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EP ELEMENTS IN RINGS AND IN SEMIGROUPS WITH INVOLUTION AND IN C^*-ALGEBRAS

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Abstract. This work includes a survey of most of the results concerning EP elements in semigroups and rings with involution and in C^*-algebras

1. Introduction. Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. It is well known that when T is an operator with closed range then its unique generalized inverse $T^\dagger$ (known also as Moore-Penrose inverse) is defined. If T has a closed range and commutes with $T^\dagger$, then T is called an EP operator. EP operators constitute a wide class of operators which includes the

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self adjoint operators with closed range, the normal operators with closed range and the invertible operators.

All the above have the origin to the notion of an EP matrix. An $n \times n$ complex matrix A is an EP matrix if the range of A coincides with the range of the adjoint A^* of A. EP matrices were introduced by Schwerdtfeger in [35] (it should be noted that Schwerdtfeger definition is not the one given in the beginning, but an equivalent one).

The notion of an EP matrix was extended by Campel and Meyer to operators on a Hilbert space in [5].

In this paper most of the results concerning EP elements in semigroups and rings with involution and in C^*-algebras are presented. For simplicity the notion of an EP operator is given first and then some more results are presented in a particular section. Following EP elements in general which are arranged in three sections for rings, for semigroups and for C^*-algebras correspondingly, are studied.

2. Preliminaries and notation. Let $\mathcal{H}$ be a complex Hilbert space. Then, $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators on $\mathcal{H}$, and for $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{R}(T)$ denotes the range of T, and $\mathcal{N}(T)$ the kernel of T.

To denote the direct sum of two subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{H}$ we will use $\mathcal{M}_1 \oplus \mathcal{M}_2$ and to denote the orthogonal sum of two orthogonal subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{H}$ we will use $\mathcal{M}_1 \overset{\perp}{\oplus} \mathcal{M}_2$. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are subspaces of $\mathcal{H}$ with $\mathcal{M}_1 \oplus \mathcal{M}_2 = \mathcal{H}$, then we will say that $\mathcal{M}_2$ is a complement of $\mathcal{M}_1$ and we will denote the projection onto $\mathcal{M}_1$ parallel to $\mathcal{M}_2$ by $P_{\mathcal{M}_1 \| \mathcal{M}_2}$. The orthogonal projection onto a subspace $\mathcal{M}$ of $\mathcal{H}$ will be denoted by $P_{\mathcal{M}}$.

The generalized inverse, known as Moore-Penrose inverse, of an operator $T \in \mathcal{B}(\mathcal{H})$ with closed range, is the unique operator $T^\dagger$ satisfying the following four conditions:

$$(1) \qquad TT^\dagger = (TT^\dagger)^*, \qquad T^\dagger T = (T^\dagger T)^*, \qquad TT^\dagger T = T, \qquad T^\dagger T T^\dagger = T^\dagger$$

where T^* denotes the adjoint operator of T.

It is easy to see that $\mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp$, $TT^\dagger$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}(T)$ and that $T^\dagger T$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{N}(T)^\perp$. It is well known that $\mathcal{R}(T^\dagger) = \mathcal{R}(T^*)$. It is also known that $T^\dagger$ is bounded if and only if T has a closed range.

An operator T with closed range is called EP if $[\,T, T^\dagger\,] = 0$. It is easy to see that we have the following equivalent conditions for T to be EP:

$$\begin{aligned} T \text{ EP} \;\;&\Leftrightarrow\;\; [T, T^\dagger] = 0 \Leftrightarrow TT^\dagger = T^\dagger T \Leftrightarrow P_T = P_{T^*} \Leftrightarrow P_{\mathcal{N}(T^*)} = P_{\mathcal{N}(T)} \\ &\Leftrightarrow\;\; \mathcal{N}(T) = \mathcal{N}(T^*) \Leftrightarrow \mathcal{R}(T) = \mathcal{R}(T^*) \Leftrightarrow \mathcal{R}(T) \overset{\perp}{\oplus} \mathcal{N}(T) = \mathcal{H}. \end{aligned}$$

It is worthwhile to notice that in finite dimensional case, since $\mathcal{N}(T)$, $\mathcal{N}(T^*)$, are equidimansional, if $\mathcal{N}(T)$ is contained in $\mathcal{N}(T^*)$ or vice versa then T is EP. This is not in general true in the infinite dimensional case (e.g. Let T be the left shift).

We note two things about the last equivalence in the characterization of EP operators:
(1) This equivalence is not true if the sum is not an orthogonal one. To see that let $P_{\mathcal{M}_1 \| \mathcal{M}_2}$ be a non-orthogonal projection. Then $P_{\mathcal{M}_1 \| \mathcal{M}_2}$ is not EP, since

$$\begin{aligned} P_{\mathcal{M}_1 \| \mathcal{M}_2} \text{ is EP} \;\;&\Leftrightarrow\;\; \mathcal{N}(P_{\mathcal{M}_1 \| \mathcal{M}_2}) = \mathcal{N}(P_{\mathcal{M}_2^\perp \| \mathcal{M}_1^\perp}) \\ &\Leftrightarrow\;\; \mathcal{M}_2 = \mathcal{M}_1^\perp \\ &\Leftrightarrow\;\; P_{\mathcal{M}_1 \| \mathcal{M}_2} = P_{\mathcal{M}_1 \| \mathcal{M}_1^\perp} \\ &\Leftrightarrow\;\; P_{\mathcal{M}_1 \| \mathcal{M}_2} = P_{\mathcal{M}_1} \\ &\Leftrightarrow\;\; P_{\mathcal{M}_1 \| \mathcal{M}_2} \text{ is an orhtogonal projection} \end{aligned}$$

whereas

$$\mathcal{R}\left(P_{\mathcal{M}_1 \| \mathcal{M}_2}\right) \oplus \mathcal{N}\left(P_{\mathcal{M}_1 \| \mathcal{M}_2}\right) = \mathcal{M}_1 \oplus \mathcal{M}_2 = \mathcal{H}.$$

(2) For matrices we have the following stronger version of this equivalence: If A is an $n \times n$ complex matrix, then A is EP if and only if there exist subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathbb{C}^n$ such that $\mathbb{C}^n = \mathcal{M}_1 \overset{\perp}{\oplus} \mathcal{M}_2$ and $A(\mathcal{M}_1) = \mathcal{M}_1$, $A(\mathcal{M}_2) = \{0\}$. This is not true if we move to operators on a Hilbert space.

Obviously T is EP if and only if T^* is EP and T is EP if and only if $T^\dagger$ is EP.

Isomorphisms are EP. Moreover we have that if T is EP, then the following are equivalent:

(1) T is an isomorphism.

(2) T is injective.

(3) T is surjective.

Normal operators with closed range are EP, since if T is normal, then

$$\mathcal{R}(T) = \mathcal{R}((TT^*)^{\frac{1}{2}}) = \mathcal{R}((T^*T)^{\frac{1}{2}}) = \mathcal{R}(T^*).$$

Elementary examples with matrices show that there exist operators with closed range which are EP, but are not normal (for example, let $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$).

For a bounded projection P we have, as we already saw, that P is EP if and only if P is orthogonal. In particular the zero operator 0 is EP. Most of the aforementioned are from [11].

Let R be a ring and $a \in R$. Then the element a is group invertible if there is $a^\sharp \in R$ such that

$$aa^\sharp a = a, \quad a^\sharp aa^\sharp = a^\sharp, \quad aa^\sharp = a^\sharp a.$$

We denote the set of all group invertible elements of R by $R^\sharp$. The element a is called regular if there exist an element $a^- \in R$ such that $aa^-a = a$. The element a^- is called inner or 1-inverse of a. A ring R is called regular if every element in R is regular or equivalently if for every $a \in R$, $a \in aRa$. Regular rings are important in many branches of Mathematics and especially in matrix theory, since regularity condition is a linear condition that solves linear equations and take the place of canonical decomposition. A non zero element $a \in R$ is called anti-regular if there exist an element $\hat{a} \in R$ such that $\hat{a}a\hat{a} = \hat{a}$. The element $\hat{a}$ is called outer or 2-inverse of a and similarly the ring R is called anti-regular if every non-zero element is anti-regular. Any element a^+ which is an inner and an outer inverse of a is called a reflexive or 1-2 inverse of a.

The right and the left annilators of $a \in R$ will be denoted by

$$a^0 = \{x \in R : \, ax = 0\} \quad {}^0a = \{x \in R : \, xa = 0\}$$

correspondingly.

An involution in R is an anti-isomorphism $(\cdot)^* : R \to R$ of degree 2, that is,

$$(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^*a^*.$$

If $a = a^*$ the element a is called Hermitian and if $a^*a = aa^*$, is called normal. A *-regular ring is a ring with involution * such that $a^*a = 0 \Rightarrow a = 0$ for all $a \in R$. An important result in a *-regular ring is the global star cancellation law: $a^*ab = a^*ac \Rightarrow ab = ac$. A ring R is called strongly regular if for every $a \in R$, $a \in a^2R$, unit regular if for every $a \in R$, there exists a unit $u \in R$ such that $aua = a$ and faithful if $aR = \{0\}$ implies $a = 0$. It is well known that a ring R is strongly regular if and only if every $a \in R$ is a group member. A fundamental theorem concerning equivalences of the notion group member is the following

Theorem 2.1 ([18], Th. 1, p. 450)**.** *Let S be a semigroup and $a \in S$. The following are equivalent.*

1. *a is a group member.*

2. *a has a group inverse $a^\sharp$ in S which satisfies $axa = a$, $xax = x$, and $ax = xa$.*

3. *a has a commutative inner inverse a^- which satisfies $axa = a$ and $ax = xa$.*

4. *$aS = eS$, $Sa = Se$ and $a \in eSe$ for some idempotent $e \in S$.*

5. *$a \in a^2S \cap Sa^2$.*

6. *$a \in a^-aSaA^=$ for some inner inverses $a^-, a^=$ in S.*

7. *$aS = Sa^+$ for some reflexive inverse a^+ in S.*

7a. *$Sa = Sa^+$ for some reflexive inverse a^+ in S.*

8. *$aS = A^-aS$ for some inner inverses $a^-, a^=$ in S.*

8a. *$Sa = Saa^-$ for some inner inverses $a^-, a^=$ in S.*

If in addition $S = R$ is a faithful ring, these are equivalent to

9. *$R = aR \oplus a^0$*

9a. *$R = Ra \oplus {}^0a$*

In any of the above cases $a^\sharp$ and $e = aa^\sharp$ are unique and the maximal subgroup containing a is given by

$$H_a = \{x \in S : x^\sharp exists, xx^\sharp = aa^\sharp = e\} = \{x \in S : xS = aS,\, Sx = Sa,\, x \in aSa\}$$

3. EP operators. The theory of EP operators has been developed considerably in the last 20 years and mainly since 1990. The first work in the subject was the paper of S. L. Campbel and C. D. Meyer in 1975, [5], followed by two papers of R. E. Hartwig [15], in 1976 and [16], in 1978, and a paper of R. E. Hartwig and I. J. Katz [17], in 1976. Then there is a gap of 12 years till the paper of K. G. Brock [4] in 1990.

The notion of an EP matrix was introduced by Schwerdtfeger in 1961. The extension of this notion to the infinite dimensional case, namely the notion of an EP operator was introduced by Campbel and Meyer in [5], in 1975 for operators with closed range, acting on a Hilbert space.

An operator $A \in \mathcal{B}(H)$ is called EP if its range $\mathcal{R}(A)$ is closed and $\mathcal{R}(A) = \mathcal{R}(A^*)$. Notice that this definition is equivalent to the one given on page 84.

In [5] several relationships between EP operators, generalised inverses, normal operators and binormal operators are given.

Theorem 3.1 ([5], Th. 2, p. 328)**.** *Let $A \in \mathcal{B}(\mathcal{H})$. Then each one of the following conditions implies the next.*

1) *A is normal.*
2) *A is EP.*
3) *A^2 is EP.*
4) *$(A^\dagger)^2 = (A^2)^\dagger$.*
5) *$AA^\dagger, A^\dagger A$ commute.*

Furthermore all the implications are proper (None of these implications reverse).

It is worthwhile to mention that the proof of the implication 2) $\Rightarrow$ 3) is based on the fact that an EP operator A, has a simple canonical form $A = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix}$ relative to the orthogonal decomposition $H = \mathcal{R}(A) \oplus \mathcal{N}(A)$, of the Hilbert space H.

With the same proof we can prove the following

Proposition 3.2. *If A^n is EP then $(A^\dagger)^n = (A^n)^\dagger$.*

Remark 3.3. Note that to each statement which characterizes a normal operator if the adjoint operation $(*)$ is replaced by the generalized inverse operation $(\dagger)$ then the resulting statement is called the dual statement and, in most cases, is a characterization for EP operators. For example the dual statement of the statement "A is a normal operator if and only if $A^*A = AA^*$" is "A is EP operator if and only if $A^\dagger A = AA^\dagger$" which is true. In [5] it is proved that Embry's result for matrices ([12]) which states that both A^*A and AA^* commute with $A + A^*$ then A is normal, it is true for operators in Hilbert space.

Theorem 3.4 ([5], Th. 6, p. 331)**.** *Let $A \in \mathcal{B}(\mathcal{H})$. If $AA^\dagger$ and $A^\dagger A$ both commute with $A + A^\dagger$, then A is EP.*

4. EP elements in rings with involution.

Hartwig in [15] defined and studied EP elements in a *-regular ring R.

Definition 4.1. *An element $a \in R$ is called EP if $aR = a^*R$.*

Hartwig in 1975 using properties of the four types of inverses (1-2,1-3,1-4, and 1-2-3-4 inverses) and techniques analogous to the complex case of EP matrices proved many equivalent formulations for an element $a \in R$ to be EP. Such formulations are seen in the following

Proposition 4.2 ([15], Prop. 25, Prop. 26, Lem. 6)**.** *Let R be a *-regular ring. The following are equivalent*

$$\begin{array}{rl}
(i) & aR = a^*R \\
(ii) & aa^\dagger = a^\dagger a \\
(iii) & Ra = Ra^* \\
(iv) & a^0 = (a^*)^0 \\
(v) & {}^0a = {}^0(a^*) \\
(vi) & a^2a^\dagger = a = a^\dagger a^2 \\
(vii) & a(aa^\dagger - a^\dagger a) = (aa^\dagger - a^\dagger a)a \\
(viii) & (a^2)^\dagger = (a^\dagger)^2, aR = a^2R, \text{ and } Ra = Ra^2 \\
(ix) & a^\dagger \in \{a\}'' = \{x|ar = ra \Rightarrow xr = rx\} \text{ (the bicommutant of } a.)
\end{array}$$

It is evident now that if a is an EP element then $a^\dagger$ is the group inverse of a and hence $R = aR \oplus a^0$.

Now since a normal element ($aa^* = a^*a$) is EP, we have that $a^\dagger \in \{a\}''$ and hence $a^\dagger a^* = a^*a^\dagger \Rightarrow (a^*)^\dagger a = a(a^*)^\dagger$.

Conversely if $aa^\dagger = a^\dagger a$ and $(a^*)^\dagger a = a(a^*)^\dagger$ then $a^\dagger \in \{a\}'' \Rightarrow a^\dagger(a^*)^\dagger = (a^*)^\dagger a^\dagger$ and hence $aa^* = a^*a$ i.e. a is a normal element. Therefore

Proposition 4.3.

(i) $aa^* = a^*a \Leftrightarrow aa^\dagger = a^\dagger a$ *and* $(a^*)^\dagger a = a(a^*)^\dagger$ ([15], Prop. 27, p. 243)

(ii) a is an EP element $\Leftrightarrow b = a(a^*a)^k$ is an EP element for some $k \geq 1$ ([15], end p. 243)

(iii) If a is an EP element so is any product of a's and a^*'s ([15], end p. 243)

The following proposition generalizes various results of Basket & Katz [1], [19] in matrices. It also characterizes the set $\mathcal{B}_a = \{x|xx^\dagger = x^\dagger x,\ aa^\dagger = xx^\dagger\}$.

Proposition 4.4 ([15], Prop. 30, and Corollary p. 245–246)**.** *Let R be a *-regular ring and let a, b be EP elements in R.*

(α) *The following conditions are equivalent*

(i) $aR = bR$

(ii) $(ab)^\dagger = b^\dagger a^\dagger$, $abR = aR$, $Rab = Rb$

(iii) ab is EP, and $abR = aR$, $Rab = Rb$

(iv) $(ab)^\dagger = b^\dagger a^\dagger$, $(ba)^\dagger = a^\dagger b^\dagger$, $abR = aR$, $baR = bR$

(v) ab, ba are EP, and $abR = aR$, $baR = bR$.

(β) If $(ab)^\dagger = a^\dagger b^\dagger$ and $abR = aR$, $Rab = Rb$ then $aR = bR$, ab is an EP element, and $ab = ba$.

(γ) If $ab = ba$, then ab, ba, a^*b, and ab^* are EP elements.

(δ) If moreover R has a unity, then there exist units u and v such that $a^* = au = va$.

Remark 4.5. Part (δ) of the above Proposition 4.4 it is proved in [16], theorem 3, p.59 under weaker conditions. The ring R does not have to be regular. It is enough R to be a ring with unity and with involution.

The following result is Corollary 2 in [17], p. 17.

Theorem 4.6. *Let R be a regular ring with involution, and suppose that a and b are EP elements of R. If $u = (1 - aa^\dagger)b$, $v = a(1 - bb^\dagger)$, $w = b(1 - aa^\dagger)$, and $z = (1 - bb^\dagger)a$, then any of the following conditions*

(α) $u^*u = 0 \Rightarrow u = 0$, $v^*v = 0 \Rightarrow v = 0$, $(ab)^\dagger = b^\dagger a^\dagger$, $abR = aR$, $Rab = Rb$

(β) $ww^* = 0 \Rightarrow w = 0$, $z^*z = 0 \Rightarrow z = 0$, $(ba)^\dagger = a^\dagger b^\dagger$, $baR = bR$, $Rab = Ra$

(γ) $vv^* = 0 \Rightarrow v = 0$, $ww^* = 0 \Rightarrow w = 0$, $(ab)^\dagger = b^\dagger a^\dagger$, $(ba)^\dagger = a^\dagger b^\dagger$, $abR = aR$, $baR = bR$.

imply the equivalent conditions:

(i) ab is EP and $abS = aS$, $Sab = Sb$,

(ii) $aS = bS$ and $aa^*S = aS$,

(iii) $aS = bS$ and $b^*bS = b^*S$,

(iv) $aS = bS$ and $aS = a^2S$,

(v) $aS = bS$ and $Sb = Sb^2$,

(vi) ab and ba are EP and $abS = aS$, $baS = bS$.

In a recent paper [31] Mosic and Djordjevic studying the reverse order law in rings with involution proved various equivalent conditions that imply that the product of two elements in a ring with involution is EP. In the following three theorems we summarize by restating these results

Theorem 4.7 ([31], Cor. 2.1, p. 277)**.** *Let $\mathcal{R}$ be a ring with involution, $a, b \in \mathcal{R}^\dagger$ and $ab \in \mathcal{R}^\sharp$. If any of the following four equivalent conditions*

(i) $b^\dagger = (ab)^\sharp a$,

(ii) $b = a^\dagger ab = baa^\dagger$ and $abb^\dagger = b^\dagger ba$,

(iii) $b\mathcal{R} \subset a^*\mathcal{R}$, $a^\dagger ab = baa^\dagger$ and $abb^\dagger = b^\dagger ba$,

(iv) $(a^*)^0 \subset b^0$, $a^\dagger ab = baa^\dagger$ *and* $abb^\dagger = b^\dagger ba$,

or any of the following four equivalent conditions

(i) $a^\dagger = b(ab)^\sharp$,

(ii) $a = abb^\dagger = b^\dagger ba$ *and* $a^\dagger ab = baa^\dagger$,

(iii) $a\mathcal{R} \subset b^*\mathcal{R}$, $a^\dagger ab = baa^\dagger$ *and* $abb^\dagger = b^\dagger ba$,

(iv) $(b^*)^0 \subset a^0$, $a^\dagger ab = baa^\dagger$ *and* $abb^\dagger = b^\dagger ba$,

is satisfied then ab *is EP and* $(ab)^\sharp = b^\dagger a^\dagger$.

Theorem 4.8 ([31], Th. 2.7, p. 279)**.** *Let* $\mathcal{R}$ *be a ring with involution. Suppose that* $a, b \in \mathcal{R}^\dagger$ *and* $ab \in \mathcal{R}^\sharp$. *Then* ab *is EP and* $(ab)^\sharp = b^\dagger a^\dagger$ *if and only if one of the following equivalent conditions holds:*

(i) $a^\dagger ab \in \mathcal{R}^\dagger$ *and* $b(ab)^\sharp = bb^\dagger a^\dagger = (abb^\dagger)^\dagger$,

(ii) $abb^\dagger \in \mathcal{R}^\dagger$ *and* $(ab)^\sharp a = b^\dagger a^\dagger a = (a^\dagger ab)^\dagger$,

(iii) $ab, a^\dagger ab^\dagger \in \mathcal{R}^\dagger$, $(ab)^\sharp = (a^\dagger ab)^\dagger a^\dagger$ *and* $(a^\dagger ab)^\dagger = b^\dagger a^\dagger a$,

(iv) $ab, abb^\dagger \in \mathcal{R}^\dagger$, $(ab)^\sharp = b^\dagger (abb^\dagger)^\dagger$ *and* $(abb^\dagger)^\dagger = bb^\dagger a^\dagger$,

(v) $ab, a^*ab \in \mathcal{R}^\dagger$, $(ab)^\sharp = (a^*ab)^\dagger a^*$ *and* $(a^*ab)^\dagger = b^\dagger (a^*a)^\sharp$

(vi) $ab, abb^* \in \mathcal{R}^\dagger$, $(ab)^\sharp = b^*(abb^*)^\dagger$ *and* $(abb^*)^\dagger = (bb^*)^\sharp a^\dagger$,

Theorem 4.9 ([31], Th. 2.8, p. 281)**.** *Let* $\mathcal{R}$ *be a ring with involution. If* $a, b \in \mathcal{R}^\dagger$, *then the following conditions are equivalent:*

(i) $ab \in \mathcal{R}^\dagger \cap \mathcal{R}^\sharp$, ab *is EP and* $(ab)^\dagger = b^\dagger a^\dagger$,

(ii) $(a^\dagger)^* b \in \mathcal{R}^\dagger \cap \mathcal{R}^\sharp$ *and* $[(a^\dagger)^* b]^\sharp = b^\dagger a^* = [(a^\dagger)^* b]^\dagger$,

(iii) $a(b^\dagger)^* \in \mathcal{R}^\dagger \cap \mathcal{R}^\sharp$ *and* $[a(b^\dagger)^*]^\sharp = b^* a^\dagger = [a(b^\dagger)^*]^\dagger$.

In [22] Koliha and Patricio used the notion of spectral idempotent of an element in a ring R with unit $1 \neq 0$ and the notion of the generalized Drazin inverse to give new characterizations of EP elements in a ring with involution. For their needs they gave the following definition: An element $\alpha \in R$ is generalized Drazin invertible (g-Drazin invertible for short) if there exists an element $\alpha^D = b \in R$ such that

$$b \in \{a\}'', \quad ab^2 = b, a^2 b - a \in R^{\mathrm{qnil}},$$

where R^{qnil} is the set of quasinilpotent elements of R. $\alpha \in R$ is quasinilpotent if for every $x \in \{\alpha\}'$, $1 + x\alpha$ is invertible. The set of g-Drazin invertible elements of R is denoted by R^{gD} and the set of Moore-Penrose invertible elements of R is denoted by $R^\dagger$.

An element $\alpha \in R$ is quasipolar if there exists $p \in R$ such that

$$p^2 = p, \quad p \in \{a\}'', \quad \alpha p \in R^{\mathrm{qnil}}, \quad \alpha + p \ \text{ is invertible}$$

Any idempotent p satisfying the above conditions is called spectral idempotent of α and is denoted by α^{π}. Any quasipolar element $\alpha \in R$ has a unique spectral idempotent.

An element α of a ring R with involution is said to be EP if $\alpha \in R^{\mathrm{gD}} \cap R^{\dagger}$ and $\alpha^{D} = \alpha^{\dagger}$. Of course this definition is equivalent to the previous one Definition 4.1 and the well known characterization of EP elements which is the condition (ii) of Proposition 4.2. An element α is generalized EP (or gEP for short) if there exists $k \in \mathbb{N}$ such that α^{k} is EP.

The main result in [22] that plays an important role to characterize EP elements in a ring with involution is the following

Theorem 4.10 ([22], Th. 7.2, p. 149)**.** *For $\alpha \in R$ the following conditions are equivalent:*

(i) α is EP

(ii) α is group invertible and $\alpha^{\pi} = (\alpha^{*})^{\pi}$

(iii) $\alpha \in R^{\mathrm{gD}} \cap R^{\dagger}$ and $\alpha^{\pi} = (\alpha^{*}\alpha)^{\pi}$

(iv) $\alpha \in R^{\mathrm{gD}} \cap R^{\dagger}$ and $\alpha^{\pi} = (\alpha\alpha^{*})^{\pi}$

(v) $\alpha \in R^{d}ag$ and $(\alpha^{*}\alpha)^{\pi} = (\alpha\alpha^{*})^{\pi}$

More over:

Theorem 4.11 ([22], Ths 7.3, 7.4 pp. 150–152)**.** *An element $\alpha \in R$ is EP if and only if*

(α) α is group invertible and one of the following equivalent conditions holds

(i) $\alpha^{\sharp}\alpha$ is symmetric

(ii) $\alpha(\alpha^{\sharp})^{*} = \alpha\alpha^{\sharp}(\alpha^{\sharp})^{*}$

(iii) $(\alpha^{\sharp})^{*} = (\alpha^{\sharp})^{*}\alpha^{\sharp}\alpha$

(iv) $\alpha^{\sharp}(\alpha^{\pi})^{*} = \alpha^{\pi}(\alpha^{\sharp})^{*}$

or

(β) α is g-Drazin invertible and one of the following equivalent conditions holds:

(i) $\alpha^{*}\alpha^{\pi}=0$

(ii) $\alpha^{\pi}\alpha^{*}=0$

(iii) $\alpha^{*} = \alpha^{*}\alpha^{D}\alpha$

(iv) $\alpha^{*} = \alpha^{D}\alpha\alpha^{*}$

From the above Theorem 4.11 the authors proved analogous results for gEP elements.

Patricio and Puystjens in [34] in order to distinguish between various conditions on an element in a ring with involution, introduced new terminology. In a ring with involution * they defined the following notions

Definition 4.12. 1. *An element α in a ring R with involution * is called *-EP if $\alpha R = \alpha^* R$.*

2. *An element α in a ring R with involution * is called *-group-Moore-Penrose (*-gMP in short) invertible if $\alpha^\dagger$ and $\alpha^\sharp$ exist and $\alpha^\dagger = \alpha^\sharp$.*

Note that in a *-regular ring an element is *-EP if and only if it is *-gMP (see also the comment following Proposition 4.2).

In a *-regular ring every element has a *-MP inverse but this is not true in an arbitrary ring R with involution * . Not every element in R has a *-MP inverse. For this reason the authors firstly characterize the elements in R which have a group inverse $\alpha^\sharp$ and a Moore-Penrose inverse $\alpha^\dagger$ such that $\alpha^\sharp = \alpha^\dagger$ and then they connected it with the notions of *-EP and *-gEP. Namely:

Proposition 4.13 ([34], Prop. 2, p. 162)**.** *Given α in a ring R with involution * the following conditions hold:*

1. *If $\alpha R = \alpha^* R$ then $\alpha^\dagger$ exists with respect to * if and only if $\alpha^\sharp$ exists, in which case $\alpha^\dagger = \alpha^\sharp$*
2. *If $\alpha^\dagger$ exists with respect to *, $\alpha^\sharp$ exists and $\alpha^\dagger = \alpha^\sharp$ then $\alpha R = \alpha^* R$*

Proposition 4.14 ([34], Corol. 3, p. 163)**.** *The following conditions are equivalent:*

1. *α is *-gMP*
2. *α is *-EP and $\alpha^\sharp$ exists*
3. *α is *-EP and $\alpha^\dagger$ exists with respect to **

In 2006, Castro-Gonzalez and Velez-Cerrada [7], used perturbations of generalized Drazin inverse and the equivalence of $\alpha \in R$ is EP if and only if α is group inverse and $\alpha^\pi = (\alpha^*)^\pi$ (see theorem 4.10(ii)) to study perturbations of EP elements in rings with involution. They gave equivalent conditions ensuring that if α is EP then an element $b = \alpha + e$ is again EP with spectral idempotent $b^\pi = \alpha^\pi + s$ where s is given. In fact we have the following

Theorem 4.15 ([7] Part of Th. 4.1, p. 391)**.** *Let R is a ring with unity 1 and with involution and $s \in R$ such that $1 - s^2$ is invertible. If α is EP and $\alpha^\pi + s$ is idempotent, then the following conditions on $b = a + e \in R$ are equivalent:*

1. *b is EP and $b^\pi = \alpha^\pi + s$*
2. *$b \in R^{gD} \cap R^\dagger$, and $b^\dagger = b^D = (1 + s + \alpha^\dagger e)^{-1} \alpha^\dagger (1 - s)$*

3. $b \in R^{gD} \cap R^{\dagger}$, *and* $b^{\dagger} = b^{D}$, *and* $(1+s)b^{\dagger} - \alpha^{\dagger}(1-s) = -\alpha^{\dagger}eb^{\dagger}$.

Finally from Theorem 4.2 in [7] the authors derived, putting $s = 0$, the new equivalent condition that α is EP if and only if α^* is group invertible and $(\alpha^*)^{\pi} = \alpha^{\pi}$.

Mosic and Djordjevic in [26] and in a recent paper (2012) [27] as far as concern EP elements in a ring R with involution gave necessary and sufficient conditions for a Moore-Penrose invertible element or for a group and Moore-Penrose invertible element in R to be a partial isometry and EP. In [26] they also gave equivalent conditions for two elements $\alpha, b \in R$ such that $\alpha b \alpha = \alpha$ and α be EP and in [27] they gave equivalent conditions for an element $\alpha \in R$ such that α be EP and satisfies the condition $(\alpha^*)^n = (\alpha^{\dagger})^n$ for some $n \in \mathbb{N}$. The next theorem summarizes the aforementioned equivalent conditions

Theorem 4.16 ([26], Th. 2.3, p. 764 and [27] Th. 2.2, p. 462)**.** *Let R be a ring with involution.*
(α) *If $\alpha \in R$ be Moore-Penrose invertible then α is a partial isometry and EP if and only if α is group invertible and one of the equivalent conditions holds:*

$$
\begin{array}{rl}
(i) & \alpha \text{ is a partial isometry and normal} \\
(ii) & \alpha^* = \alpha^{\sharp} \\
(iii) & \alpha\alpha^* = \alpha^{\dagger}\alpha \\
(iv) & \alpha^*\alpha = \alpha\alpha^{\dagger} \\
(v) & \alpha\alpha^* = \alpha\alpha^{\sharp} \\
(vi) & \alpha^*\alpha = \alpha\alpha^{\sharp} \\
(vii) & \alpha^*\alpha^{\dagger} = \alpha^{\dagger}\alpha^{\sharp} \\
(viii) & \alpha^{\dagger}\alpha^* = \alpha^{\sharp}\alpha^{\dagger} \\
(ix) & \alpha^{\dagger}\alpha^* = \alpha^{\dagger}\alpha^{\sharp} \\
(x) & \alpha^*\alpha^{\dagger} = \alpha^{\sharp}\alpha^{\dagger} \\
(xi) & \alpha^*\alpha^{\sharp} = \alpha^{\sharp}\alpha^{\dagger} \\
(xii) & \alpha^*\alpha^{\dagger} = \alpha^{\sharp}\alpha^{\sharp} \\
(xiii) & \alpha^*\alpha^{\sharp} = \alpha^{\dagger}\alpha^{\dagger} \\
(xiv) & \alpha^*\alpha^{\sharp} = \alpha^{\sharp}\alpha^{\sharp} \\
(xv) & \alpha\alpha^*\alpha^{\dagger} = \alpha^{\dagger} \\
(xvi) & \alpha\alpha^*\alpha^{\dagger} = \alpha^{\sharp} \\
(xvii) & \alpha\alpha^*\alpha^{\sharp} = \alpha^{\dagger} \\
(xviii) & \alpha\alpha^{\dagger}\alpha^* = \alpha^{\dagger}
\end{array}
$$

$$
\begin{array}{rl}
(xix) & \alpha^*\alpha^2 = \alpha \\
(xx) & \alpha^2\alpha^* = \alpha \\
(xxi) & \alpha\alpha^\dagger\alpha^* = \alpha^\sharp \\
(xxii) & \alpha^*\alpha^\dagger\alpha = \alpha^\sharp
\end{array}
$$

(β) *If $\alpha \in R$ is Moore-Penrose invertible and group invertible and $n \in \mathbb{N}$ then α is a partial isometry and EP if and only if one of the equivalent conditions holds:*

$$
\begin{array}{rl}
(i) & \alpha \ \textit{ is a partial isometry and } \alpha^*\alpha^n = \alpha^n\alpha^* \\
(ii) & \alpha^n\alpha^* = \alpha^\dagger\alpha^n \\
(iii) & \alpha^*\alpha^n = \alpha^n\alpha^\dagger \\
(iv) & \alpha^n\alpha^* = \alpha^n\alpha^\sharp \\
(v) & \alpha^*\alpha^n = \alpha^n\alpha^\sharp \\
(vi) & \alpha^*(\alpha^\dagger)^n = \alpha^\dagger(\alpha^\sharp)^n \\
(vii) & (\alpha^\dagger)^n\alpha^* = (\alpha^\sharp)^n\alpha^\dagger \\
(viii) & (\alpha^\dagger)^n\alpha^* = \alpha^\dagger(\alpha^\sharp)^n \\
(ix) & \alpha^*(\alpha^\dagger)^n = (\alpha^\sharp)^n\alpha^\dagger \\
(x) & \alpha^*(\alpha^\sharp)^n = (\alpha^\sharp)^n\alpha^\dagger \\
(xi) & \alpha^*(\alpha^\dagger)^n = (\alpha^\sharp)^{n+1} \\
(xii) & \alpha^*(\alpha^\sharp)^n = (\alpha^\dagger)^{n+1} \\
(xiii) & \alpha^*(\alpha^\sharp)^n = (\alpha^\sharp)^{n+1} \\
(xiv) & \alpha\alpha^*(\alpha^\dagger)^n = (\alpha^\sharp)^n \\
(xv) & \alpha\alpha^*(\alpha^\sharp)^n = (\alpha^\dagger)^n \\
(xvi) & \alpha^*\alpha^{n+1} = \alpha^n \\
(xvii) & \alpha^{n+1}\alpha^* = \alpha^n \\
(xviii) & \alpha(\alpha^\dagger)^n\alpha^* = (\alpha^\sharp)^n \\
(xix) & \alpha^*(\alpha^\dagger)^n\alpha = (\alpha^\sharp)^n
\end{array}
$$

In 2009, [32] Mosic, Djordjevic and Koliha generalized well known necessary and sufficient conditions in matrices and operators on Hilbert space in the setting of a ring with involution characterizing EP elements. They introduced 34 necessary and sufficient conditions for an element α of a ring with involution to be EP. Very recently 2012, in [30] Mosic and Djordjevic gave 20 more necessary and sufficient conditions involving powers of their group and Moore-Penrose inverse characterizing EP elements. These 54 conditions are included in the next theorem

Theorem 4.17 ([32], Th. 2.1, p. 529 and [30], Th. 2.1, p. 6703)**.** *Let R be a ring with involution and let $n, m \in \mathbb{N}$. An element $a \in R$ is EP if and only if α is group invertible and Moore-Penrose invertible and one of the following equivalent conditions holds:*

1. $\alpha\alpha^\dagger\alpha^\sharp = \alpha^\dagger\alpha^\sharp\alpha$
2. $\alpha\alpha^\dagger\alpha^\sharp = \alpha^\sharp\alpha\alpha^\dagger$
3. $\alpha^*\alpha\alpha^\sharp = \alpha^*$
4. $\alpha\alpha^\sharp\alpha^* = \alpha^*\alpha\alpha^\sharp$
5. $\alpha\alpha^\sharp\alpha^\dagger = \alpha^\dagger\alpha\alpha^\sharp$
6. $\alpha\alpha^\sharp\alpha^\dagger = \alpha^\sharp\alpha^\dagger\alpha$
7. $\alpha^\dagger\alpha\alpha^\sharp = \alpha^\sharp\alpha^\dagger\alpha$
8. $(\alpha^\dagger)^2\alpha^\sharp = \alpha^\dagger\alpha^\sharp\alpha^\dagger$
9. $\alpha^\dagger\alpha^\sharp\alpha^\dagger = \alpha^\sharp(\alpha^\dagger)^2$
10. $\alpha^\dagger(\alpha^\sharp)^2 = \alpha^\sharp\alpha^\dagger\alpha^\sharp$
11. $\alpha^\dagger(\alpha^\sharp)^2 = (\alpha^\sharp)^2\alpha^\dagger$
12. $(\alpha^\sharp)^2\alpha^\dagger = \alpha^\sharp\alpha^\dagger\alpha^\sharp$
13. $\alpha(\alpha^\dagger)^2 = \alpha^\sharp$
14. $\alpha^*\alpha^\dagger = \alpha^*\alpha^\sharp$
15. $\alpha^\dagger\alpha^* = \alpha^\sharp\alpha^*$
16. $\alpha^\dagger\alpha^\dagger = \alpha^\sharp\alpha^\dagger$
17. $\alpha^\dagger\alpha^\dagger = \alpha^\dagger\alpha^\sharp$
18. $(\alpha^\dagger)^2 = (\alpha^\sharp)^2$
19. $\alpha\alpha^\sharp\alpha^\dagger = \alpha^\sharp$
20. $\alpha^\sharp\alpha^\dagger = (\alpha^\sharp)^2$
21. $\alpha^\dagger\alpha^\sharp = (\alpha^\sharp)^2$
22. $\alpha^\dagger\alpha\alpha^\sharp = \alpha^\dagger$
23. $\alpha^\sharp\alpha^\dagger\alpha = \alpha^\dagger$
24. $\alpha\alpha^\dagger\alpha^*\alpha = \alpha^*\alpha\alpha\alpha^\dagger$
25. $\alpha^\dagger\alpha\alpha\alpha^* = \alpha\alpha^*\alpha^\dagger\alpha$
26. $\alpha\alpha^\dagger(\alpha\alpha^* - \alpha^*\alpha) = (\alpha\alpha^* - \alpha^*\alpha)\alpha\alpha^\dagger$
27. $\alpha^\dagger\alpha(\alpha\alpha^* - \alpha^*\alpha) = (\alpha\alpha^* - \alpha^*\alpha)\alpha^\dagger\alpha$
28. $\alpha^*\alpha\sharp\alpha + \alpha\alpha\sharp\alpha^* = 2\alpha^*$
29. $\alpha^\dagger\alpha\sharp\alpha + \alpha\alpha\sharp\alpha^\dagger = 2\alpha^\dagger$
30. $\alpha\alpha\alpha^\dagger + \alpha^\dagger\alpha\alpha = 2\alpha$
31. $\alpha\alpha\alpha^\dagger + (\alpha\alpha\alpha^\dagger)^* = \alpha + \alpha^*$
32. $\alpha^\dagger\alpha\alpha + (\alpha^\dagger\alpha\alpha)^* = \alpha + \alpha^*$
33. $\alpha\alpha^\dagger\alpha^* = \alpha^*\alpha\alpha^\dagger$
34. $\alpha^*\alpha^\dagger\alpha = \alpha^\dagger\alpha\alpha^*$

35. $(\alpha^\sharp)^{n+m-1} = (\alpha^\dagger)^m(\alpha^\sharp)^{n-1}$ (*or* $(\alpha^\sharp)^{n+m-1} = (\alpha^\sharp)^{n-1}(\alpha^\dagger)^m$)
36. $(\alpha^*)^n\alpha\alpha^\sharp = (\alpha^*)^n$
37. $\alpha\alpha^\sharp(\alpha^*)^n = (\alpha^*)^n\alpha\alpha^\sharp$
38. $\alpha(\alpha^\sharp)^n(\alpha^\dagger)^m = \alpha^\dagger\alpha(\alpha^\sharp)^{n+m-1}$
39. $(\alpha^\dagger)^2(\alpha^\sharp)^n = \alpha^\dagger(\alpha^\sharp)^n\alpha^\dagger$ (*or* $(\alpha^\dagger)(\alpha^\sharp)^n\alpha^\dagger = (\alpha^\sharp)^n(\alpha^\dagger)^2$)
40. $\alpha^\dagger(\alpha^\sharp)^n = (\alpha^\sharp)^n\alpha^\dagger$
41. $\alpha(\alpha^\dagger)^{n+1} = (\alpha^\sharp)^n$
42. $\alpha^*(\alpha^\dagger)^n = \alpha^*(\alpha^\sharp)^n$ (*or* $(\alpha^\dagger)^n\alpha^* = (\alpha^\sharp)^n\alpha^*$)
43. $(\alpha^\dagger)^{n+1} = (\alpha^\sharp)^n\alpha^\dagger$(*or*$(\alpha^\dagger)^{n+1} = \alpha^\dagger(\alpha^\sharp)^n$)
44. $(\alpha^\dagger)^n = (\alpha^\sharp)^n$
45. $\alpha\alpha^\dagger(\alpha^*)^n = (\alpha^*)^n\alpha\alpha^\dagger$ (*or* $(\alpha^*)^n\alpha^\dagger\alpha = \alpha^\dagger\alpha(\alpha^*)^n$)
46. $\alpha\alpha^\dagger(\alpha^*)^n\alpha^m = (\alpha^*)^n\alpha^m\alpha\alpha^\dagger$ (*or* $\alpha^\dagger\alpha\alpha^m(\alpha^*)^n = \alpha^m(\alpha^*)^n\alpha^\dagger\alpha$)
47. $\alpha\alpha^\dagger(\alpha^m(\alpha^*)^n - (\alpha^*)^n\alpha^m) = (\alpha^m(\alpha^*)^n - (\alpha^*)^n\alpha^m)\alpha\alpha^\dagger$
    (*or* $\alpha^\dagger\alpha(\alpha^m(\alpha^*)^n - (\alpha^*)^n\alpha^m) = (\alpha^m(\alpha^*)^n - (\alpha^*)^n\alpha^m)\alpha^\dagger\alpha$)
48. $(\alpha^*)^n\alpha^\sharp\alpha + \alpha\alpha^\sharp(\alpha^*)^n = 2(\alpha^*)^n$
49. $\alpha^\dagger(\alpha^\sharp)^n\alpha + \alpha\alpha^\sharp(\alpha^\dagger)^n = 2(\alpha^\dagger)^n$
50. $\alpha^n\alpha\alpha^\dagger + \alpha^\dagger\alpha\alpha^n = 2\alpha^n$
51. $\alpha^n\alpha\alpha^\dagger + (\alpha^n\alpha\alpha^\dagger)^* = \alpha^n + (\alpha^*)$ (*or* $\alpha^\dagger\alpha\alpha^n + (\alpha^\dagger\alpha\alpha^n)^* = \alpha^n + (\alpha^*)$)
52. $\alpha^n = \alpha^n\alpha\alpha^\dagger$(*or* $\alpha^n = \alpha^\dagger\alpha\alpha^n$)
53. $\alpha^n\alpha^\dagger = \alpha^\dagger\alpha^n$
54. $[(\alpha^\sharp)^*]^n = \alpha\alpha^\sharp[(\alpha^\sharp)^*]^n$(*or* $[(\alpha^\sharp)^*]^n = [(\alpha^\sharp)^*]^n\alpha^\sharp\alpha$)

Moreover

Theorem 4.18 ([30], Th. 2.3, p. 6706)**.** *Let $n \in \mathbb{N}$ and $\alpha \in R$ such that α is group and Moore-Penrose invertible and α^n is Moore-Penrose invertible. Then α is EP if and only if $\alpha(\alpha^n)^\dagger = (\alpha^n)^\dagger\alpha$.*

Patricio and Araujo in 2010, in [33], Th. 2.3, p. 449, studied necessary and sufficient conditions for $\alpha\alpha^\dagger = bb^\dagger$ where α, b are Moore-Penrose invertible and then used them to get new characterizations of EP elements in a ring R with involution. Note that in the condition $\alpha\alpha^\dagger = bb^\dagger$ if we take $b = \alpha^\dagger$ we have $\alpha\alpha^\dagger = \alpha^\dagger\alpha$ which means that α is EP. From their Theorem 2.3 for $b = \alpha^\dagger$ we get the following

Proposition 4.19. *Let $\alpha \in R$ be Moore-Penrose invertible. Then α is EP if and only if one of the following conditions holds:*
(i) $\alpha = \alpha^\dagger\alpha^2$ and $v = \alpha\alpha^* + 1 - \alpha^\dagger\alpha$ is invertible
(ii) $\alpha = \alpha^\dagger\alpha^2$ and $w = \alpha\alpha^* + 1 - \alpha^\dagger(\alpha^\dagger)^-$ is a unit for one and hence for all choices of $(\alpha^\dagger)^-$

(iii) $[\alpha\alpha^\dagger, \alpha^\dagger\alpha] = 0$ *and both* $k = \alpha^\dagger(\alpha^\dagger)^* + 1 - \alpha^\dagger\alpha$ *and* $v = \alpha\alpha^* + 1 - \alpha^\dagger\alpha$ *are units.*

Similar considerations for EP elements could be drawn correspondingly, by using in Theorem 2.3, [33], p. 449 $b = \alpha^*$, $\alpha = b^\dagger$ and $\alpha = b^*$.

Very recently, in June 2012 Chen in [8] gives new characterizations of EP elements in rings with involution. These characterizations are presented in the following three statements.

Proposition 4.20 ([8], Prop. 2.3, p. 556). *Let $\mathcal{R}$ be a ring with involution. Then $a \in \mathcal{R}$ is EP if and only if $a \in \mathcal{R}^\sharp \cap \mathcal{R}^\dagger$ and satisfies the following two conditions:*

(i) $(a^\dagger)^2 a^\sharp = a^\sharp (a^\dagger)^2$,

(ii) $aa^\dagger = a^2(a^\dagger)^2$

To prove this Chen uses also Theorem 7.3 of [22].

Theorem 4.21 ([8], Th. 2.4, p. 556). *Let $\mathcal{R}$ be a ring with involution. Then $a \in \mathcal{R}$ is EP if and only if $a \in \mathcal{R}^\sharp \cap \mathcal{R}^\dagger$ and satisfies one of the following conditions:*

(i) $a^n a^\dagger = a^\dagger a^n$, *for some* $n \geq 1$,

(ii) $(a^\sharp)^n a^\dagger = a^\dagger (a^\sharp)^n$, *for some* $n \geq 1$,

(iii) $(a^\dagger)^n = (a^\sharp)^n$, *for some* $n \geq 1$.

Corollary 4.22 ([8], Cor. 2.5, p. 557). *Let $\mathcal{R}$ be a ring with involution. Then $a \in \mathcal{R}$ is EP if and only if $a \in \mathcal{R}^\sharp \cap \mathcal{R}^\dagger$ and satisfies the following two conditions:*

(i) $a^\dagger \in \mathcal{R}^\sharp$,

(ii) $(a^\dagger)^2 a^\sharp = a^\sharp (a^\dagger)^2$.

It is well known that $1 - \alpha b$ is EP does not imply in general that $1 - b\alpha$ is EP. Castro-Gonzalez, Mendes-Araujo and Petro Patricio in [6], Cor. 3.8, 3.9, p. 163 gave necessary and sufficient conditions for if $1 - \alpha b$ is EP (correspondingly gEP) then $1 - b\alpha$ is EP (correspondingly gEP).

Remark 4.23. For any regular ring, we could define an element a to be EP if there exists a 1-2 inverse a^+ such that either $a^+R = aR$ or $Ra^+ = Ra$ (see Hartwig, [16], p. 60). For *-regular rings (or when conditions $a^*aR = a^*R$, $aa^*R = aR$ are valid), the two definitions coincide since the Moore-Penrose inverse $a^\dagger$ exists and so $a^\dagger R = a^*R$.

Hartwig and Luh in [18] showed that an element of a ring R with unity is regular if and only if there exists a unit $u \in R$ and a group G such that $a \in uG$

and derived that an EP element α is a group member in R which means that α is contained in a subgroup of R with respect to multiplication.

In the next section, we will see some of the result in Proposition 4.4 are valid under much more weaker conditions, in the more abstract setting of semigroups. Moreover, Meenakshi and Anbalagan in [25] defined and studied EP elements in the more abstract setting of an incline.

Now we will see an extension of the definition of an EP element to an E^kP, $k \geq 1$ element ([15], p. 243 and [24]).

Definition 4.24. *Let R be a *-regular ring. Hartwig defined the notion of E^kP, $k \geq 1$ elements. An element $a \in R$ is E^kP, $k \geq 1$, if the following hold*
(i) $a^k a^\dagger = a^\dagger a^k$, (ii) $(a^\dagger)^k a = a(a^\dagger)^k$.

The following relations are a consequence of this definition.
$a^{k+1}a^\dagger = a^k = a^\dagger a^{k+1} \Rightarrow$
$a^k(aa^\dagger - a^\dagger a) = (aa^\dagger - a^\dagger a)a^k = a(a^k a^\dagger - a^\dagger a^k) = (a^k a^\dagger - a^\dagger a^k)a.$
Also $a^{k+1}R = a^kR$, $Ra^{k+1} = Ra^k$, $a^k(a^\dagger)^k a^k = a^k$.

By symmetry, using relation (ii) of the Definition 4.24, we get analogous relations with a and $a^\dagger$ interchanged.

It is easy to see that if a is E^kP element then $(a^\dagger)^k = (a^k)^\#$ is the group inverse of a^k and hence $R = a^kR \oplus (a^k)^0$.
Also Meenakshi in [24] studied E^kP elements in *-regular rings. He investigated connections between E^kP and EP elements and applied them to give sufficient conditions for two E^kP elements to satisfy the reverse order law for the Moore-Penrose inverse.

5. EP elements in semigroups with involution. Let S be a regular semigroup with two-sided zero 0 and with involution. Moreover we assume S to be reflexive, that is, for all $a \in S$,

$$^0((Sa)^0) \subseteq Sa, \tag{2}$$

but we do not assume the global star-cancellation law. In such semigroups the Moore-Penrose inverse does not exist in general, but in the study of EP elements its existence is very helpful. For this reason we give first a list of equivalent relations which ensure the existence of Moore-Penrose inverse.

Theorem 5.1 ([17], Th. 1, & Corollary p. 14–15)**.** *Let S be a regular reflexive semigroup with involution and let a be a fixed element of S. The following are equivalent:*

(α)

(i) there is a solution a^{13} *to* $axa = a$, $(ax)^* = ax$,

(ii) there is a solution to $a^*ax = a^*$

(iii) $a^*aS = a^*S$

(iv) $(a^*a)^0 = a^0$,

(v) $aS \cap (aS)^\perp = (0)$.

Moreover

(β)

(i) there is a solution a^{14} *to* $axa = a$, $(xa)^* = xa$,

(ii) there is a solution to $aa^*x^* = a$

(iii) $aa^*S = aS$

(iv) $(aa^*)^0 = (a^*)^0$,

(v) $a^0 \cap (a^0)^\perp = (0)$.

(γ) *The Moore-Penrose inverse* $a^\dagger$ *exists if and only if* (α) *and* (β) *hold and is then given by* $a^\dagger = a^{14}aa^{13}$.

Definition 5.2. *An element* a *of a regular semigroup with involution is called EP if* $aS = a^*S$.

For the next two results S will be a regular reflexive semigroup with involution.

Theorem 5.3 ([17], Lem. 1, Lem. 3, p. 15)**.** *If* a *and* b *are EP elements in* S, *then*

(α) $aS = bS \Leftrightarrow a^*S = b^*S \Leftrightarrow Sa = Sb \Leftrightarrow a^0 = b^0$

(β) *If* $a \in S$ *is EP then the following are equivalent:*

(i) $a^2S = aS$

(ii) $aa^*S = aS$

(iii) $Sa^2 = Sa$

(iv) $a^*aS = aS$

In any case $a^\dagger$ *exists and* $aa^\dagger = a^\dagger a$.

Theorem 5.4 ([17], Th. 2, p. 16)**.** *Let* a *and* b *be EP elements of a regular reflexive semigroup* S *with involution. Then the following are equivalent:*

(i) ab *is EP and* $abS = aS, Sab = Sb$, (*ii*) $aS = bS$ *and* $aa^*S = aS$,

(*iii*) $aS = bS$ *and* $b^*bS = b^*S$,

(*iv*) $aS = bS$ *and* $aS = a^2S$,

(*v*) $aS = bS$ *and* $Sb = Sb^2$,

(*vi*) ab *and* ba *are EP and* $abS = aS, baS = bS$.

In any case $a^\dagger$, $b^\dagger$, $(ab)^\dagger$, *and* $(ba)^\dagger$ *exist, each commutes with its commutes with its Moore-Penrose inverse, and*
$(ab)^\dagger = b^\dagger a^\dagger$, $(ba)^\dagger = a^\dagger b^\dagger$.

The Theorem 5.4 remains true if we omit the hypothesis that b is EP. Also a consequence of the Theorem 5.4 is the following

Corollary 5.5. *Let* S *be as in Theorem* 5.4 *and* a *and* b *EP elements of* S. *If* $abS = aS = a^2S$ *and* $Sab = Sb = Sb^2$, *then* $a^\dagger$, $b^\dagger$, *and* $(ab)^\dagger$ *exist. If, addition,* $(ab)^\dagger = a^\dagger b^\dagger$, *then* $aS = bS$, ab *and* ba *are EP, and* $ab = ba$.

X. Mary in [23] studies generalized inverses introducing a new generalized inverse that is the inverse along an element of a semigroup based on Green's suitable relations. From this point of view someone could find (indirectly) in his work reformulations in the new setting of equivalent conditions concerning EP elements (see for example proposition 13 and theorem 14, [23], pp. 1842-1843 and compare them with Propositions 4.13 and 4.14).

6. EP elements in C^*-algebras. Let $\mathcal{A}$ be a C^*-algebra. Then every regular element $\alpha \in \mathcal{A}$ has a Moore-Penrose inverse (see [13], Theorem 6) which is denoted by $\alpha^\dagger$, it is uniquely determined and $\alpha^\dagger \in \{\alpha, \alpha^*\}''$. By uniqueness it is clear that $(\alpha^*)^\dagger = (\alpha^\dagger)^*$. It is also well known that (see [13], Theorem 2 and 8) a necessary and sufficient condition for an element $\alpha \in \mathcal{A}$ to be regular is that the range ideal $\alpha\mathcal{A}$ be closed i.e. $\alpha\mathcal{A} = \mathrm{cl}\alpha\mathcal{A}$. An element $\alpha \in \mathcal{A}$ is called EP if it commutes with its Moore-Penrose inverse $\alpha^\dagger$.
Harte and Mbekhta in [14], referred to EP elements in a C^*-algebra indirectly through their Theorem 10, which is the following

Theorem 6.1 ([14], Th. 10, p. 137). *If* $\alpha \in \alpha\mathcal{A}\alpha$ *is a regular element of a* C^**-algebra* $\mathcal{A}$ *then the following are equivalent:*

(*i*) $\alpha\alpha^\dagger = \alpha^\dagger\alpha$

(*ii*) $\alpha^0 = (\alpha^*)^0$

(*iii*) ${}^0\alpha ={}^0 (\alpha^*)$

$(iv) \qquad \alpha \in \mathcal{A}^{-1}\alpha^*$

$(v) \qquad \alpha \in \alpha^*\mathcal{A}^{-1}$

Compare the above Theorem 6.1 with Proposition 4.2 which is referred to *-regular rings.

We remark here that if α is a regular element in a C^*-algebra $\mathcal{A}$ then there exists $b \in \mathcal{A}$ such that $\alpha = \alpha b\alpha$, $b = b\alpha b$. Indeed the generalized inverse b of α ($\alpha = \alpha b\alpha$) can be normalized: If we put $c = b\alpha b$ then $\alpha = \alpha c\alpha$ and $c = c\alpha c$, and the passage from b to c does not change the idempotents $p = p^2 = b\alpha$ and $q = q^2 = ab$.

Let $\mathcal{A}$ be a Banach algebra with unit e. We say that an element $\alpha \in \mathcal{A}$ is Drazin invertible if there exists an element $x \in \mathcal{A}$ such that

$$x \in \{\alpha\}', \quad \alpha x^2 = x, \quad \alpha(e - \alpha x) \in QN(\mathcal{A}),$$

where $QN(\mathcal{A})$ denotes the set of quasinilpotent elements of $\mathcal{A}$. Such x, when it exists, it is unique and it is denoted by $x = \alpha^D$. We also denote the set of all Drazin invertible elements of $\mathcal{A}$ by $\mathcal{A}^D$ (see e.g [20]).
Koliha in [20] generalizes a well known result for matrices to C^*-algebras which is another equivalent condition for an element $\alpha \in \mathcal{A}$ to be EP. Let $\mathcal{A}$ be a unital C^*-algebra. Then

Proposition 6.2 ([20], Prop. 2.2, p. 19)**.** *Let $\alpha \in \mathcal{A}$ be Moore-Penrose invertible. Then*

$$\alpha^\dagger\alpha = \alpha\alpha^\dagger \Longleftrightarrow [\alpha \in \mathcal{A}^D \quad \textit{and} \quad \alpha^\dagger = \alpha^D] \Longrightarrow \alpha \quad \textit{is simply polar.}$$

It is proved in [20] that an element α in $\mathcal{A}$ is Moore-Penrose invertible if and only if it is regular and α is Drazin invertible if and only if it is quasipolar. Taking these into account $\mathcal{A}^D$, $\mathcal{A}^\dagger$ denote also the sets of all quasipolar and regular elements of $\mathcal{A}$ respectively. $\mathcal{A}^{-1}$ denotes the set of all invertible elements of $\mathcal{A}$. $\alpha \in \mathcal{A}$ is quasipolar if 0 is isolated (possibly removable) singularity of the resolvent of α and polar if 0 is at most a pole of the resolvent. If α is quasipolar, then so is α^* and $(\alpha^*)^\pi = (\alpha^\pi)^*$.

It is also well known (see [21], proposition 1.1) that

$$\begin{aligned} \alpha \in \mathcal{A}^D &\Leftrightarrow \text{there exists } x \in \mathcal{A} \text{ s.t. } \alpha x = x\alpha,\ x\alpha x = x,\ \alpha x\alpha = \alpha + u,\ u \in QN(\mathcal{A}) \\ &\Leftrightarrow \text{there exists } p = p^2 \in \mathcal{A} \text{ s.t. } \alpha p = p\alpha \in QN(\mathcal{A}) \text{ and } \alpha + p \in \mathcal{A}^{-1} \end{aligned}$$

The idempotent p above is the spectral idempotent of $\alpha \in \mathcal{A}^D$ at 0, denoted by $p = \alpha^\pi$

α is polar $\Leftrightarrow \alpha^k \alpha^\pi = 0$ for some non negative integer k.

If $\alpha\alpha^\pi = 0$ then α is simply polar. It is known that $\alpha^\pi = e - \alpha^D \alpha$ and $\alpha^D = (\alpha + \alpha^\pi)^{-1}(e - \alpha^\pi)$.

In [21], a number of new characterizations is given, in terms of spectral idempotents, of the EP elements of a C^*-algebra.

Theorem 6.3 ([21], Th. 2.1, p. 83)**.** *Let $\alpha \in \mathcal{A}^\dagger$. Then $\alpha^\dagger \alpha = \alpha\alpha^\dagger$ if and only if α is simply polar with a selfadjoint spectral idempotent at* 0*. In this case*

$$\alpha^\pi = (\alpha^*)^\pi = (\alpha^*\alpha)^\pi = (\alpha\alpha^*)^\pi.$$

A consequence of the above Theorem 6.3 and its proof is the following

Corollary 6.4 ([21], Cor. 2.2, p. 84)**.** *The following conditions are equivalent*

(*i*) $\alpha \in \mathcal{A}^\dagger$ *and* $\alpha^\dagger \alpha = \alpha\alpha^\dagger$

(*ii*) $\alpha \in \mathcal{A}^\dagger \cap \mathcal{A}^D$ *and* $\alpha^\dagger = \alpha^D$

(*iii*) α *is simply polar and* $(\alpha^*)^\pi = \alpha^\pi$

(*iv*) α *is simply polar and* $\alpha^\pi = (\alpha^*\alpha)^\pi$ (*respectively* $\alpha^\pi = (\alpha\alpha^*)^\pi$)

(*v*) $\alpha \in \mathcal{A}^\dagger$ *and* $(\alpha^*\alpha)^\pi = (\alpha\alpha^*)^\pi$

Another interesting result is the following corollary in which it is given one more necessary and sufficient condition for an element $\alpha \in \mathcal{A}^\dagger$ to be EP.

Corollary 6.5 ([21], Cor. 2.3, p. 84)**.** *Let $\alpha \in \mathcal{A}^\dagger$. Then α is EP (equivalently $\alpha^\dagger\alpha = \alpha\alpha^\dagger$) if and only $\alpha^\dagger = f(\alpha)$ for some function f holomorphic in a neighbourhood of the spectrum $\sigma(\alpha)$.*

The Theorem 3.1 in [21] extends and generalizes well known characterizations of EP elements to C^*-algebras. Since some of these characterizations have already been stated in previous theorems, we will state only the new ones in the next theorem.

Theorem 6.6 ([21], Th. 3.1, p. 84–85)**.** *If $\alpha \in \mathcal{A}^\dagger$, then the following are equivalent.*

(*i*) $\alpha \in \mathcal{A}$ *is EP.*

(*ii*) $\alpha^2\alpha^\dagger = \alpha = \alpha^\dagger\alpha^2$

(*iii*) $(\alpha^*\alpha)^\pi\alpha = 0 = \alpha(\alpha\alpha^*)^\pi$

(*iv*) $\alpha\mathcal{A} = \alpha^*\mathcal{A}$

(v) $\mathcal{A}\alpha = \mathcal{A}\alpha^*$

(vi) $\alpha \in \alpha^\dagger \mathcal{A} \cap \mathcal{A}\alpha^\dagger$

(vii) $\alpha \in \alpha^\dagger \mathcal{A}^{-1} \cap \mathcal{A}^{-1}\alpha^\dagger$

As we have seen in Proposition 4.2 some of these relations are valid in the more abstract situation of *-regular rings.

In Proposition 4.4 and Theorem 4.6 we have seen conditions where the product of two EP elements is again EP. In the following theorem we have analogous results in C^*-algebras.

Theorem 6.7 ([21], Th. 4.3, p. 87). *Let $\mathcal{A}^\dagger_{com}$ be the set of all EP elements of $\mathcal{A}$, $\alpha, b \in \mathcal{A}^\dagger_{com}$ and α^0, b^0 be finite dimensional vector subspaces of $\mathcal{A}$. Then the following conditions are equivalent:*

(i) $\alpha b \in \mathcal{A}^\dagger_{com}$

(ii) $(\alpha b)\alpha^\pi = 0$ *and* $b^\pi(\alpha b) = 0$

(iii) $\alpha^0 \subset (\alpha b)^0$ *and* ${}^0 b \subset {}^0(\alpha b)$

(iv) $(\alpha b)^0 = \alpha^0 + b^0$ *and* ${}^0(\alpha b) = {}^0\alpha + {}^0 b$

Benítez in [2] gives a new characterization for an element $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is a C^*-algebra with unit, to be EP and uses it to study the commutativity of two elements of $\mathcal{A}$ in the case when one of them is EP. Moreover he uses a representation of $\mathcal{A}$ with respect to a projection $p \in \mathcal{A}$ and the aforementioned results to get a kind of "simultaneous diagonalization" when $\alpha \in \mathcal{A}$ is EP, $b \in \mathcal{A}$, $\alpha b = b\alpha$ and α, b is written according to this representation.

Theorem 6.8 ([2], Th. 2.1 and Cor. 2.3, pp. 766–767). *Let $\mathcal{A}$ be a C^*-algebra with unit 1 and $\alpha \in \mathcal{A}$*

(α) *The following conditions are equivalent*

(i) *α is EP*

(ii) *There exists a unique projection $p \in \mathcal{A}$ such that $\alpha + p \in \mathcal{A}$ and $\alpha p = p\alpha$. (Note here that the unique projection is $p = \alpha^\pi = 1 - \alpha\alpha^\dagger$)*

(β) *If α has a Moore-Penrose inverse and $k \in \mathbb{N}$, then the following conditions are equivalent*

(i) $\alpha^k = \alpha^\dagger$

(ii) *α is EP and* $\alpha^{k+1} + \alpha^\pi = 1$

Now if $p \in \mathcal{A}$ is a projection, then every $x \in \mathcal{A}$ has the following matrix representation which preserves the involution in $\mathcal{A}$

$$x = \begin{bmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-p) \end{bmatrix} \tag{3}$$

According to this representation when $\alpha \in \mathcal{A}$ is EP, $b \in \mathcal{A}$ and $p = \alpha^\pi$ we get

$$\alpha = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}, \qquad b = \begin{bmatrix} \alpha^\pi b \alpha^\pi & \alpha^\pi b(1-\alpha^\pi) \\ (1-\alpha^\pi) b \alpha^\pi & (1-\alpha^\pi) b (1-\alpha^\pi) \end{bmatrix}$$

Then:

Theorem 6.9 ([2], Th. 3.1 and Cor. 3.3 and Th. 3.6, p. 768)**.** *Let $\mathcal{A}$ be a C^*-algebra with unit 1, $\alpha \in \mathcal{A}$ is EP and $b \in \mathcal{A}$. Then*
(α) $\|(1-\alpha^\pi)b\alpha^\pi\| \leq \|\alpha b - b\alpha\| \, \|\alpha^\dagger\|$, $\quad \|\alpha^\pi b(1-\alpha^\pi)\| \leq \|\alpha b - b\alpha\| \, \|\alpha^\dagger\|$
(β) *If α, b commute then $b\alpha^\pi = \beta^\pi b$ and if moreover b is also EP then*

(*i*) $\quad \alpha^\pi b^\pi = b^\pi \alpha^\pi$

(*ii*) $\quad \alpha b^\dagger = b^\dagger \alpha$ *and* $b\alpha^\dagger = \alpha^\dagger b$

(*iii*) $\quad \alpha^\dagger b^\dagger = b^\dagger \alpha^\dagger = (\alpha b)^\dagger$

(γ) *If $\alpha b = b\alpha = 0$ then*

(*i*) $\quad \alpha^\pi b = b = b\alpha^\pi$

(*ii*) $\quad \alpha^\dagger b = b\alpha^\dagger = 0$

(*iii*) $\quad b \in \mathcal{A}^\dagger$ *implies* $\alpha b^\dagger = b\alpha^\dagger = 0$

(*iv*) $\quad b \in \mathcal{A}^\dagger$ *implies* $\alpha + b \in \mathcal{A}^\dagger$ *and* $(\alpha + b)^\dagger = \alpha^\dagger + b^\dagger$

(*v*) $\quad b$ *is EP implies* $\alpha + b$ *is also EP and* $(\alpha + b)^\pi = \alpha^\pi + b^\pi - 1$

In [9], characterizations of EP elements of a C^*-algebra $\mathcal{A}$ through different kind of factorizations of $\alpha \in \mathcal{A}$ (for operators in $\mathcal{B}(H)$ see [11]) such as $\alpha = b\alpha^*$, $\alpha = ucw$ and $a = bc$ where the factors satisfy certain conditions, are introduced. Boasso also studied in [3] EP Banach space operators and EP Banach algebra elements using factorizations. In this work, he derived new results for EP elements in the frame of unital C^*-algebras. All these results referred to C^*-algebras from both papers [9] and [3] we will be presented later on.

Firstly we will see a characterization from [9] in terms of the existence of projections in $\mathcal{A}$, namely

Theorem 6.10 ([9], Th. 1.4 p. 588)**.** *An element $\alpha \in \mathcal{A}$ is EP if and only if there exists a projection $p \in \mathcal{A}$ such that $p\alpha = \alpha = \alpha p$, $\alpha \in (p\mathcal{A}p)^{-1}$*

The characterizations of EP elements using factorizations are summarized in the following three theorems

Theorem 6.11 ([9], Ths 1.5, 1.7, 1.8, 1.10 and Lems 1.6, 1.9 pp 589–591)**.** *Let $\alpha \in \mathcal{A}$ be regular. The following conditions are equivalent:*

(*i*) $\quad \alpha$ *is EP*

(ii)	$\alpha = b\alpha^* = \alpha^* c$ *for some* $\alpha, c \in \mathcal{A}$
(iii)	$\alpha^*\alpha = b_1\alpha^*$ *and* $\alpha\alpha^* - c_1\alpha$ *for some* $b_1, c_1 \in \mathcal{A}$
(iv)	$\alpha^*\alpha = b_2\alpha^\dagger$ *and* $\alpha^\dagger = c_2\alpha$ *for some* $b_2, c_2 \in \mathcal{A}$
(v)	$\alpha = ucu^*$ *for some* $c, u \in \mathcal{A}$ *satisfying* $c^0 = (c^*)^0$ *and* $u^0 = \{0\}$
(vi)	$\alpha = ucw = w^*d^*v^*$ *for some* $c, d, u, v, w \in \mathcal{A}$ *satisfying* $c^0 = d^0$ *and* $u^0 = v^0 = \{0\}$
(vii)	$\alpha^*\alpha = u_1c_1w_1$ *and* $\alpha\alpha^* = v_1d_1w_1$ *for some* $c_1, d_1, u_1, v_1, w_1 \in \mathcal{A}$ *satisfying* $c_1^0 = d_1^0$ *and* $u_1^0 = v_1^0 = \{0\}$
$(viii)$	$\alpha = u_2c_2w_2$ *and* $\alpha^\dagger = v_2d_2w_2$ *for some* $c_2, d_2, u_2, v_2, w_2 \in \mathcal{A}$ *satisfying* $c_2^0 = d_2^0$ *and* $u_2^0 = v_2^0 = \{0\}$
(ix)	$\alpha = ucw = w^*d^*u^*$ *for some* $c, d, u, w \in \mathcal{A}$ *with* $c^0 \subset d^0$, $(c^*)^0 \subset (d^*)^0$ *and* $u^0 = (w^*)^0 = \{0\}$
(x)	$\alpha = bc, b * \mathcal{A} = \mathcal{A} = c\mathcal{A}$ *for some* $b, c \in \mathcal{A}$ *and one of the following conditions holds:* $1.bb^\dagger = c^\dagger c$, $2.b(b^*b)^{-1}b^* = c^*(cc^*)^{-1}c$ $3.(b^*)^0 = c^0$

Theorem 6.12 ([3], Ths 3.10, 3.11 pp. 250–251)**.** *Let* $\mathcal{A}$ *be a unital* C^**-algebra and consider* $\alpha \in \mathcal{A}$ *such that* $\alpha^\dagger$ *exists. Suppose that there exist* $b, c \in \mathcal{A}$ *such that* $\alpha = bc$, $b^0 = 0$ *and* $c\mathcal{A} = \mathcal{A}$. *Then, the following statements are equivalent.*

(i)	$\alpha \in \mathcal{A}$*is EP*,
(ii)	$\alpha \in c^*\mathcal{A} \cap \mathcal{A}b^*$,
(iii)	$(b^*)^0 = c^0$,
(iv)	$b\mathcal{A} = c^*\mathcal{A}$,
(v)	${}^0b = {}^0(c^*)$,
(vi)	$\mathcal{A}c = \mathcal{A}b^*$,
(vii)	$b\mathcal{A}^{-1} = c^*\mathcal{A}^{-1}$,
$(viii)$	$\mathcal{A}^{-1}c = \mathcal{A}^{-1}b^*$,
(ix)	$\exists x \in \mathcal{A}^{-1} : \; c = xb^*$,
(x)	$\exists y \in \mathcal{A} : \; y^0 = 0,$ *and* $c = yb^*$,
(xi)	$\exists z_1, z_2 \in \mathcal{A} : c = z_1b^*$ *and* $b^* = z_2c$,
(xii)	$\exists v \in \mathcal{A} : v\mathcal{A} = \mathcal{A}$ *and* $b = c^*v$,

(*xiii*) $\exists s_1 \in \mathcal{A} : s^0 = 0 \quad and \quad b^* = s_1 c,$

(*xiv*) $\exists s_2 \in \mathcal{A} : s_2\mathcal{A} = \mathcal{A} \quad and \quad c^* = bs_2,$

(*xv*) $\alpha^*\alpha = c^*b^*bcbb^\dagger \quad and \quad \alpha^*\alpha = c^*b^*c^\dagger cbc,$

(*xvi*) $\alpha\alpha^* = bcc^*b^*c^*(c^*)^\dagger \quad and \quad \alpha\alpha^* = bcbb^\dagger c^*b^*,$

(*xvii*) $\alpha^*\alpha = c^*b^*bcbb^\dagger \quad and \quad \alpha^*\alpha = c^*b^*bcbb^\dagger,$

(*xviii*) $\alpha\alpha^* = c^\dagger b^\dagger bcbcc^*b^* \quad and \quad \alpha^*\alpha = c^*b^*bcbb^\dagger,$

(*xix*) $\alpha^*\alpha = bcc^\dagger b^\dagger b^*c^*bc \quad and \quad \alpha\alpha^* = bcc^*b^*c^*(c^*)^\dagger,$

(*xx*) $\alpha\alpha^* = c\dagger b^\dagger bcbcc^*b^* \quad and \quad \alpha^*\alpha = bcc^\dagger b^\dagger b^*c^*bc$

Theorem 6.13 ([3], Th. 4.2 p. 252)**.** *Let $\mathcal{A}$ be a unital C^*-algebra and consider $\alpha \in \mathcal{A}$ such that $\alpha^\dagger$ exists. Then, the following statements are equivalent.*

(*i*) $\alpha \in \mathcal{A}$ *is EP,*

(*ii*) $\exists\, s \in \mathcal{A}$ *such that* $s^0 = 0$ *and* $\alpha^* = s\alpha,$

(*iii*) $\exists\, s_1, s_2 \in \mathcal{A}$ *such that* $\alpha^* = s_1\alpha$ *and* $\alpha = s_2\alpha^*,$

(*iv*) $\exists\, u \in \mathcal{A}$ *such that* $u\mathcal{A} = \mathcal{A}$ *and* $\alpha^* = \alpha u,$

(*v*) $\exists\, u_1, u_2 \in \mathcal{A}$ *such that* $\alpha^* = \alpha u_1$ *and* $\alpha = \alpha^* u_2,$

(*vi*) $\exists\, t \in \mathcal{A}$ *such that* ${}^0t = 0$ *and* $\alpha^* = \alpha t,$

(*vii*) $\exists\, x \in \mathcal{A}$ *such that* $\mathcal{A}x = \mathcal{A}$ *and* $\alpha^* = x\alpha,$

(*viii*) $\exists\, v \in \mathcal{A}^{-1}$ *such that* $\alpha^*\alpha = v\alpha\alpha^*,$

(*ix*) $\exists\, v_1 \in \mathcal{A}$ *such that* $\mathrm{v}_1^0 = 0$ *and* $\alpha^*\alpha = v_1\alpha\alpha^*,$

(*x*) $\exists\, v_2$ *and* $\mathrm{v}_3 \in \mathcal{A}$ *such that* $\alpha^*\alpha = v_2\alpha\alpha^*$ *and* $\alpha\alpha^* = \mathrm{v}_3\alpha^*\alpha,$

(*xi*) $\exists\, w \in \mathcal{A}^{-1}$ *such that* $\alpha^*\alpha = \alpha\alpha^* w,$

(*xii*) $\exists\, w_1 \in \mathcal{A}$ *such that* $w_1\mathcal{A} = \mathcal{A}$ *and* $\alpha^*\alpha = \alpha\alpha^* w_1,$

(*xiii*) $\exists\, w_2$ *and* $w_3 \in \mathcal{A}$ *such that* $\alpha^*\alpha = \alpha\alpha^* w_2$ *and* $\alpha\alpha^* = \alpha^*\alpha w_3,$

(*xiv*) $\exists\, z_1$ *and* $z_2 \in \mathcal{A}$ *such that* $\alpha^*\alpha = \alpha z_1\alpha^*$ *and* $\alpha\alpha^* = \alpha^* z_2\alpha,$

(*xv*) $\exists\, h_1 \in \mathcal{A}^{-1}$ *such that* $\alpha^*\alpha = \alpha h_1 h_1^*\alpha^*,$

(*xvi*) $\exists\, h_2 \in \mathcal{A}$ *such that* $h_2^0 = 0$ *and* $\alpha^*\alpha = \alpha h_2 h_2^*\alpha^*,$

(*xii*) $\exists\, h_3 \in \mathcal{A}$ *such that* $h_3\mathcal{A} = \mathcal{A}$ *and* $\alpha^*\alpha = \alpha h_3 h_3^*\alpha^* w_1.$

Note that in [10] it is deduced (Theorem 4.1, p.6) the well known result that selfadjoint C^*-elements are EP.

Mosić and Djordjević in two recent papers [28],[29] defined and studied weighted-EP elements in C^*-algebras. In [28], sixty six equivalent conditions for a regular element $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is a unital C^*-algebra, to be weighted-EP, are given. These conditions are based on analogous conditions for weighted-EP complex square matrices which have been studied by Tian and Wang in [36]. In [29], weighted-EP elements in C^*-algebras in terms of factorizations following the motivations of the papers [3], [9] and [11], are studied. For completeness we will give the definition of weighted-EP element and of the weighted Moore-Penrose inverse and we will present all these results in the sequel.

Definition 6.14. *Let $\mathcal{A}$ be a unital C^*-algebra.*

(i) An element $\alpha \in \mathcal{A}$ is said to be weighted-EP with respect to two invertible positive elements $e, f \in \mathcal{A}$ (or weighted-EP w.r.t. (e,f)) if both $e\alpha$ and αf^{-1} are EP, that is α is regular, $e\alpha\mathcal{A} = (e\alpha)^*\mathcal{A}$, and $\alpha f^{-1}\mathcal{A} = (\alpha f^{-1})^*\mathcal{A}$.

(ii) The element $\alpha \in \mathcal{A}$ has the weighted MP-inverse with weights e and f if there exists $b \in \mathcal{A}$ such that

$$\alpha b\alpha = \alpha, \quad b\alpha b = b, \quad (e\alpha b)^* = e\alpha b, \quad (fb\alpha)^* = fb\alpha.$$

The unique weighted MP-inverse with weights e and f will be denoted by $\alpha^{\dagger}_{e,f}$

We also need the definition of the following mapping: $x \to x^{*e,f} = e^{-1}x^*f$, for all $x \in \mathcal{A}$. Notice that this map is not an involution, because in general $(xy)^{*e,f} \neq y^{*e,f}x^{*e,f}$.

Theorem 6.15 ([28], Th. 2.2, p. 9.16 and Cor. 2.3, p. 929)**.** *Let $\mathcal{A}$ be a unital C^*-algebra, and let e and f be invertible positive elements in $\mathcal{A}$. If $\alpha \in \mathcal{A}$ is regular, then the following statements are equivalent:*

1. α *is weighted-EP w.r.t.* (e, f),
2. α *is weighted-EP w.r.t.* (f, e),
3. α *is both weighted-EP w.r.t.* (e, e) *and* (f, f),
4. $e\alpha\mathcal{A} = f\alpha\mathcal{A} = \alpha^*\mathcal{A}$,
5. $e^{-1}\alpha^*\mathcal{A} = f^{-1}\alpha^*\mathcal{A} = \alpha\mathcal{A}$,
6. $\alpha^{\dagger}_{e,f}\mathcal{A} = \alpha\mathcal{A}$ *and* $(\alpha^{\dagger}_{e,f})^*\mathcal{A} = \alpha^*\mathcal{A}$,
7. α^* *is weighted-EP w.r.t.* (e^{-1}, f^{-1}),
8. $\alpha\alpha^{\dagger}_{e,f} = \alpha^{\dagger}_{e,f}\alpha$,
9. $\alpha \in \mathcal{A}^{\sharp}$ *and* $\alpha^k = \alpha^{\dagger}_{e,f}\alpha\alpha^k = \alpha^k\alpha\alpha^{\dagger}_{e,f}$, *for any/some integer* $k \geq 1$,
10. $\alpha^{\dagger}_{e,f} = \alpha(\alpha^{\dagger}_{e,f})^2 = (\alpha^{\dagger}_{e,f})^2\alpha$,

11. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha^\sharp = \alpha^\dagger_{e,f}$,
12. $\alpha \in \mathcal{A}^\sharp$ *and both* $e\alpha\alpha^\sharp$ *and* $f\alpha\alpha^\sharp$ *are Hermitian,*
13. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha^\sharp\alpha^\dagger_{e,f} = \alpha^\dagger_{e,f}\alpha^\sharp$,
14. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha\alpha^\sharp\alpha^\dagger_{e,f} = \alpha^\dagger_{e,f}\alpha^\sharp\alpha$,
15. $\alpha \in \alpha^\dagger_{e,f}\mathcal{A}^{-1} \cap \mathcal{A}^{-1}\alpha^\dagger_{e,f}$,
16. $\alpha \in \alpha^\dagger_{e,f}\mathcal{A} \cap \mathcal{A}\alpha^\dagger_{e,f}$,
17. $\alpha\mathcal{A}^{-1} = f^{-1}\alpha^*\mathcal{A}^{-1}$ *and* $\mathcal{A}^{-1}\alpha = \mathcal{A}^{-1}\alpha^* e$,
18. $\mathcal{A}^{-1}\alpha^* = \mathcal{A}^{-1}\alpha f^{-1}$ *and* $\alpha^*\mathcal{A}^{-1} = e\alpha\mathcal{A}^{-1}$,
19. $\exists\, x \in \mathcal{A}$ *such that* $\alpha = e^{-1}\alpha^* x \alpha^* f$,
20. $\alpha = (\alpha e^{-1})^\dagger \alpha e^{-1} \alpha f \alpha (f\alpha)^\dagger$,
21. $\alpha \in \mathcal{A}^\sharp$ *and* α^k *is weighted-EP w.r.t.* (e, f), *for any/some integer* $k \geq 1$,
22. $\alpha\alpha^*\alpha$ *is weighted-EP w.r.t.* (e, f),
23. $\alpha^0 = [(e\alpha)^*]^0$ *and* $(\alpha^*)^0 = (\alpha f^{-1})^0$,
24. $\mathcal{A} = e^{-1}\alpha^*\mathcal{A} \oplus (\alpha^*)^0 = \alpha^*\mathcal{A} \oplus (\alpha^* f)^0$,
25. $\alpha^\dagger$ *is weighted-EP w.r.t.* (e^{-1}, f^{-1}),
26. $\alpha^\dagger_{e,f}$ *is weighted-EP w.r.t.* (e, f),
27. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha^{2k-1} = \alpha^\dagger_{e,f}\alpha^{2k+1}\alpha^\dagger_{e,f}$, *for any/some integer* $k \geq 1$,
28. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha\alpha^\dagger_{e,f}\alpha^\dagger_{e,f}\alpha = \alpha^\dagger_{e,f}\alpha\alpha\alpha^\dagger_{e,f}$,
29. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha^\sharp$ *is weighted-EP w.r.t.* (e, f),
30. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha\alpha^\sharp = \alpha\alpha^\dagger_{e,e} = \alpha\alpha^\dagger_{f,f}$ *(or* $\alpha\alpha^\sharp = \alpha^\dagger_{e,e}\alpha = \alpha^\dagger_{f,f}\alpha$),
31. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha\alpha^\sharp = \alpha\alpha^\dagger_{e,f} = \alpha\alpha^\dagger_{f,e}$ *(or* $\alpha\alpha^\sharp = \alpha^\dagger_{f,e}\alpha = \alpha^\dagger_{e,f}\alpha$),
32. $\alpha \in \mathcal{A}^\sharp$, $\alpha\alpha^\dagger_{e,e}e^{-1}\alpha^*\alpha = e^{-1}\alpha^*\alpha\alpha\alpha^\dagger_{e,e}$ *and* $\alpha\alpha^\dagger_{f,f}f^{-1}\alpha^*\alpha = f^{-1}\alpha^*\alpha\alpha\alpha^\dagger_{f,f}$,
33. $\alpha \in \mathcal{A}^\sharp$, $\alpha\alpha^\dagger_{e,f}e^{-1}\alpha^*\alpha = e^{-1}\alpha^*\alpha\alpha\alpha^\dagger_{e,f}$ *and* $\alpha\alpha^\dagger_{f,e}f^{-1}\alpha^*\alpha = f^{-1}\alpha^*\alpha\alpha\alpha^\dagger_{f,e}$,
34. $\alpha \in \mathcal{A}^\sharp$, $\alpha^\dagger_{e,e}\alpha\alpha\alpha^* e = \alpha\alpha^* e\alpha^\dagger_{e,e}\alpha$ *and* $\alpha^\dagger_{f,f}\alpha\alpha\alpha^* f = \alpha\alpha^* f\alpha^\dagger_{f,f}\alpha$,
35. $\alpha \in \mathcal{A}^\sharp$, $\alpha^\dagger_{f,e}\alpha\alpha\alpha^* e = \alpha\alpha^* e\alpha^\dagger_{f,e}\alpha$ *and* $\alpha^\dagger_{e,f}\alpha\alpha\alpha^* f = \alpha\alpha^* f\alpha^\dagger_{e,f}\alpha$,
36. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha^k\alpha\alpha^\dagger_{e,f} + \alpha^\dagger_{e,f}\alpha\alpha^k = 2\alpha^k$ *for any/some integer* $k \geq 1$,
37. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha^\dagger_{e,f}\alpha^\sharp\alpha + \alpha\alpha^\sharp\alpha^\dagger_{e,f} = 2\alpha^\dagger_{e,f}$,

38. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha^{*e,f} = \alpha^{*e,f}\alpha\alpha^\sharp = \alpha^\sharp\alpha\alpha^{*e,f}$,

39. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha^{*e,f}\alpha\alpha^\sharp + \alpha^\sharp\alpha\alpha^{*e,f} = 2\alpha^{*e,f}$,

40. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha^k\alpha\alpha^\dagger_{e,f} + (\alpha^k\alpha\alpha^\dagger_{e,f})^* = \alpha^\dagger_{e,f}\alpha\alpha^k + (\alpha^\dagger_{e,f}\alpha\alpha^k)^* = \alpha^k + (\alpha^k)^*$, *for any/some integer* $k \geq 1$,

41. $\alpha\alpha^\dagger_{e,f}(\alpha + \lambda\alpha^\dagger_{e,f}) = (\alpha + \lambda\alpha^\dagger_{e,f})\alpha\alpha^\dagger_{e,f}$ *and* $\alpha^\dagger_{e,f}\alpha(\alpha + \lambda\alpha^\dagger_{e,f}) = (\alpha + \lambda\alpha^\dagger_{e,f})\alpha^\dagger_{e,f}\alpha$, *for any/some complex number* $\lambda \neq 0$,

42. $\alpha b = b\alpha \Rightarrow \alpha^\dagger_{e,f}b = b\alpha^\dagger_{e,f}$,

43. $\alpha^\dagger_{e,f} = \varphi(\alpha)$, *for some function* φ *holomorphic in a neighborhood of* $\sigma(\alpha)$,

44. $(\alpha + \lambda\alpha^\dagger_{e,e})\mathcal{A} = (\alpha + \lambda\alpha^\dagger_{f,f})\mathcal{A} = (\lambda\alpha + \alpha^3)\mathcal{A}$ *and* $\mathcal{A}(\alpha + \lambda\alpha^\dagger_{e,e}) = (\alpha + \lambda\alpha^\dagger_{f,f}) = \mathcal{A}(\lambda\alpha + \alpha^3)$, *for any/some complex number* $\lambda \neq 0$,

45. $(\alpha + \lambda\alpha^\dagger_{e,f})\mathcal{A} = (\lambda\alpha + \alpha^3)\mathcal{A}$ *and* $\mathcal{A}(\alpha + \lambda\alpha^\dagger_{e,f}) = \mathcal{A}(\lambda\alpha + \alpha^3)$, *for any/some complex number* $\lambda \neq 0$,

46. $(\alpha + \lambda\alpha^\dagger_{e,e})^0 = (\alpha + \lambda\alpha^\dagger_{f,f})^0 = (\lambda\alpha + \alpha^3)^0$ *and* ${}^0(\alpha + \lambda\alpha^\dagger_{e,e}) = {}^0(\alpha + \lambda\alpha^\dagger_{f,f}) ={}^0 (\lambda\alpha + \alpha^3)$, *for any/some complex number* $\lambda \neq 0$,

47. $(\alpha + \lambda\alpha^\dagger_{e,f})^0 = (\lambda\alpha + \alpha^3)^0$ *and* ${}^0(\alpha + \lambda\alpha^\dagger_{f,e}) = {}^0(\lambda\alpha + \alpha^3)$, *for any/some complex number* $\lambda \neq 0$,

48. $\alpha \in \mathcal{A}^\sharp$ *and* $(\alpha^\dagger_{e,f})^2\alpha^\sharp = \alpha^\dagger_{e,f}\alpha^\sharp\alpha^\dagger_{e,f} = \alpha^\sharp(\alpha^\dagger_{e,f})^2$,

49. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha(\alpha^\dagger_{e,f})^2 = \alpha^\sharp = (\alpha^\dagger_{e,f})^2\alpha$,

50. $\alpha \in \mathcal{A}^\sharp$, $\alpha^{*f,e}\alpha^\dagger_{e,f} = \alpha^{*f,e}\alpha^\sharp$ *and* $\alpha^\dagger_{e,f}\alpha^{*f,e} = \alpha^\sharp\alpha^{*f,e}$,

51. $\alpha \in \mathcal{A}^\sharp$ *and* $(\alpha^\dagger_{e,f})^2 = (\alpha^\sharp)^2$,

52. $\alpha^{*e,f} = \alpha^{*e,f}\alpha^\dagger_{e,f}\alpha = \alpha\alpha^\dagger_{e,f}\alpha^{*e,f}$,

53. $\alpha \in \mathcal{A}^\sharp$ *and* $(\alpha^\sharp)^{*e,f} = \alpha\alpha^\sharp(\alpha^\sharp)^{*e,f} = (\alpha^\sharp)^{*e,f}\alpha^\sharp\alpha$ *(or* $(\alpha^\sharp)^{*e,f} = \alpha\alpha^\sharp(\alpha^\sharp)^{*e,f} = (\alpha^\sharp)^{*e,f}\alpha^\sharp\alpha)$,

54. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha^\dagger_{e,f}(\alpha^\sharp)^2 = (\alpha^\sharp)^2\alpha^\dagger_{e,f}$,

55. $\alpha \in \mathcal{A}^\sharp$ *and* $\alpha^k\alpha^\dagger_{e,f} = \alpha^\dagger_{e,f}\alpha^k$, *for any/some integer* $k \geq 1$,

56. $\alpha\alpha^\dagger_{e,f}(\alpha + \lambda\alpha^{*e,f}) = (\alpha + \lambda\alpha^{*e,f})\alpha\alpha^\dagger_{e,f}$ *and* $\alpha^\dagger_{e,f}\alpha(\alpha + \lambda\alpha^{*e,f}) = (\alpha + \lambda\alpha^{*e,f})\alpha^\dagger_{e,f}\alpha$, *for any/some complex number* $\lambda \neq 0$,

57. $\alpha \in \mathcal{A}^\sharp$, $\alpha\alpha^\dagger_{e,e}(\alpha\alpha^* e - e^{-1}\alpha^*\alpha) = (\alpha\alpha^* e - e^{-1}\alpha^*\alpha)\alpha\alpha^\dagger_{e,e}$ *and* $\alpha\alpha^\dagger_{f,f}(\alpha\alpha^* f - f^{-1}\alpha^*\alpha) = (\alpha\alpha^* f - f^{-1}\alpha^*\alpha)\alpha\alpha^\dagger_{f,f}$,
58. $\alpha \in \mathcal{A}^\sharp$, $\alpha\alpha^\dagger_{e,f}(\alpha\alpha^* e - e^{-1}\alpha^*\alpha) = (\alpha\alpha^* e - e^{-1}\alpha^*\alpha)\alpha\alpha^\dagger_{e,f}$ *and* $\alpha\alpha^\dagger_{f,e}(\alpha\alpha^* f - f^{-1}\alpha^*\alpha) = (\alpha\alpha^* f - f^{-1}\alpha^*\alpha)\alpha\alpha^\dagger_{f,e}$,
59. $\alpha \in \mathcal{A}^\sharp$, $\alpha^\dagger_{e,e}\alpha(\alpha\alpha^* e - e^{-1}\alpha^*\alpha) = (\alpha\alpha^* e - e^{-1}\alpha^*\alpha)\alpha^\dagger_{e,e}\alpha$ *and* $\alpha^\dagger_{f,f}\alpha(\alpha\alpha^* f - f^{-1}\alpha^*\alpha) = (\alpha\alpha^* f - f^{-1}\alpha^*\alpha)\alpha^\dagger_{f,f}\alpha$,
60. $\alpha \in \mathcal{A}^\sharp$, $\alpha^\dagger_{f,e}\alpha(\alpha\alpha^* e - e^{-1}\alpha^*\alpha) = (\alpha\alpha^* e - e^{-1}\alpha^*\alpha)\alpha^\dagger_{f,e}\alpha$ *and* $\alpha^\dagger_{e,f}\alpha(\alpha\alpha^* f - f^{-1}\alpha^*\alpha) = (\alpha\alpha^* f - f^{-1}\alpha^*\alpha)\alpha^\dagger_{e,f}\alpha$,
61. $\alpha \in \mathcal{A}^\sharp$ *and* $(a^{s+t})^\dagger = (\alpha^s)^\dagger_{e,1}(\alpha^t)^\dagger_{1,e} = (\alpha^s)^\dagger_{f,1}(\alpha^t)^\dagger_{1,f}$, *for any/some integers* $s, t \geq 1$,
62. $\alpha \in \mathcal{A}^\sharp$ *and* $(a^{s+t})^\dagger_{e,f} = (\alpha^s)^\dagger_{f,f}(\alpha^t)^\dagger_{e,f} = (\alpha^s)^\dagger_{e,f}(\alpha^t)^\dagger_{e,e}$, *for any/some integers*
63. $\alpha^{*f,e}\mathcal{A} = \alpha^{*e,f}\mathcal{A} = \alpha\mathcal{A}$ *(or* $\alpha^{*f,e}\mathcal{A} = \alpha\mathcal{A}$ *and* $\mathcal{A}\alpha^{*f,e} = \mathcal{A}\alpha$,
64. $(\alpha^{*f,e})^0 = \alpha^0$ *and* ${}^0(\alpha^{*f,e}) = {}^0\alpha$,
65. $\alpha\mathcal{A}^{-1} = \alpha^{*f,e}\mathcal{A}^{-1}$ *and* $\mathcal{A}^{-1}\alpha = \mathcal{A}^{-1}\alpha^{*f,e}$,
66. $\mathcal{A}^{-1}\alpha^* = \mathcal{A}^{-1}(\alpha^{*f,e})^*$ *and* $\alpha^*\mathcal{A}^{-1} = (\alpha^{*f,e})^*\mathcal{A}^{-1}$

In the next theorem, weighted-EP elements through the factorizations $\alpha = b\alpha^{*e,f}$, $\alpha^{*f,e} = s\alpha$ and $\alpha = e^{-1}ucvf$, are characterized.

Theorem 6.16 ([29], Theorems 2.1, 3.1, 3.2, 4.1, p. 5385–5387)**.** *Let $\mathcal{A}$ be a unital C^*-algebra, and let e and f be invertible positive elements in $\mathcal{A}$. If $\alpha \in \mathcal{A}$ is regular, then the following statements are equivalent:*

1. α *is weighted* $-$ *EP w.r.t.* (e, f),
2. $\alpha = b\alpha^{*f,e} = \alpha^{*f,e}c$ *for some* $b, c \in \mathcal{A}$,
3. $\alpha^{*f,e}\alpha = b_1\alpha^{*f,e} = \alpha c_1$ *and* $\alpha\alpha^{*f,e} = \alpha^{*f,e}b_2 = c_2\alpha$ *for some* $b_1, b_2, c_1, c_2 \in \mathcal{A}$,
4. $\alpha^{*f,e}\alpha = b_3\alpha^\dagger_{e,f}$, $\alpha\alpha^{*f,e} = \alpha^\dagger_{e,f}b_4$ *and* $\alpha^\dagger_{e,f} = c_3\alpha = \alpha c_4$ *for some* $b_3, b_4, c_3, c_4 \in \mathcal{A}$,
5. $\exists\, s, t \in \mathcal{A} :\ s^0 = {}^0t = \{0\}$ *and* $\alpha^{*f,e} = s\alpha = \alpha t$,

6. $\exists\, s_1, s_2, t_1, t_2 \in \mathcal{A}:\ \alpha^{*f,e} = s_1\alpha = \alpha t_1$ *and* $\alpha = s_2\alpha^{*f,e} = \alpha^{*f,e}t_2$,
7. $\exists\, u, \mathrm{v} \in \mathcal{A}:\ u\mathcal{A} = \mathcal{A} = \mathcal{A}\mathrm{v}$ *and* $\alpha^{*f,e} = \alpha u = \mathrm{v}\alpha$,
8. $\exists\, x, y \in \mathcal{A}^{-1}:\ \alpha^{*f,e}\alpha = x\alpha\alpha^{*f,e} = \alpha\alpha^{*f,e}y$,
9. $\exists\, x_1, y_1 \in \mathcal{A}:\ x_1^0 = {}^0y_1 = \{0\}$ *and* $\alpha^{*f,e}\alpha = x_1\alpha\alpha^{*f,e} = \alpha\alpha^{*f,e}y_1$,
10. $\exists\, x_2, y_2 \in \mathcal{A}:\ \mathcal{A}x_2 = \mathcal{A} = y_2\mathcal{A}$ *and* $\alpha^{*f,e}\alpha = x_2\alpha\alpha^{*f,e} = \alpha\alpha^{*f,e}y_2$,
11. $\exists\, x_3, x_4, y_3, y_4 \in \mathcal{A}:\ \alpha^{*f,e}\alpha = x_3\alpha\alpha^{*f,e} = \alpha\alpha^{*f,e}y_3$
    and $\alpha\alpha^{*f,e} = x_4\alpha^{*f,e}\alpha = \alpha^{*f,e}\alpha y_4$,
12. $\exists\, z_1, z_2 \in \mathcal{A}:\ \alpha^{*f,e}\alpha = \alpha z_1\alpha^{*f,e}$ *and* $\alpha\alpha^{*f,e} = \alpha^{*f,e}z_2\alpha$,
13. $\exists\, g_1, h_1 \in \mathcal{A}^{-1}:\ \alpha^{*f,e}\alpha = \alpha h_1 h_1^{*f,e}\alpha^{*f,f}$ *and* $\alpha\alpha^{*f,e} = \alpha^{*e,f}g_1^{*f,f}g_1\alpha$,
14. $\exists\, g_2, h_2 \in \mathcal{A}:\ g_2^0 = {}^0h_2 = \{0\},\ \alpha^{*f,e}\alpha = \alpha h_2 h_2^{*e,f}\alpha^{*f,f}$
    and $\alpha\alpha^{*f,e} = \alpha^{*e,f}g_2^{*f,f}g_2\alpha$,
15. $\exists\, g_3, h_3 \in \mathcal{A}:\ \mathcal{A}g_3 = \mathcal{A} = h_3\mathcal{A},\ \alpha^{*f,e}\alpha = \alpha h_3 h_3^{*e,f}\alpha^{*f,f}$
    and $\alpha\alpha^{*f,e} = \alpha^{*e,f}g_3^{*f,f}g_3\alpha$,
16. $\exists\, s, t \in \mathcal{A}:\ s^0 = {}^0t = \{0\}$ *and* $\alpha^{\dagger}_{e,f} = s\alpha = \alpha t$,
17. $\exists\, s_1, s_2, t_1, t_2 \in \mathcal{A}:\ \alpha^{\dagger}_{e,f} = s_1\alpha = \alpha t_1$ *and* $\alpha = s_2\alpha^{\dagger}_{e,f} = \alpha^{\dagger}_{e,f}t_2$,
18. $\exists\, u, \mathrm{v} \in \mathcal{A}:\ u\mathcal{A} = \mathcal{A} = \mathcal{A}\mathrm{v}$ *and* $\alpha^{\dagger}_{e,f} = \alpha u = \mathrm{v}\alpha$,
19. $\exists\, x, y \in \mathcal{A}^{-1}:\ \alpha^{\dagger}_{e,f}\alpha = x\alpha\alpha^{\dagger}_{e,f} = \alpha\alpha^{\dagger}_{e,f}y$,
20. $\exists\, x_1, y_1 \in \mathcal{A}:\ x_1^0 = {}^0y_1 = \{0\}$ *and* $\alpha^{\dagger}_{e,f}\alpha = x_1\alpha\alpha^{\dagger}_{e,f} = \alpha\alpha^{\dagger}_{e,f}y_1$,
21. $\exists\, x_2, y_2 \in \mathcal{A}:\ \mathcal{A}x_2 = \mathcal{A} = y_2\mathcal{A}$ *and* $\alpha^{\dagger}_{e,f}\alpha = x_2\alpha\alpha^{\dagger}_{e,f} = \alpha\alpha^{\dagger}_{e,f}y_2$,
22. $\exists\, x_3, x_4, y_3, y_4 \in \mathcal{A}:\ \alpha^{\dagger}_{e,f}\alpha = x_3\alpha\alpha^{\dagger}_{e,f} = \alpha\alpha^{\dagger}_{e,f}y_3$
    and $\alpha\alpha^{\dagger}_{e,f} = x_4\alpha^{\dagger}_{e,f}\alpha = \alpha^{\dagger}_{e,f}\alpha y_4$,
23. $\exists\, z_1, z_2 \in \mathcal{A}:\ \alpha^{\dagger}_{e,f}\alpha = \alpha z_1\alpha^{\dagger}_{e,f}$ *and* $\alpha\alpha^{\dagger}_{e,f} = \alpha^{\dagger}_{e,f}z_2\alpha$,
24. $\exists\, c, d, u, \mathrm{v} \in \mathcal{A}:\ \alpha = e^{-1}uc\mathrm{v}f = e^{-1}f\mathrm{v}^*d^*u^*e^{-1}f,\ \mathrm{v}\mathcal{A} = \mathcal{A} = \mathcal{A}u$,
    $c\mathcal{A} = d\mathcal{A}$ *and* $\mathcal{A}c = \mathcal{A}d$,
25. $\exists\, c, d, u, \mathrm{v} \in \mathcal{A}:\ \alpha = e^{-1}uc\mathrm{v}f = e^{-1}f\mathrm{v}^*d^*u^*e^{-1}f,\ u^0 = \{0\} = {}^0\mathrm{v},\ c^0 = d^0$
    and ${}^0c = {}^0d$,
26. $\exists\, c, d, u, \mathrm{v} \in \mathcal{A}:\ \alpha = e^{-1}uc\mathrm{v}f,\ \alpha^{\dagger}_{e,f} = e^{-1}ud\mathrm{v}f,\ \mathrm{v}\mathcal{A} = \mathcal{A} = \mathcal{A}u,\ c\mathcal{A} = d\mathcal{A}$
    and $\mathcal{A}c = \mathcal{A}d$,
27. $\exists\, c, d, u, \mathrm{v} \in \mathcal{A}:\ \alpha = e^{-1}uc\mathrm{v}f,\ \alpha^{\dagger}_{e,f} = e^{-1}ud\mathrm{v}f,\ u^0 = \{0\} = {}^0\mathrm{v},\ c^0 = d^0$

and ${}^0c = {}^0d$,

28. $\exists\, c, d, u, \mathrm{v} \in \mathcal{A} :\ \alpha^{*f,e}\alpha = uc\mathrm{v},\ \alpha\alpha^{*f,e} = ud\mathrm{v},\ \mathrm{v}\mathcal{A} = \mathcal{A} = \mathcal{A}u,\ c\mathcal{A} = d\mathcal{A}$ *and* $\mathcal{A}c = \mathcal{A}d$,

29. $\exists\, c, d, u, \mathrm{v} \in \mathcal{A} :\ \alpha^{*f,e}\alpha = uc\mathrm{v},\ \alpha\alpha^{*f,e} = ud\mathrm{v},\ u^0 = \{0\} = {}^0\mathrm{v},\ c^0 = d^0$ *and* ${}^0c = {}^0d$.

Characterizations of weighted-EP elements in terms of the factorization of $\alpha \in \mathcal{A}$ of the form

$$\alpha = bc,\ f^{-1}b^*\mathcal{A} = \mathcal{A} = c\mathcal{A} \tag{4}$$

where $b, c, f \in \mathcal{A}$ and f is a positive and invertible element, are included in the following

Theorem 6.17 ([29], Th. 5.1 p. 5389)**.** *Let e, f, h be invertible positive elements in $\mathcal{A}$. If $\alpha \in \mathcal{A}$ has a factorization (4) then α is regular and the following conditions are equivalent*

1. α *is weighted* $-$ *EP w.r.t.* (e, h),
2. $bb^\dagger_{e,f} = c^\dagger_{f,h}c$,
3. $c^0 = [(eb)^*]^0$ *and* $(b^*)^0 = (ch^{-1})^0$,
4. ${}^0c^* = {}^0(eb)$ *and* ${}^0b = {}^0(h^{-1}c^*)$,
5. $c^*\mathcal{A} = eb\mathcal{A}$ *and* $b\mathcal{A} = h^{-1}c^*\mathcal{A}$,
6. $\mathcal{A}c = \mathcal{A}b^*e$ *and* $\mathcal{A}b^* = \mathcal{A}ch^{-1}$,
7. $\exists\, u \in \mathcal{A}^{-1} :\ c = ub^\dagger_{e,f}$ *and* $b = c^\dagger_{f,h}u$,
8. $\exists\, x, y \in \mathcal{A}^{-1} :\ c = xb^*e$ *and* b^*ych^{-1},
9. $\mathcal{A}^{-1}c = \mathcal{A}^{-1}b^*e$ *and* $\mathcal{A}^{-1}b^* = \mathcal{A}^{-1}ch^{-1}$,
10. $c^*\mathcal{A}^{-1} = eb\mathcal{A}^{-1}$ *and* $b\mathcal{A}^{-1} = h^{-1}c^*\mathcal{A}^{-1}$,
11. $\exists\, x, y \in \mathcal{A} :\ x^0 = y^0 = \{0\},\ c = xb^*e$ *and* $b^* = ych^{-1}$,
12. $\exists\, x, x_1, y, y_1 \in \mathcal{A} :\ c = xb^*e,\ b^*e = x_1c,\ b^* = ych^{-1}$ *and* $ch^{-1} = y_1b^*$,
13. $\exists\, x, y \in \mathcal{A} :\ x\mathcal{A} = y\mathcal{A} = \mathcal{A},\ c^* = ebx$ *and* $b = h^{-1}c^*y$,
14. $\alpha \in h^{-1}c^*\mathcal{A} \cap \mathcal{A}b^*e$ (*or* $\alpha \in c^\dagger_{f,h}\mathcal{A} \cap \mathcal{A}b^\dagger_{e,f}$),
15. $\alpha^\dagger_{e,h} \in b\mathcal{A} \cap \mathcal{A}c$,
16. $b(b^*eb)^{-1}b^*e = h^{-1}c^*(ch^{-1}c^*)^{-1}c$,

17. $b = c^{\dagger}_{f,h}cb,\ c = cbb^{\dagger}_{e,f},\ b^{\dagger}_{e,f} = b^{\dagger}_{e,f}c^{\dagger}_{f,h}c$ *and* $c^{\dagger}_{f,h} = bb^{\dagger}_{e,f}c^{\dagger}_{f,h}$,
18. $\mathcal{A}^{-1}c = \mathcal{A}^{-1}b^{\dagger}_{e,f}$ *and* $b\mathcal{A}^{-1} = c^{\dagger}_{f,h}\mathcal{A}^{-1}$,
19. $\exists\, u \in \mathcal{A}:\ u^0 = {}^0u = \{0\},\ c = ub^{\dagger}_{e,f}$ *and* $b = c^{\dagger}_{f,h}u$,
20. $\exists\, u \in \mathcal{A}:\ \mathcal{A}u = u\mathcal{A} = \mathcal{A},\ c = ub^{\dagger}_{e,f}$ *and* $b = c^{\dagger}_{f,h}u$,
21. $\exists\, \mathrm{v} \in \mathcal{A}:\ \mathrm{v}^0 = {}^0\mathrm{v} = \{0\},\ b^{\dagger}_{e,f} = \mathrm{v}c$ *and* $c^{\dagger}_{f,h} = b\mathrm{v}$,
22. $\exists\, \mathrm{v} \in \mathcal{A}:\ \mathcal{A}\mathrm{v} = \mathrm{v}\mathcal{A} = \mathcal{A},\ b^{\dagger}_{e,f} = \mathrm{v}c$ *and* $c^{\dagger}_{f,h} = b\mathrm{v}$,
23. $\exists\, u, u_1, \mathrm{v}, \mathrm{v}_1 \in \mathcal{A}:\ c = ub^{\dagger}_{e,f},\ b^{\dagger}_{e,f} = \mathrm{v}c,\ b = c^{\dagger}_{f,h}u_1$ *and* $c^{\dagger}_{f,h} = b\mathrm{v}_1$.

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