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EVOLUTION EQUATIONS FOR THE STEFAN PROBLEM

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Abstract. We study particular kind of Stefan problem and use the theory of abstract quasilinear evolution equations for its solution.

Introduction. The Stefan problem is a special kind of a free boundary problem which models phase transition phenomena of two or more materials, for example melting of ice and freezing of water. It is named after the physicist J. Stefan who has originally designed a model that describes ice formation in polar seas, see [11], [12]. Lam and Clapeyron treated similar problem in [5]. For detailed historical account and older results see Rubinstein [9], and for more recent results see the monographs by Meirmanov [8] and Visintin [13]. We study a quasy-steady variant and propose in our model a boundary condition with surface tension and kinetic undercooling that reflects the relaxation dynamics. Our approach to the problem is by using the theory of abstract parabolic evolution equations.

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The Stefan problem consists in finding the unknown free boundary Γ_t and the temperature u in the following set of equations

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_t \\ V + \partial_\nu u = 0 & \text{on } \Gamma_t \\ u = aV + \kappa & \text{on } \Gamma_t \\ \Gamma(0) = \Gamma_0 \end{cases}$$

The boundary condition $u = aV + \kappa$ with positive function $a > 0$ expresses the temperature as a function of the local normal velocity V and the normal curvature κ of the phase boundary.

1. The oblique derivative problem. Using the so called Hanzawa transformation [3], the free boundary problem can be transformed to a fixed domain $D \subset \mathbb{R}^n$ with boundary $\partial D = \Sigma$.

$$\begin{cases} \mathcal{A}(\rho)v = 0 & \text{in } D \\ v + \delta\mathcal{B}(\rho)v = H(\rho) & \text{on } \Sigma \\ \partial_t\rho + L_\rho\mathcal{B}(\rho)v = 0 & \text{on } \Sigma \\ \rho(0) = \rho_0 & \text{on } \Sigma. \end{cases}$$

The first two equations form a boundary value problem, known as Oblique Derivative Problem. We assume that the initial geometry Γ_0 is in the class $C^{3,\alpha}$. Then we have that the boundary describing function ρ is in the same class. $\mathcal{A}(\rho)$ is a second order uniformly elliptic operator, $\mathcal{A}(\rho) : C^{2,\alpha}(D) \to C^{0,\alpha}(D)$ and it has the representation

$$\mathcal{A}(\rho)v = \sum_{i,j} a_{ij}(\rho)\partial^2_{ij}v + \sum_i a_i(\rho)\partial_i v,$$

with $a_{ij}(\rho) \in C^{2,\alpha}(D)$, $a_i(\rho) \in C^{1,\alpha}(D)$. The boundary operator $\mathcal{B}(\rho) : C^{2,\alpha}(D) \to C^{0,\alpha}(\Sigma)$ has the representation

$$\mathcal{B}(\rho)v = \overrightarrow{b_\rho} \cdot \nabla v,$$

for a nowhere tangential and nowhere vanishing vector field

$$\overrightarrow{b_\rho} : \Sigma \to \mathbb{R}^n.$$

L_ρ is a strictly positive function. The novel thing is that δ is not a constant but also a strictly positive function. $H(\rho)$ is the transformed mean curvature and its regularity is decisive for the solvability of the problem. In our model $H(\rho)$ is a

$C^{1,\alpha}(\Sigma)$ function. For more details the reader is referred to [6].
Let $\mathcal{S}$ be the formal solution operator for the Oblique Derivative Problem, depending on g.

Theorem 1.1. *There is a unique solution $v = \mathcal{S}(g) \in C^{2,\alpha}(D)$, and there exist a constant $C > 0$ such that*

$$\|\mathcal{S}(\rho)g\|_{2,\alpha} \leq C(\|\mathcal{S}(\rho)g\|_\infty + \|g\|_{1,\alpha}).$$

It holds $\delta\mathcal{B}(\rho)\mathcal{S}(\rho) = I - \gamma\mathcal{S}(\rho)$.

Proof. For the first assertition see [2, Theorem 6.31]. For the second

$$\begin{aligned}\delta\mathcal{B}(\rho)\mathcal{S}(\rho)g &= g - \gamma\mathcal{S}(\rho)g \\ &= [I - \gamma\mathcal{S}(\rho)]g.\end{aligned}$$ □

We assume that the distance function ρ belongs to the set $\mathcal{U} := \{\rho \in C^{3,\alpha}(\Sigma) : \|\rho\|_{C^{3,\alpha}(\Sigma)} < b\}$ and show that the solution operator is Lipschitz continuous on this set.

Theorem 1.2. *The solution operator for sufficiently small $b > 0$*

$$\mathcal{S} : \mathcal{U} \to \mathcal{L}(C^{1,\alpha}(\Sigma), C^{2,\alpha}(D)),$$

$$\rho \mapsto \mathcal{S}(\rho)g$$

of the Oblique Derivative Problem

$$\begin{aligned}\mathcal{A}(\rho)v &= 0 \\ v + \delta\mathcal{B}(\rho)v &= g\end{aligned}$$

is Lipschitz continuous. There is a constant $C > 0$ such that

$$\|\mathcal{S}(\rho_1) - \mathcal{S}(\rho_2)\|_{\mathcal{L}(C^{1,\alpha}(\Sigma),C^{2,\alpha}(D))} \leq C\|\rho_1 - \rho_2\|_{C^{1,\alpha}(\Sigma)},$$

for all $\rho_1, \rho_2 \in \mathcal{U}$.

Proof. We introduce $\widetilde{\mathcal{B}}(\rho)v := \gamma v + \delta\mathcal{B}(\rho)v$ and denote with $\tilde{\mathcal{S}}(\rho)$ the inverse to $\begin{pmatrix}\mathcal{A}(\rho) \\ \widetilde{\mathcal{B}}(\rho)\end{pmatrix}$. Then the operator $\begin{pmatrix}\mathcal{A} \\ \widetilde{\mathcal{B}}\end{pmatrix}$ is in $C^\infty(\mathcal{U}, \mathcal{L}(C^{2,\alpha}(D), C^\alpha(D) \oplus C^{1,\alpha}(\Sigma)))$, especially is Lipschitz continuous and for some constant $C > 0$

$$\left\| \begin{pmatrix}\mathcal{A}(\rho_1) \\ \widetilde{\mathcal{B}}(\rho_1)\end{pmatrix} - \begin{pmatrix}\mathcal{A}(\rho_2) \\ \widetilde{\mathcal{B}}(\rho_2)\end{pmatrix} \right\|_{\mathcal{L}(C^{2,\alpha}(D),C^\alpha(D)\oplus C^{1,\alpha}(\Sigma))} \leq C\|\rho_1 - \rho_2\|_{C^{1,\alpha}(\Sigma)}.$$

We have

$$\|\tilde{\mathcal{S}}(\rho_1)-\tilde{\mathcal{S}}(\rho_2)\|_{\mathcal{L}(C^\alpha(D)\oplus C^{1,\alpha}(\Sigma),C^{2,\alpha}(D))}$$

$$=\left\|\tilde{\mathcal{S}}(\rho_1)\left[\begin{pmatrix}\mathcal{A}(\rho_2)\\ \widetilde{\mathcal{B}}(\rho_2)\end{pmatrix}-\begin{pmatrix}\mathcal{A}(\rho_1)\\ \widetilde{\mathcal{B}}(\rho_1)\end{pmatrix}\right]\tilde{\mathcal{S}}(\rho_2)\right\|_{\mathcal{L}(C^\alpha(D)\oplus C^{1,\alpha}(\Sigma),C^{2,\alpha}(D))}$$

$$\leq\|\tilde{\mathcal{S}}(\rho_1)\|_{\mathcal{L}(C^\alpha(D)\oplus C^{1,\alpha}(\Sigma),C^{2,\alpha}(D))}\left\|\begin{pmatrix}\mathcal{A}(\rho_2)\\ \widetilde{\mathcal{B}}(\rho_2)\end{pmatrix}-\begin{pmatrix}\mathcal{A}(\rho_1)\\ \widetilde{\mathcal{B}}(\rho_1)\end{pmatrix}\right\|_{\mathcal{L}(C^{2,\alpha}(D),C^\alpha(D)\oplus C^{1,\alpha}(\Sigma))}$$

$$\times\|\tilde{\mathcal{S}}(\rho_2)\|_{\mathcal{L}(C^\alpha(D)\oplus C^{1,\alpha}(\Sigma),C^{2,\alpha}(D))}$$

$$\leq C\|\rho_1-\rho_2\|_{C^{1,\alpha}(\Sigma)}.$$

It follows that $\mathcal{S}(\rho)=\tilde{\mathcal{S}}(\rho)\Big|_{\{0\}\oplus C^{1,\alpha}(\Sigma)}$ is also Lipschitz continuous. □

2. The evolution equation. The last two equations in the Stefan problem give the evolution equation

$$\begin{cases}\partial_t\rho+L_\rho\mathcal{B}(\rho)v=0 & \text{on } \Sigma\\ \rho(0)=\rho_0 & \text{on } \Sigma.\end{cases}$$

Plugging the solution $v=\mathcal{S}(\rho)H(\rho)$ into the evolution equation gives fully nonlinear evolution equation

$$\begin{cases}\partial_t\rho+L_\rho\mathcal{B}(\rho)\mathcal{S}(\rho)H(\rho)=0\\ \rho(0)=\rho_0.\end{cases}$$

This evolution equation can be linearized by using the quasilnear structure of the mean curvature operator $H(\rho)$. It can be written as $H(\rho)=P(\rho)\rho+Q(\rho)$, where $P(\rho)$ is a second order uniformly elliptic differential operator and $Q(\rho)$ is an analytic function depending on the first and second order derivatives of ρ, see Escher and Simonett [1].

Then the evolution equation becomes quasilnear

$$\begin{aligned}\partial_t\rho+A(\rho)\rho &= F(\rho)\\ \rho(0) &= \rho_0,\end{aligned}\tag{1}$$

with the operator

$$A(\rho):=L_\rho\mathcal{B}(\rho)\mathcal{S}(\rho)P(\rho),\quad\text{and the function}\quad F(\rho):=-L_\rho\mathcal{B}(\rho)\mathcal{S}(\rho)Q(\rho).$$

3. The existence and uniqueness theorem. We use the following interpolation spaces of Da Prato and Grisvard which are defined for two Banach spaces E_0 and E_1 with $E_1 \hookrightarrow E_0$, E_1 densely embedded in E_0 and $0 < \theta \leq 1$.

$$\mathbb{E}_0^\theta(J) := \{u \in C(\dot{J}, E_0) : \lim_{t \to 0+} \|t^{1-\theta} u(t)\|_{E_0} = 0\}$$

$$\mathbb{E}_1^\theta(J) := \{u \in C^1(\dot{J}, E_0) \cap C(\dot{J}, E_1) : \lim_{t \to 0+} t^{1-\theta}(\|u'(t)\|_{E_0} + \|u(t)\|_{E_1}) = 0\}$$

$$\gamma \mathbb{E}_1^\theta(J) := \{u(0) : u \in \mathbb{E}_1^\theta(J)\}$$

For more information on these and other interpolation spaces we refer the reader to Lunardi's book [7].

With $\mathcal{H}(E_1, E_0)$ we denote the set of all analytic generators from E_1 to E_0. In the following, the concept of continuous maximal regularity is used.

Definition 3.1. *The operator $A \in \mathcal{H}(E_1, E_0)$ has continuous maximal regularity if the inhomogeneous Cauchy problem*

$$\begin{cases} \dot{u}(t) = Au(t) + f(t), & \text{on } J \\ u(0) = u_0. \end{cases} \tag{2}$$

has a unique solution $u \in \mathbb{E}_1^\theta$ for all $f \in \mathbb{E}_0^\theta$ and $u_0 \in \gamma \mathbb{E}_1^\theta$.

For maximal regularity in the context of L^p spaces see the excellent survey by Kunstmann and Weis [4].

Remark 3.2. Uniformly elliptic operators have maximal regularity. This property is inherited by operators which are lower order perturbations of operators with maximal regularity.

The set of all operators with continuous maximal regularity is denoted by $\mathcal{M}_\theta(E_1, E_0)$. For the solution of the Stefan problem we need the following existence and uniqueness theorem which is a version of [10, Theorem 3.1]

Theorem 3.3. *Let E_0 and E_1 be two Banach spaces such that $E_1 \hookrightarrow E_0$ and for $0 < \theta \leq 1$ let $V_\theta \subset \gamma \mathbb{E}_1^\theta(J)$ be an open neighbourhood of $u_0 \in V_\theta$. Assume in addition that $F \in \mathrm{Lip}(V_\theta, E_0)$ and $A \in \mathrm{Lip}(V_\theta, \mathcal{M}_\theta(E_1, E_0))$. Then there is $\tau > 0$, such that the quasilinear evolution equation*

$$\begin{cases} \dot{u} + A(u)u = F(u) & \text{on } J, \\ u(0) = u_0 \end{cases} \tag{3}$$

has a unique solution $u \in \mathbb{E}_1^\theta(J_\tau)$.

Proof. We define A_0 and $B = B(u)$ with $A_0 := A(u_0)$, $B(u) := A_0 - A(u)$. Then (3) can be rewritten as

$$\begin{cases} \dot{u} + A_0 u = B(u)u + F(u), & \text{on } J \\ u(0) = u_0, \end{cases}$$

which is a inhomogeneous Cauchy problem. Since $A(u_0) \in \mathcal{M}_\theta(E_1, E_0)$, the solution is given by

$$u = y_0 + J_{A_0}(B(u)u + F(u)) \tag{4}$$

where J_A is the operator $J_A(f)(t) := \int_0^t e^{(t-s)A} f(s)\, ds$ and $y_0(t) := e^{-tA_0}u_0$. The solutions of (4) are the fixed points of the mapping Φ,

$$\Phi(u) := y_0 + J_{A_0}(B(u)u + F(u)).$$

For a small $\varepsilon > 0$ which we determine later, we choose neighborhood $U \subset V_\theta$ of u_0, such that $\|B(u)\|_{\mathcal{L}(E_1,E_0)} \le \varepsilon$ for all $u \in U$. Next we introduce the subset W of $\mathbb{E}_1^\theta(J_\tau)$,

$$W := \{u \in \mathbb{E}_1^\theta(J_\tau) : u(0) = u_0, u(J_\tau) \subset U, \|u\|_{E_0} \le 2\|y_0\|_{\mathbb{E}_1^\theta(J_\tau)}\},$$

and claim that Φ is a contraction on W. There is a constant $C > 0$ such that

$$\|\Phi(u) - \Phi(v)\|_{\mathbb{E}_1^\theta(J_\tau)} \le C\{\|B(u)u - B(v)v\|_{\mathbb{E}_0^\theta(J_\tau)} + \|F(u) - F(v)\|_{\mathbb{E}_0^\theta(J_\tau)}\} \tag{5}$$

for all $u, v \in W$, which follows from the boundedness of the operator norm of J_{A_0}. For σ with $0 < \sigma < \theta$, one can show the embedding

$$\mathbb{E}_1^\theta(J) \hookrightarrow C^{0,\theta-\sigma}(J, \gamma\mathbb{E}_1^\sigma(J)).$$

Then for all $u, v \in W$

$$\|u(t) - v(t)\|_{\gamma\mathbb{E}_1^\sigma(J_\tau)} \le Ct^{\theta-\sigma}\|u - v\|_{\mathbb{E}_1^\theta(J_\tau)},$$

and it follows that

$$\begin{aligned} \|F(u) - F(v)\|_{\mathbb{E}_0^\theta(J_\tau)} &= \sup_{t\in\dot{J}_\tau} t^{1-\theta}\|F(u(t)) - F(v(t))\|_{E_0} \\ &\le M \sup_{t\in\dot{J}_\tau} t^{1-\theta}\|u(t) - v(t)\|_{\gamma\mathbb{E}_1^\sigma(J_\tau)} \\ &\le MC \sup_{t\in\dot{J}_\tau} t^{1-\theta}t^{\theta-\sigma}\|u - v\|_{\mathbb{E}_1^\theta(J_\tau)} \\ &= MC\tau^{1-\sigma}\|u - v\|_{\mathbb{E}_1^\theta(J_\tau)}. \end{aligned}$$

With the assumption $\|B(u)\|_{\mathcal{L}(E_1,E_0)} \leq \varepsilon$, the first term in (5) can be estimated as

$$\|B(u)u - B(v)v\|_{\mathbb{E}_0^\theta(J_\tau)} \leq \underbrace{\|B(u)(u-v)\|_{\mathbb{E}_0^\theta(J_\tau)}}_{I} + \underbrace{\|(B(u)-B(v))v\|_{\mathbb{E}_0^\theta(J_\tau)}}_{II}.$$

$$\begin{aligned} I &= \sup_{t\in\dot{J}_\tau} t^{1-\theta}\|B(u(t))(u(t)-v(t))\|_{E_0} \\ &\leq \sup_{t\in\dot{J}_\tau} t^{1-\theta}\|B(u)\|_{\mathcal{L}(E_1,E_0)}\|u(t)-v(t)\|_{E_0} \\ &\leq \varepsilon\|u-v\|_{\mathbb{E}_1^\theta(J_\tau)}. \end{aligned}$$

From the embedding it follows

$$\begin{aligned} II &= \sup_{t\in\dot{J}_\tau} t^{1-\theta}\|(B(u(t))-B(v(t)))v(t)\|_{E_0} \\ &\leq \sup_{t\in\dot{J}_\tau} t^{1-\theta}M\|u(t)-v(t)\|_{\gamma\mathbb{E}_1^\sigma(J_\tau)}\|v(t)\|_{E_0} \\ &\leq MC\sup_{t\in\dot{J}_\tau} t^{1-\theta}t^{\theta-\sigma}\|u-v\|_{\mathbb{E}_1^\theta(J_\tau)}\|v(t)\|_{\mathbb{E}_1^\theta(J_\tau)} \\ &\leq MC\tau^{\theta-\sigma}\|u-v\|_{\mathbb{E}_1^\theta(J_\tau)}. \end{aligned}$$

So we have

$$\|B(u)u - B(v)v\|_{\mathbb{E}_0^\theta(J_\tau)} \leq \varepsilon\|u-v\|_{\mathbb{E}_1^\theta(J_\tau)} + MC\tau^{\theta-\sigma}\|u-v\|_{\mathbb{E}_1^\theta(J_\tau)}.$$

With this and the previous estimate:

$$\|\Phi(u)-\Phi(v)\|_{\mathbb{E}_1^\theta(J_\tau)} \leq C\{\tau^{1-\sigma}\|u-v\|_{\mathbb{E}_1^\theta(J_\tau)} + \varepsilon\|u-v\|_{\mathbb{E}_1^\theta(J_\tau)} + \tau^{\theta-\sigma}\|u-v\|_{\mathbb{E}_1^\theta(J_\tau)}\}.$$

If we now choose ε and τ sufficiently small, say

$$\varepsilon < 1/2C, \text{ und } \tau < (1/2C)^{1/max\{1-\sigma,\theta-\sigma\}},$$

then Φ is indeed a contraction. Applying the Banach Fixed Point Theorem, Φ possesses a unique fixed point which is the solution of (4). Hence the evolution solution has also unique solution $u \in \mathbb{E}_1^\theta(J_\tau)$ and the proof is complete. □

4. Solution of the Stefan problem. We have shown that the Stefan problem can be reduced to a single quasilinear evolution equation (1). In order to apply the abstract existence theorem, we proceed to show that (1) satisfies all its conditions.

Theorem 4.1. *The function $F : \mathcal{U} \to C^{1,\alpha}(\Sigma)$ in the evolution equation*

$$\begin{aligned} \partial_t \rho + A(\rho)\rho &= F(\rho), \\ \rho(0) &= \rho_0 \end{aligned} \tag{6}$$

with

$$A(\rho) := L_\rho \mathcal{B}(\rho)\mathcal{S}(\rho)P(\rho)$$

and

$$F(\rho) := -L\rho\mathcal{B}(\rho)\mathcal{S}(\rho)Q(\rho)$$

is Lipschitz continuous. It holds

$$\|F(\rho_1) - F(\rho_2)\|_{C^{1,\alpha}(\Sigma)} \leq C\|\rho_1 - \rho_2\|_{C^{1,\alpha}(\Sigma)}.$$

P r o o f. We set $M(\rho) := L_\rho/\delta$ and recall that $\delta \geq c > 0$. Then $M(\rho)$ is Lipschitz continuous. Similarly $M(\rho)Q(\rho)$ and $M(\rho)\gamma\mathcal{S}(\rho)Q(\rho)$ are also Lipschitz continuous. Hence

$$\|M(\rho_1)Q(\rho_1) - M(\rho_2)Q(\rho_2)\|_{C^{1,\alpha}(\Sigma)} \leq C\|\rho_1 - \rho_2\|_{C^{1,\alpha}(\Sigma)}$$

and

$$\|M(\rho_1)\gamma\mathcal{S}(\rho_1)Q(\rho_1) - M(\rho_2)\gamma\mathcal{S}(\rho_2)Q(\rho_2)\|_{C^{1,\alpha}(\Sigma)} \leq C\|\rho_1 - \rho_2\|_{C^{1,\alpha}(\Sigma)}.$$

By Theorem 1.1

$$\begin{aligned} &\|F(\rho_1) - F(\rho_2)\|_{C^{1,\alpha}(\Sigma)} \\ &= \|L_{\rho_1}\mathcal{B}(\rho_1)\mathcal{S}(\rho_1)Q(\rho_1) - L_{\rho_2}\mathcal{B}(\rho_2)\mathcal{S}(\rho_2)Q(\rho_2)\|_{C^{1,\alpha}(\Sigma)} \\ &= \|M(\rho_1)Q(\rho_1) - M(\rho_2)Q(\rho_2) - M(\rho_1)\gamma\mathcal{S}(\rho_1)Q(\rho_1) + M(\rho_2)\gamma\mathcal{S}(\rho_2)Q(\rho_2)\|_{C^{1,\alpha}(\Sigma)} \\ &\leq C_1\|\rho_1 - \rho_2\|_{C^{1,\alpha}(\Sigma)} + C_2\|\rho_1 - \rho_2\|_{C^{1,\alpha}(\Sigma)} \\ &\leq C\|\rho_1 - \rho_2\|_{C^{1,\alpha}(\Sigma)}. \end{aligned}$$

□

Analogously one can show using $P \in C^\infty(\mathcal{U}, \mathcal{L}(C^{3,\alpha}(\Sigma), C^{1,\alpha}(\Sigma)))$,

Theorem 4.2. *The operator* $A(\rho) = \mathcal{B}(\rho)\mathcal{S}(\rho)P(\rho)$*,*

$$A : \mathcal{U} \to \mathcal{L}(C^{3,\alpha}(\Sigma), C^{1,\alpha}(\Sigma))$$

is Lipschitz continuous.

The maximal regularity of $A(\rho)$ follows from its representation as a lower order perturbation of $P(\rho)$ which we give next, and the Remark 3.2.

Theorem 4.3. *It holds*

$$A(\rho) = \frac{L_\rho}{\delta}P(\rho) - \frac{L_\rho}{\delta}\gamma\mathcal{S}(\rho)P(\rho)$$

Proof. Indeed, by Theorem 1.1

$$\begin{aligned} A(\rho) &= \frac{L_\rho}{\delta}[\delta\mathcal{B}(\rho)\mathcal{S}(\rho)P(\rho)] \\ &= \frac{L_\rho}{\delta}(I - \gamma\mathcal{S}(\rho))P(\rho) \\ &= \frac{L_\rho}{\delta}P(\rho) - \frac{L_\rho}{\delta}\gamma\mathcal{S}(\rho)P(\rho). \end{aligned}$$ □

Summarizing all the previous results we have reached our goal of solving the Stefan problem.

Theorem 4.4. *For any initial geometry* $\rho(0,.) = \Gamma_0$*,* $\rho(0,.) \in C^{3,\alpha}(\Sigma)$*,* $0 < \alpha < 1$*, the Stefan Problem has a unique local solution* (v, ρ) *on a sufficiently small time interval* $J_\tau = [0, \tau)$*, such that*

$$v \in C(J_\tau, C^{2,\alpha}(D))$$

and

$$\rho \in C(J_\tau; C^{3,\alpha}(\Sigma)) \cap C^1(J_\tau; C^{1,\alpha}(\Sigma)).$$

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