namely the multiplication operator corresponding to χ_A.

Definition 9. *We say that a set of operators $\mathcal{T} \subseteq \mathcal{B}(\mathcal{H})$ is* supported *in a set $\Omega \subseteq X \times X$ if $P(B)TP(A) = 0$ for all $T \in \mathcal{T}$ whenever $\Omega \cap (A \times B) = \varnothing$.*

If Ω is a measurable set of positive product measure, then it supports nonzero operators, for example any Hilbert-Schmidt operator whose kernel vanishes almost everywhere (with respect to product measure) on Ω^c. However even sets of product measure zero can support nonzero operators: for example the diagonal $\Delta = \{(x,x) : x \in [0,1]\}$ supports the identity operator, as well as any multiplication operator M_f with $f \in L^\infty([0,1])$.

A set $\Omega \subseteq X \times X$ is said to be *marginally null* if it is contained in a union $(N \times X) \cup (X \times M)$, where $\mu(N) = \mu(M) = 0$. Such a set cannot support a nonzero operator T, because $(N^c \times M^c) \cap \Omega = \varnothing$ whereas $P(M^c)TP(N^c) = T \neq 0$.

One would like to define 'the support' of a set $\mathcal{T}$ of operators to be the complement of the union of the family $\mathcal{E}$ all Borel rectangles on which $\mathcal{T}$ vanishes. However this union is in general non-measurable. The way around this difficulty is the following: there exists a countable set $\{E_n\} \subseteq \mathcal{E}$ whose union E (a measurable set) 'almost contains' every Borel rectangle $A \times B \in \mathcal{E}$, in the sense that $(A \times B) \backslash E$ is *marginally null*. Thus E^c 'almost contains' every subset of $X \times X$ supporting $\mathcal{T}$.

Definition 10. *The complement E^c of the union of the rectangles in $\{E_n\}$ is defined to be* the support *of $\mathcal{T}$ and is denoted* $\operatorname{supp} \mathcal{T}$.

Let us call two subsets E, F of $X \times X$ *marginally equivalent* (written $E \simeq F$) if their symmetric difference is marginally null; let us call a subset E of $X \times X$ ω-open if it is marginally equivalent to a *countable* union of Borel rectangles; the complements of ω-open sets are of course called ω-*closed* sets. Thus $\operatorname{supp} \mathcal{T}$ is ω-closed; it is uniquely determined (up to marginal equivalence) and is the smallest ω-closed set supporting $\mathcal{T}$.

6.3. Masa bimodules, reflexivity and operator synthesis. Fix any set $\Omega \subseteq X \times Y$. The set of all operators which are supported in Ω is denoted $\mathfrak{M}_{\max}(\Omega)$.

This is easily seen to be a weak-* closed linear space. Also, it is a *bimodule over the masa* $\mathcal{M}$: indeed if T is supported in Ω, then so is $M_f T M_g$, for every M_f, M_g in the masa $\mathcal{M}$.

It is not hard to see that $\mathfrak{M}_{\max}(\Omega)$ is *reflexive* in the sense of Loginov-Shulman [17]; that is, $\mathfrak{M}_{\max}(\Omega)$ is equal to the annihilator of a set of *rank one functionals*.[19] The support of $\mathfrak{M}_{\max}(\Omega)$ is an ω-closed set, it contains Ω and is, up to marginally null sets, the smallest ω-closed set containing Ω; it is called *the* ω-*closure* of Ω.

It can be shown conversely that if an $\mathcal{M}$-bimodule $\mathcal{T}$ is reflexive, then it is necessarily of the form $\mathcal{T} = \mathfrak{M}_{\max}(\Omega)$, where Ω can be chosen ω-closed; in fact, Ω is the support of $\mathcal{T}$.

Thus there is a bijective correspondence between *reflexive* $\mathcal{M}$-bimodules and ω-closed subsets of $X \times X$.

Note that, in case X comes equipped with a topology, the support of a masa bimodule cannot always be chosen to be topologically closed. For example, there is a reflexive $\mathcal{M}$-bimodule $\mathcal{U} \subset \mathcal{B}(L^2([0,1])$ (where $\mathcal{M}$ is the masa of $L^\infty([0,1]))$ such that the smallest *closed* subset of $[0,1] \times [0,1]$ supporting $\mathcal{U}$ is the whole of $[0,1] \times [0,1]$, although $\mathcal{U} \neq \mathcal{B}(L^2([0,1])$.[20]

We have seen that for unital and selfadjoint operator algebras, closure in the weak-* topology automatically implies reflexivity (von Neumann's bicommutant theorem, 6.2). For non-selfadjoint algebras this is no longer true: the simplest example is the algebra of all 2×2 complex matrices of the form $\left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right]$; but this algebra is not a masa bimodule. What happens in the masa bimodule case?

[19] Indeed, $\mathfrak{M}_{\max}(\Omega)$ is the annihilator of the set of all functionals $\omega_{P(A)f,P(B)g}$, where $f, g \in L^2(X, \mu)$ are arbitrary and A, B are Borel subsets of X satisfying $(A \times B) \cap \Omega = \varnothing$.

[20] One can take $\mathcal{U} = \{M + PXP : M \in \mathcal{M},\ X \in \mathcal{B}(\mathcal{H})\}$ where $P = P(A)$ and $A \subseteq [0,1]$ is chosen so that both A and A^c intersect every open set in a set of nonzero Lebesgue measure.

If $\mathcal{U} \subseteq \mathcal{B}(\mathcal{H})$ is a weak-* closed subspace which is a bimodule over a *discrete* masa $\mathcal{D}$ (so that $\mathcal{H}$ may be realized as ℓ^2 and $\mathcal{D}$ as the algebra of all diagonal matrices), then it is automatically (and trivially) reflexive: its support Ω is the complement of the set of all pairs $(m,n) \in \mathbb{N} \times \mathbb{N}$ such that $u_{m,n} = 0$ for every $U = [u_{m,n}]$ in $\mathcal{U}$; hence, since $\omega_{e_n,e_m}(U) = u_{m,n}$, $\mathcal{U}$ is the annihilator of all rank one functionals $\{\omega_{e_n,e_m} : (m,n) \in \Omega^c\}$; equivalently, every matrix which vanishes in Ω^c must be in $\mathcal{U}$, and so $\mathcal{U} = \mathfrak{M}_{\max}(\Omega)$.

When the masa $\mathcal{M}$ is not generated by its minimal projections, the situation is more complex. Arveson [1] was the first to exhibit a weak-* closed masa bimodule $\mathcal{U}$ with support Ω for which $\mathcal{U} \neq \mathfrak{M}_{\max}(\Omega)$. He called this phenomenon *failure of operator synthesis*, as his example was based on the failure of spectral synthesis in the group algebra $L^1(\mathbb{R}^3)$.

He proved[21] that any weak-* closed masa bimodule $\mathcal{U}$ with support Ω lies between two extremal weak-* closed masa bimodules: $\mathfrak{M}_{\min}(\Omega) \subseteq \mathcal{U} \subseteq \mathfrak{M}_{\max}(\Omega)$.

The predual approach [22]. When the masa $\mathcal{M}$ is identified with the multiplication algebra of $L^\infty(X,\mu)$ acting on $\mathcal{H} = L^2(X,\mu)$, every element $\omega \in \mathcal{B}(\mathcal{H})_\sim$ is identified with a function on $X \times X$; indeed $\omega = \sum_{k=1}^{n} \omega_{f_k,g_k}$ corresponds to the function

$$F_\omega(s,t) = \sum_{k=1}^{n} f_k(s)\bar{g}_k(t), \quad (s,t) \in X \times X.$$

In fact every element $\omega \in \mathcal{B}(\mathcal{H})_*$ admits a representation $\omega = \sum_{k=1}^{\infty} \omega_{f_k,g_k}$ with $\sum_k \|f_k\| \, \|g_k\| = \|\omega\| < \infty$ and hence defines the function

$$F_\omega(s,t) = \sum_{k=1}^{\infty} f_k(s)\bar{g}_k(t), \quad (s,t) \in X \times X$$

where the series converges *marginally almost everywhere on* $X \times X$, that is for all $(s,t) \in X \times X$ outside a marginally null set. Two functions define the same element of $\mathcal{B}(\mathcal{H})_*$ if and only if they agree *marginally almost everywhere.*

The space $T(X)$ of all (marginal equivalence classes of) functions on $X \times X$ of the above form, equipped with the norm $\|\cdot\|_t$ inherited from $\mathcal{B}(\mathcal{H})_*$ coincides

[21] for the separably acting unital algebra case

with the *projective tensor product* $L^1(X,\mu)\widehat{\otimes}L^1(X,\mu)$. Given any ω-closed set $\Omega \subseteq X \times X$ we consider the subspaces

$$\begin{aligned}\Phi(\Omega) &= \{h \in T(X) : h = 0 \text{ marginally a.e. on } \Omega\} \\ \Psi_0(\Omega) &= \{h \in T(X) : h = 0 \text{ marginally a.e. in an } \omega\text{-open neighbourhood of } \Omega\}\end{aligned}$$

where an ω-open neighbourhood of Ω is a countable cover of Ω by Borel rectangles. It can be shown that the annihilator of $\Psi_0(\Omega)$ in $\mathcal{B}(\mathcal{H})$ is $\mathfrak{M}_{\max}(\Omega)$, while the annihilator of $\Phi(\Omega)$ is the minimal weak-* closed bimodule $\mathfrak{M}_{\min}(\Omega)$ having support Ω. This leads to the

Definition 11. *An ω-closed set Ω is said to satisfy* operator synthesis *if $\mathfrak{M}_{\min}(\Omega) = \mathfrak{M}_{\max}(\Omega)$, equivalently is every $h \in T(X)$ that vanishes (marginally a.e.) on Ω can be approximated (in the norm $\|\cdot\|_t$) by elements of $T(X)$ vanishing in an ω-open neighbourhood of Ω.*

The investigation of conditions that imply operator synthesis is an active area of research, with close connections to harmonic analysis. We refer the reader to the contribution of I. G. Todorov [27] in these proceedings.

REFERENCES

[1] W. Arveson. Operator algebras and invariant subspaces. *Ann. of Math.* (2) **100** (1974), 433–532.

[2] K. R. Davidson. C^*-algebras by example. Fields Institute Monographs, vol. **6**. Providence, RI, American Mathematical Society, 1996.

[3] J. Dixmier. C^*-algebras. Translated from the French by Francis Jellett. North-Holland Mathematical Library, vol. **15**. Amsterdam, North-Holland Publishing Co., 1977.

[4] J. Dixmier. von Neumann algebras. Translated from the second French edition by F. Jellett. North-Holland Mathematical Library, vol. **27**. Amsterdam-New York, North-Holland Publishing Co., 1981.

[5] J. Dixmier. Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann). Reprint of the second (1969) edition. Les Grands Classiques Gauthier-Villars. Paris, Éditions Jacques Gabay, 1996.

[6] J. Dixmier. Les C^*-algèbres et leurs représentations. Reprint of the second (1969) edition. Les Grands Classiques Gauthier-Villars. Paris, Éditions Jacques Gabay, 1996.

[7] J. A. Erdos, A. Katavolos, V. S. Shulman. Rank one subspaces of bimodules over maximal abelian selfadjoint algebras. *J. Funct. Anal.* **157**, *2* (1998), 554–587.

[8] P. A. Fillmore. Notes on Operator Theory. Van Nostrand Reinhold Mathematical Studies, No. **30**. New York, Van Nostrand Reinhold Co., 1970.

[9] P. A. Fillmore. A User's Guide to Operator Algebras. Canadian Mathematical Society Series of Monographs and Advanced Texts. A Wiley-Interscience Publication. New York, John Wiley & Sons Inc., 1996.

[10] I. Gel'fand, M. Neumark. On the imbedding of normed rings into the ring of operators in Hilbert space. In: C^*-algebras: 1943–1993 (San Antonio, TX, 1993) *Contemp. Math.* **167**, 2–19, Providence, RI, Amer. Math. Soc. (Corrected reprint of the 1943 original).

[11] R. V. Kadison, J. R. Ringrose. Fundamentals of the theory of operator algebras. Vol. I: Elementary theory. Pure and Applied Mathematics, vol. **100**. New York, Academic Press Inc., 1983.

[12] R. V. Kadison, J. R. Ringrose. Fundamentals of the theory of operator algebras. Vol. II: Advanced theory. Pure and Applied Mathematics, vol. **100**. Orlando, FL, Academic Press Inc., 1986.

[13] R. V. Kadison, J. R. Ringrose. Fundamentals of the theory of operator algebras. Vol. III: Special topics, Elementary theory—an exercise approach. Boston, MA, Birkhäuser Boston Inc., 1991.

[14] R. V. Kadison, J. R. Ringrose. Fundamentals of the theory of operator algebras. Vol. IV: Special topics, Advanced theory—an exercise approach. Boston, MA, Birkhäuser Boston Inc., 1992.

[15] R. V. Kadison, J. R. Ringrose. Fundamentals of the theory of operator algebras. Vol. I: Elementary theory, Reprint of the 1983 original. Graduate Studies in Mathematics, vol. **15**. Providence, RI, Amer. Math. Soc., 1997.

[16] R. V. Kadison, J. R. Ringrose. Fundamentals of the theory of operator algebras. Vol. II: Advanced theory, Corrected reprint of the 1986 original. Graduate Studies in Mathematics, vol. 16. Providence, RI, Amer. Math. Soc., 1997.

[17] A. I. Loginov, V. S. Sul'man. Hereditary and intermediate reflexivity of W^*-algebras. *Math. USSR, Izv.* **9** (1977), 1189–1201.

[18] G. J. Murphy. C^*-algebras and Operator Theory. Boston, MA, Academic Press Inc., 1990.

[19] F. J. Murray, J. von Neumann. On rings of operators. *Ann. of Math.* (2) **37**, *1* (1936), 116–229.

[20] G. K. Pedersen. C^*-algebras and their Automorphism Groups. London Mathematical Society Monographs, vol. **14**. London - New York -San Francisco, Academic Press Inc., 1979.

[21] S. Sakai. C^*-algebras and W^*-algebras. Reprint of the 1971 edition. Classics in Mathematics. Berlin,Springer-Verlag, 1998.

[22] V. S. Shulman, L. Turowska. Operator synthesis. I: Synthetic sets, bilattices and tensor algebras. *J. Funct. Anal.* **209**, *2* (2004), 293–331.

[23] V. S. Shulman, L. Turowska. Operator synthesis. II: Individual synthesis and linear operator equations. *J. Reine Angew. Math.* 590 (2006), 143–187.

[24] M. Takesaki. Theory of operator algebras, I. Reprint of the first (1979) edition. Encyclopaedia of Mathematical Sciences, vol. 124. Operator Algebras and Non-commutative Geometry, **5**. Berlin, Springer-Verlag, 2002.

[25] M. Takesaki. Theory of operator algebras, II. Encyclopaedia of Mathematical Sciences, vol. **125**. Operator Algebras and Non-commutative Geometry, **6**. Berlin, Springer-Verlag, 2003.

[26] M. Takesaki. Theory of operator algebras, III. Encyclopaedia of Mathematical Sciences, vol. **127**. Operator Algebras and Non-commutative Geometry, **8**. Berlin, Springer-Verlag, 2003.

[27] I. G. Todorov. Interactions between harmonic analysis and operator theory. *Serdica Math. J.* **41**, *1* (2015), 13–34.

[28] J. von Neumann. Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren. *Math. Ann.* **102** (1929), 370–427.

[29] N. E. Wegge-Olsen. K-theory and C^*-algebras. A friendly approach. Oxford Science Publications. New York, The Clarendon Press Oxford University Press, 1993.

Department of Mathematics
University of Athens
Panepistimioupolis
157 84 Athens, Greece
e-mail: `akatavol@math.uoa.gr`

Received August 25, 2013