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EPW SEXTICS AND HILBERT SQUARES OF K3 SURFACES*

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Abstract. We prove that the Hilbert square $S^{[2]}$ of a very general primitively polarized K3 surface S of degree $d(n) = 2(4n^2 + 8n + 5)$, $n \geq 1$ is birational to a double Eisenbud–Popescu–Walter sextic. Our result implies a positive answer, in the case when r is even, to a conjecture of O'Grady: On the Hilbert square of a very general K3 surface of genus $r^2 + 2$, $r \geq 1$ there is an antisymplectic birational involution. We explicitly give this involution on $S^{[2]}$ in terms of the corresponding EPW polarization on it.

1. Introduction and motivations. O'Grady conjectured in [13] that on the Hilbert square of a K3 surface of genus $g = r^2 + 2, r \geq 0$ there exists an antisymplectic involution (see (4.3.3) in [13]).

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We show here the following theorem, that in particular implies that O'Grady conjecture is true in the case when r is even.

Main Theorem. *The Hilbert square of a very general $K3$ surface of degree $d(n) = 2(4n^2 + 8n + 5)$, $n \geq 1$ is birational to a double EPW sextic.*

Indeed, for $d(n) = 8n^2 + 16n + 10 = 2(4(n+1)^2 + 1)$, the genus of S is $g(n) = d(n)/2 + 1 = 4n^2 + 8n + 6 = (2n+2)^2 + 2$, and for $n \geq 1$, $r = 2n + 2$ covers all even numbers $r \geq 4$, and the antisymplectic birational involution is determined in terms of the EPW polarization (see Section 3).

Notice that while the case $r = 0$ is well known, and the case $r = 2$ is studied in detail by O'Grady (see e.g. §4.3 in [13]), in the cases of odd r very little is known: only the case $r = 1$ is studied in [5] and [8].

To show our main result we will follow O'Grady's study of the case $r = 2$, considering a double EPW sextic associated to a special K3 surface of degree 10, together with the methods used by Hassett in [10].

The proof of our main theorem is given in Section 3 while notations and basic facts and properties of double EPW sextics are recalled in Section 2.

2. Fano fourfolds X_{10}, EPW sextics and K3 surfaces.

2.1. Fano fourfolds X_{10}. By X_{10} we denote a prime Fano fourfold of index two and degree 10. By [12] and [9], any smooth X_{10} is either a complete intersection of the Grassmannian $G(2,5) = G(2, \mathbf{C}^5) \subset \mathbb{P}^9 = \mathbb{P}(\wedge^2 \mathbf{C}^5)$ with a hyperplane and a quadric (the 1-st, or the Mukai's type), or a double covering of the smooth Fano fourfold $W_5 = G(2,5) \cap \mathbb{P}^7$ branched along a quadratic section of W_5 (the 2-nd, or the Gushel's type). Both Mukai's and Gushel's types appear as complete intersections

$$CG(2,5) \cap \mathbb{P}^8 \cap Q \subset \mathbb{P}^{10} = \mathbb{P}(\mathbf{C} \oplus \wedge^2 \mathbf{C}^5)$$

of the cone $CG(2,5) \subset \mathbb{P}^{10}$ over the grassmannian $G(2,5)$ with a subspace $\mathbb{P}^8$ and a quadric Q in $\mathbb{P}^{10}$, and the two types differ by whether the vertex of the cone $CG(2,5)$ belongs to $\mathbb{P}^8$ (the Gushel's type) or not (the Mukai's type).

The moduli stack $\mathscr{X}_{10}$ of smooth X_{10} is of dimension 24, and the general X_{10} in $\mathscr{X}_{10}$ is from the first type. The condition for X_{10} to be of the second type is of codimension 2, and the general X_{10} of the second type is a smooth deformation from X_{10} of the first type.

Let X be a fourfold of type $\mathscr{X}_{10}$. By the Hodge–Riemann bilinear relations, the Hodge structure on $H^4(X, \mathbf{Z})$ has weight 2, and the intersection form on X endows the 4-th integral cohomology of X with a structure of the lattice $\Lambda = H^4(X, \mathbf{Z}) = I_{22,2}$, where $I_{22,2}$ denotes the lattice $22\langle 1\rangle \oplus 2\langle -1\rangle$.

By [6], the lattice $H^4(X, \mathbf{Z})$ contains the fixed rank two polarization sublattice $\Lambda_2 := H^4(G, \mathbf{Z})|_X$ spanned on the restrictions to X of the two Schubert cycles $\sigma_{1,1}$ and σ_2 on $G = G(2,5)$. In the basis $(u, v) = (\sigma_{1,1}|_X, \sigma_2|_X - \sigma_{1,1}|_X)$, the intersection form of the lattice

$$\Lambda_2 = H^4(G, \mathbf{Z})|_X = \mathbf{Z}u + \mathbf{Z}v$$

is given by

$$u^2 = v^2 = 2, \quad uv = 0.$$

For $X = X_{10}$, the primitive cohomology lattice with respect to the lattice polarization Λ_2, or the *vanishing cohomology lattice* is

$$\Lambda_0 = H^4(X, \mathbf{Z})_{\mathrm{van}} = \Lambda_2^{\perp} = 2E_8 \oplus 2U \oplus 2\langle 2\rangle,$$

ibid. Λ_0 is even of signature $(20, 2)$.

2.2. EPW sextics. Eisenbud–Popescu–Walter sextics, or in short EPW sextics, are special hypersurfaces of degree six in $\mathbb{P}^5$, first introduced in [7] as examples of Lagrangian degeneracy loci. These hypersurfaces are singular in codimension two, but O'Grady realized in [13] [14] that they admit smooth double covers which are irreducible holomorphic symplectic fourfolds. We will refer to this double covering as *double EPW sextic*. In fact, the first examples of such double covers were discovered by Mukai in [12], who constructed them as moduli spaces of stable rank two vector bundles on a polarized K3 surface of degree 10.

Moreover O'Grady showed in [14] that the generic such double cover is a deformation of the Hilbert square of a K3 and that the family of double EPW sextics is a locally versal family of projective deformations of such a Hilbert square of a $K3$ surface.

Let V be a 6-dimensional complex vector space and let us choose a volume-form on V

$$\mathrm{vol} : \wedge^6 V \to \mathbf{C}$$

and let us equip $\wedge^3 V$ with the symplectic form $(\alpha, \beta)_V := \mathrm{vol}(\alpha \wedge \beta)$.

Let $LG(\wedge^3 V)$ be the symplectic Grassmannian parametrizing Lagrangian subspaces of $\wedge^3 V$. Given a non-zero $v \in V$ let

$$F_v := \{\alpha \in \wedge^3 V | v \wedge \alpha = 0\}$$

be the sub-space of $\wedge^3 V$ consisting of multiples of v. The form $(\ ,\ )_V$ is zero on F_v and $\dim(F_v) = 10$, thus $F_v \in LG(\wedge^3 V)$. Let

$$F \subset \wedge^3 V \otimes \mathscr{O}_{\mathbb{P}(V)} \tag{1}$$

be the sub-vector-bundle with fiber F_v over $[v] \in \mathbb{P}(V)$. Then

$$\det F \cong \mathscr{O}_{\mathbb{P}(V)}(-6). \tag{2}$$

Given $A \in LG(\wedge^3 V)$ we let $Y_A = \{[v] \in \mathbb{P}(V) | F_v \cap A \neq \{0\}\}$. Thus Y_A is the degeneracy locus of the map $\lambda_A : F \to (\wedge^3 V/A) \otimes \mathscr{O}_{\mathbb{P}(V)}$ where λ_A is given by inclusion (1) followed by the quotient map

$$\wedge^3 V \otimes \mathscr{O}_{\mathbb{P}(V)} \to (\wedge^3 V/A) \otimes \mathscr{O}_{\mathbb{P}(V)}. \tag{3}$$

Since the vector bundles appearing in (3) have equal rank, the determinant of λ_A makes sense and Y_A is the zero-scheme of $\det \lambda_A$ in $\mathbb{P}(V)$; in particular Y_A has a natural structure of a closed subscheme of $\mathbb{P}(V)$. By (2) we have $\det \lambda_A \in H^0(\mathscr{O}_{\mathbb{P}(V)}(6))$, and hence Y_A is either a sextic hypersurface or $\mathbb{P}(V)$. An *EPW sextic* is a sextic hypersurface in $\mathbb{P}^5$ which is projectively equivalent to Y_A for some $A \in LG(\wedge^3 V)$, and a *double EPW sextic* is its associated double covering studied by O'Grady.

2.3. Necessary conditions and negative Pell's equations. Next we will look for necessary conditions to have a birational map between a Hilbert square of a K3 surface of degree d and a double EPW sextic. We will follow Mukai ([12]).

Proposition 2.1. *Let $\widetilde{Y} \to Y$ be a double EPW sextic which is smooth and birational to $S^{[2]}$ for a primitively polarized K3 surface S of degree $d = 2g-2 \geq 10$ and Picard number 1. Then the negative Pell's equation*

$$y^2 - (g-1)x^2 = -1$$

has an integer solution.

P r o o f. By a result of Mukai (see [12, Corollary 5.9]), if Y is birational to $S^{[2]}$, then there exists an isometry between the Neron–Severi lattices $NS(Y) \cong NS(S^{[2]})$. Recall that $NS(S^{[2]}) = \mathbf{Z}h + \mathbf{Z}\delta$, where $(h,h) = d = 2g-2$, $(h,\delta) = 0$, and $(\delta,\delta) = -2$.

Let $\pi : \widetilde{Y} \to Y$ be the double covering defined by the antisymplectic involution, as in [13], [14]. The EPW polarization γ on $\widetilde{Y}$ is the preimage of the hyperplane class on the EPW sextic $Y \subset \mathbb{P}^5$. Therefore the intersection index

$$\gamma^4 = \deg \pi \cdot \deg(Y) = 2 \cdot 6 = 12.$$

Since the double EPW sextic $\widetilde{Y}$ is a deformation of a Hilbert square of a K3 surface (see [14]) then the Fujiki constant $c(\widetilde{Y}) = c(S^{[2]}) = 3$, see (1.0.1) and (4.1.4) in [13]. Therefore for the Beauville form $(\cdot,\cdot)$ on $\mathrm{NS}(\widetilde{Y})$ one will have:

$$12 = \gamma^4 = c(\widetilde{Y})(\gamma,\gamma)^2 = 3(\gamma,\gamma)^2,$$

which yields

$$(\gamma,\gamma) = 2.$$

By the isometry $NS(\widetilde{Y}) \cong NS(S^{[2]})$ we can identify γ with an element of $NS(S^{[2]})$, i.e. the birationality of $\widetilde{Y}$ with the Hilbert square of a K3 surface as above implies that there exist integers x, y such that $\gamma = xh - y\delta$. Then

$$2 = (\gamma,\gamma) = (xh - y\delta, xh - y\delta) = dx^2 - 2y^2 = (2g-2)x^2 - 2y^2,$$

from where

$$y^2 - (g-1)x^2 = -1. \qquad \square$$

Remark 2.2. It is well known that if p is prime then the negative Pell's equation $y^2 - px^2 = -1$ has a solution if and only if $p = 2$ or $p \equiv 1 \pmod 4$, see e.g. Theorem 3.4.2 in [1].

Below we use the case when $p = 5$ which corresponds to a double EPW sextic birational to the Hilbert square of a K3 surface of degree 10, see §4.3 in [13]. For $p = 5$, the minimal solution of $y^2 - 5x^2 = -1$ is $(y,x) = (2,1)$. All solutions (y_n, x_n), $n \geq 0$ to $y^2 - 5x^2 = -1$ are given by

$$2y_n = (1 + 2\sqrt{5})(2 + \sqrt{5})^{2n} + (1 - 2\sqrt{5})(2 - \sqrt{5})^{2n},$$

$$2x_n = (2 + 1/\sqrt{5})(2 + \sqrt{5})^{2n} + (2 - 1/\sqrt{5})(2 - \sqrt{5})^{2n},$$

the minimal solution being $(2,1)$, see e.g. Theorem 3.4.1 on p.141 and the formulas on p.305 in [1].

3. Double EPW sextics and Hilbert squares of K3 surfaces.

We can now state main result of the paper, which is the following:

Main Theorem. *The Hilbert square of a very general $K3$ surface of degree $d = d(n) = 2(4n^2 + 8n + 5)$, $n \geq 1$ is birational to a double EPW sextic.*

The proof of the Main Theorem uses methods, similar to those used by Hassett in [10] to show a (stronger, in some sense) similar result for the variety of lines on a cubic fourfold. Our main observation is that the same approach can be used also in the case of double EPW sextics. We divide the proof into several parts:

3.1. The birational involution on $S^{[2]}$ for a K3 surface S of degree 10. For a K3 surface S with a polarization f of degree $f^2 = d = 2g-2$ and Picard number 1, any curve $C \in |f|$ defines a divisor $F_C = \{\xi \in S^{[2]} : \mathrm{Supp}(\xi) \cap C \neq \emptyset\}$ on $S^{[2]}$. All divisors F_C belong to the same class $f \in \mathrm{NS}(S^{[2]})$. We use the same notation for the class f and for the polarization f on S. The class of the diagonal $\Delta = \{\xi \in S^{[2]} : \mathrm{Supp}(\xi) = \text{point}\}$ is divisible by two in $\mathrm{NS}(S^{[2]})$, and if $\Delta = 2\delta$ then

$$\mathrm{NS}(S^{[2]}) = \mathbf{Z}f + \mathbf{Z}\delta.$$

If $(\cdot,\cdot)$ is the Beauville form on $\mathrm{NS}(S^{[2]})$, then

$$(f,f) = d,\ (f,\delta) = 0,\ (\delta,\delta) = -2.$$

If on S there is a polarization f of degree $d = 10$, then there exists a birational involution

$$j : S^{[2]} \to S^{[2]}.$$

For the general pair (x,y) of points on the general S the involution j can be described geometrically as follows (for more detail see [13]):

Let $G = G(2,5) = G(1 : \mathbb{P}^4) \subset \mathbb{P}^9$ be the grassmannian of lines in $\mathbb{P}^4$. By [12], the general smooth K3 surface S of degree 10 is a quadratic section $S = V_5 \cap Q$ of the unique smooth del Pezzo threefold $V_5 = G \cap \mathbb{P}^6$, which is a prime Fano threefold of index 2 and degree 5. By the general choice of $S \subset V_5$, the general non-ordered pair of points (x,y) on $S \subset V_5$ is a general pair of points on V_5. The del Pezzo threefold V_5 has the property that through the general pair of points on V_5 passes a unique conic $q = q_{x,y}$. Indeed, let l_x, l_y be the two lines in $\mathbb{P}^4$ representing the points $x, y \in V_5 \subset G = G(1 : \mathbb{P}^4)$. By the general choice of x, y, the lines l_x and l_y do not intersect each other and span a 3-space $\mathbb{P}^3_{x,y} \subset \mathbb{P}^4$. Any conic $q \subset G$ which passes through x and y lies in the Plücker quadric $G(2,4)_{x,y} = G(1 : \mathbb{P}^3_{x,y}) \subset G$. In addition, since $V_5 = G \cap \mathbb{P}^6$ then any conic on V_5 which passes through x and y lies on the codimension 3 subspace $\mathbb{P}^6 \subset \mathbb{P}^9 = \mathrm{Span}(G)$. Therefore the set of conics on V_5 which pass through x and y sweept out the intersection $q_{x,y} = G(2,4)_{x,y} \cap \mathbb{P}^6$, which by the general choice of x, y is a codimension 3 linear section of the 4-dimensional quadric $G(2,4)_{x,y}$, i.e. a conic. Since $S = V_5 \cap Q$ is a quadratic section of V_5, the conic $q_{x,y}$ intersects S at x, y and a pair of other 2 points x', y'. This defines a birational involution

$$j : S^{[2]} \longrightarrow S^{[2]},\quad j(x,y) = (x',y').$$

Let $r = f - 2\delta \in \mathrm{NS}(S^{[2]})$. Then

$$(r, r) = (f - 2\delta, f - 2\delta) = (f, f) + 4(\delta, \delta) = 2.$$

By Propositions 4.1 and 4.21 in [13], on $\mathrm{NS}(S^{[2]})$ the involution j is given by the reflection with respect to r

$$j : z \mapsto j(z) = -z + (z, r)r = -z + (z, f - 2\delta)(f - 2\delta).$$

We keep the same notation for the involution j on $S^{[2]}$ and the involution j on $\mathrm{NS}(S^{[2]})$. In particular,

$$\begin{aligned} j(f) &= -f + (f, f - 2\delta)(f - 2\delta) = -f + 10(f - 2\delta) = 9f - 20\delta, \\ j(\delta) &= -\delta + (\delta, f - 2\delta)(f - 2\delta) = -\delta + 4(f - 2\delta) = 4f - 9\delta. \end{aligned}$$

3.2. The Hilbert square of a K3 surface of degree 10 as a double EPW sextic. Let $S \subset V_5 \subset G = G(1 : \mathbb{P}^4)$ be a very general K3 surface with a polarization h of degree 10, where $V_5 = G \cap \mathbb{P}^6$ is as above. By [13], [14], the Hilbert square $S^{[2]}$ is a special case (as a birational equivalence class) of a double EPW sextic. The double covering is defined by the involution j on $S^{[2]}$, and can be described as follows.

Let $\mathbb{P}^5 = |I_S(2)|$ be the projective space of quadrics in $\mathbb{P}^6$ which contain $S \in |\mathscr{O}_{V_5}(2)|$. In $\mathbb{P}^5$, the quadrics which contain V_5 form a hyperplane identified with the space of Pfaffian quadrics. Let $\xi \in S^{[2]}$, and let $\mathbb{P}^1_\xi = \mathrm{Span}(\xi)$. Then ξ defines a hyperplane

$$\mathbb{P}^4_\xi = |I_{S \cup \mathbb{P}^1_\xi}(2)| \subset |I_S(2)| = \mathbb{P}^5.$$

If S does not contain lines, which is the general case, then the map

$$\pi : S^{[2]} \to \check{\mathbb{P}}^5, \ \xi \mapsto \mathbb{P}^4_\xi$$

is well defined for any $\xi \in S^{[2]}$. The map π is (generically) the double covering defined by the involution j. We shall show only that the images of two involutive elements by π coincide; for more detail see [13] and [12]. Indeed, if $j(\xi)$ is the involutive of ξ, then the lines $\mathbb{P}^1_\xi = \mathrm{Span}(\xi)$ and $\mathbb{P}^1_{j(\xi)} = \mathrm{Span}(j(\xi))$ intersect each other, since by construction of $j(\xi)$, $\xi + j(\xi)$ lie on a conic – see above. Since the lines $\mathbb{P}^1_\xi$ and $\mathbb{P}^1_{j(\xi)}$ are bisecant or tangent to S and intersect each other, any quadric which contains S together with one of these two lines contains also the other line. By the definition of π, the latter yields that the images $\pi(\xi)$ and $\pi(j(\xi))$ coincide.

By §4.3 in [13], the image $Y_0 \subset \check{\mathbb{P}}^5$ of the double covering π is an EPW sextic, defining the double EPW sextic

$$\widetilde{Y}_0 \to Y_0,$$

which is birational to the Hilbert square $S^{[2]}$, see also Theorem 4.15 in [15].

In the sequel we will also need the following result of O'Grady:

Lemma 3.1 (see Proposition 4.21 and Corollary 5.21 in [13])**.** *The class γ of the EPW polarization $\pi^*(\mathscr{O}_{Y_0}(1)) \in \mathrm{NS}(S^{[2]}) = \mathbf{Z}h + \mathbf{Z}\delta$ is $\gamma = h - 2\delta$.*

Remark 3.2. Here we assume that $NS(S) \cong \mathbf{Z}$ and denote by h the ample generator. The EPW-polarization $\gamma = xh - y\delta$ is j-invariant, i.e. $j(\gamma) = \gamma$, where

$$j : z \mapsto -z + (z, r)r$$

is the involution defined by $r = h - 2\delta$, which interchanges the two preimages of the general point $p \in Y_0$, see Subsection 3.2. The equality $\gamma = j(\gamma) = -\gamma + (r, \gamma)r$ yields $2\gamma = (r, \gamma)r$, i.e. γ is proportional to $r = h - 2\delta$. Since γ is primitive, i.e. not divisible by an integer, $\gamma = r$.

3.3. K3 surfaces with two polarizations of degree 10.

Lemma 3.3. *Let R be the rank two lattice $R = \mathbf{Z}f + \mathbf{Z}h$ with intersection form*

	f	h
f	10	$n+10$
h	$n+10$	10

where $n \geq 1$. Then there exists a K3 surface S with $NS(S) = \mathbf{Z}f + \mathbf{Z}h$, such that f and h are two very ample polarizations on S.

P r o o f. Let $\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ be the K3 cohomology lattice. By Theorem 2.4 in [11], there exists an embedding $R \subset \Lambda$. By the surjectivity of the period map for K3 surfaces one can assume that e.g. f is a very ample polarization on a K3 surface S. Since $(f, h) > 0$ then the divisor class h is effective, and one needs to see that h is very ample. If h is not ample then on S will exist a (-2)-curve E such that $(h, E) \leq 0$. If then $k = (h, E) = 0$ then $R_0 = \mathbf{Z}h + \mathbf{Z}E$ will be a sublattice of R of discriminant $d(R_0) = -20$. Since $R_0 \subset R$ then $d(R) = -n(n + 20)$ divides $d(R_0) = -20$, which is not possible. There remains the possibility when $(h, E) = -k < 0$. Since $E^2 = -2$, then E defines a reflection

$$r_E : x \mapsto \bar{x} = x + (x, E)E,$$

$x \in NS(S) \supset R$. In particular, $\bar{h} = h - kE$, $(\bar{h}, \bar{h}) = (h, h) = 10$, and $(f, \bar{h}) = (f, h - kE) = (f, h) - k(f, E) < (f, h)$ since f is (very) ample and E is effective. Since $\bar{h} \in R$ then $R' = \mathbf{Z}f + \mathbf{Z}\bar{h}$ is a sublattice of $R = \mathbf{Z}f + \mathbf{Z}h$. Therefore $d(R)$ divides $d(R')$, and since both $d(R)$ and $d(R')$ are negative, then $d(R') \leq d(R)$. But

$$d(R') = (f, f)(\bar{h}, \bar{h}) - (f, \bar{h})^2 =$$

$$= (f, f)(h, h) - (f, \bar{h})^2 > (f, f)(h, h) - (f, h)^2 = d(R),$$

contradiction. This proves the Lemma. For more detail see Lemma 4.3.3 and §6 in [10]. □

3.4. Proof of the Main Theorem. Let S be a very general K3 surface with a primitive polarization h of degree 10 as in Subsection 3.2. Denote by $\widetilde{Y}_0$ the EPW sextic, corresponding to $S^{[2]}$. Let $\widetilde{Y}_t$ be a local deformation of $\widetilde{Y}_0$ in the polarization $\gamma = h - 2\delta$ as a double EPW sextic $\pi_t : \widetilde{Y}_t \to Y_t$. Since $\widetilde{Y}_t$ is a deformation of a Hilbert square of a K3 surface, the Fujiki constant $c(\widetilde{Y}_t) = c(S^{[2]}) = 3$, and as in the proof of Proposition 2.1, we get $(\gamma, \gamma) = 2$.

Let S be a very general K3 surface with two polarizations f and h (generating the Neron–Severi lattice) as in Lemma 3.3. By above, e.g. in the polarization h, the Hilbert square $S^{[2]}$ is birational to a double EPW sextic $\widetilde{Y}_0$. By Proposition 2.2 and Theorem 4.15 in [15], $Y_0 = Y_A$, $A \in \Delta - \Sigma$ (ibid. (0.0.7)-(0.0.8)), and by Proposition 6.1 of [14] has a unique singular point p_0 of multiplicity three. The Hilbert square $S^{[2]} \to \widetilde{Y}_0$ is a small resolution of p_0 which is a contraction of a Lagrangian plane on $S^{[2]}$ to the point p_0.

Next, we proceed as in the proof of Theorem 6.1.4 in [10] for families of lines on cubic fourfolds, adapted to the case of double EPW sextics.

By [15] the period map for double EPW sextics extends regularly around the period point of $S^{[2]}$. By the surjectivity of the period map for K3 surfaces (see [11]), one can consider $h_2 = \gamma + (2n+2)\delta_2 \in \Pi$ as the quasi-polarization of a K3 surface of genus $g(n) = d(n)/2 + 1$ (see below for the definition of the lattice Π), with $\delta_2 = 4f - 9\delta \in \Pi$ the class of the half-diagonal on its Hilbert square, see also the proof of Proposition 7 in [4]. By Proposition 10, Theorem 6 and Remark 2 on p. 779–780 of [2] (see also Theorem 6.1.2 in [10]) in the 20-dimensional local moduli space $\mathscr{M}$ of double EPW sextics $\widetilde{Y}_t$ around $\widetilde{Y}_0$ the condition that $\delta_2 = 4f - 9\delta$ remains algebraic, i.e. an element of $NS(\widetilde{Y}_t)$, describes locally a smooth component of the divisor in $\mathscr{M}$ on which $\widetilde{Y}_t$ remains birational to a Hilbert square of a K3 surface S_t of genus $g(n)$.

For the general double EPW sextic $\widetilde{Y}_t$ as above, the lattice $\mathrm{NS}(\widetilde{Y}_t)$ has rank two, and is the saturation of the rank two sublattice

$$\Pi = \mathbf{Z}\gamma + \mathbf{Z}\delta_2 = \mathbf{Z}(h - 2\delta) + \mathbf{Z}(4f - 9\delta).$$

Since Π is saturated, then $\mathrm{NS}(\widetilde{Y}_t)$ coincides with Π, in particular the discriminant $d(\mathrm{NS}(\widetilde{Y}_t))$ is the discriminant of Π. By using $(\gamma,\gamma) = 2$ and $(\delta_2, \delta_2) = -2$, and the intersection table from Lemma 3.3, we compute

$$(\gamma, \delta_2) = (h - 2\delta, 4f - 9\delta) = 4(h, f) + 18(\delta, \delta) = 4(n + 10) - 36 = 4n + 4.$$

Therefore

$$d(\mathrm{NS}(\widetilde{Y}_t)) = d(\Pi) = \det \begin{pmatrix} (\gamma,\gamma) & (\gamma,\delta_2) \\ (\gamma,\delta_2) & (\delta_2,\delta_2) \end{pmatrix} =$$
$$= (\gamma,\gamma)(\delta_2,\delta_2) - (\gamma,\delta_2)^2 = 2(-2) - (4n+4)^2 =$$
$$= (-2)(8n^2 + 16n + 10).$$

Therefore $\widetilde{Y}_t$ is birational to the Hilbert square of a K3 surface S_t of degree

$$d(n) = 2g(n) - 2 = 8n^2 + 16n + 10 = 2(4(n + 1)^2 + 1).$$

This proves the Main Theorem. □

Remark 3.4. Let $NS(S_t^{[2]}) = \mathbf{Z}h_2 + \mathbf{Z}\delta_2$, where h_2 is the primitive polarization class on $S_t^{[2]}$. By using that

$$NS(S_t^{[2]}) \cong NS(\widetilde{Y}_t) \cong \Pi,$$

we can compute directly the degree $d(n) = (h_2, h_2)$ of the K3 surface S_t. Since h_2 is primitive and orthogonal to the half-diagonal class δ_2, and since

$$\Pi \cap \delta_2^{\perp} = \mathbf{Z}(\gamma + (2n + 2)\delta_2),$$

then $h_2 = \gamma + (2n + 2)\delta_2$. From here, and by the intersection table from Lemma 3.3, we get again

$$d(n) = (h_2, h_2) = (\gamma + (2n + 2)\delta_2, \gamma + (2n + 2)\delta_2) =$$
$$= (\gamma,\gamma) + 2(2n + 2)(\gamma,\delta_2) + (2n + 2)^2(\delta_2,\delta_2) =$$
$$= 2 + 2(2n + 2)(4n + 4) + (2n + 2)^2(-2) =$$
$$= 2 + 8(n + 1)^2 = 8n^2 + 16n + 10.$$

The intersection matrix of Π in the base h_2, δ_2, is

	h_2	δ_2
h_2	$d(n)$	0
δ_2	0	$-2.$

Remark 3.5. The Main Theorem implies that on the Hilbert square $S^{[2]}$ of a general K3 surface S of degree $d(n) = 8n^2 + 16n + 10$, $n \geq 1$ the EPW polarization $\gamma = h_2 - (2n+2)\delta_2$ defines an antisymplectic birational involution.

This proves the O'Grady conjecture that on the Hilbert square of a K3 surface of genus $g = r^2 + 2, r \geq 0$ there exists an antisymplectic involution (see (4.3.3) in [13]), in the case when r is even. Indeed, for $d(n) = 8n^2 + 16n + 10 = 2(4(n+1)^2+1)$, the genus of S is $g(n) = d(n)/2+1 = 4n^2+8n+6 = (2n+2)^2+2$, and for $n \geq 1$, $r = 2n+2$ covers all even numbers $r \geq 4$. The case $r = 0$ is well known, and the case $r = 2$ is studied in detail by O'Grady, see e.g. §4.3 in [13]. The odd case $r = 1$ is studied in [5] and [8].

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