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# SUMMABILITY METHODS IN WEIGHTED APPROXIMATION TO DERIVATIVES OF FUNCTIONS

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Abstract. In this paper, we use summability methods on the approximation to derivatives of functions by a family of linear operators acting on weighted spaces. This point of view enables us to overcome the lack of ordinary convergence in the approximation. To support this idea, at the end of the paper, we will give a sequence of positive linear operators obeying the arithmetic mean approximation (or, approximation with respect to the Cesàro method) although it is impossible in the usual sense. Some graphical illustrations are also provided.

**Introduction.** Approximation to functions and their derivatives by a family of linear operators acting on weighted spaces was studied by Èfendiev [10], whose results improve the classical Korovkin theory (see [2, 15]) based on positive linear operators defined on the space of all continuous and real-valued

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functions on a compact subset of the real line. In this paper, we investigate Èfendiev's results in the framework of summability theory. More precisely, we obtain various weighted approximation theorems in the sense of summability process. This point of view enables us to overcome the lack of the ordinary convergence of approximating operators. For example, by using our results it is possible to construct a class of linear operators approximating in the Cesàro sense a given function and its derivatives (but not in the usual sense). We mainly use the $\mathcal{A}$-summability method (process) introduced by Bell [8].

Let a sequence $\mathcal{A} = \{A^v\} = \{[a^v_{nk}]\}$ $(k, n, v \in \mathbb{N})$ be a non-negative regular summability method, that is, if

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a^v_{nk} x_k = L, \text{ uniformly in } v \in \mathbb{N}, \tag{1.1}$$

for any sequence $x = \{x_k\}$ converging to a number $L$. Recall that a sequence $x$ is called $\mathcal{A}$-summable to $L$ if (1.1) holds. A characterization on the regularity of $\mathcal{A}$ is similar to the well-known Silverman-Toeplitz conditions. We should note that this general summability method $\mathcal{A}$ contains many well-known (regular) convergence methods. For example, the case of $\mathcal{A} = \{A^v\} = \{I\}$, the identity matrix, for each $v \in \mathbb{N}$, coincides with the ordinary convergence in Cauchy's sense. Another interesting method is the almost convergence introduced by Lorentz [16], which is the case of $\mathcal{A} = \mathcal{F} = \{[c^v_{nk}]\}$ defined by

$$c^v_{nk} := \begin{cases} \dfrac{1}{n}, & \text{if } v \le k \le n + v - 1 \\ 0, & \text{otherwise,} \end{cases} \tag{1.2}$$

Furthermore, the Cesàro convergence method (or, the arithmetic mean convergence), which is the case of $\mathcal{A} = \{C_1\} = \{[c^1_{nk}]\}$ is easily obtained from (1.2) by taking $v = 1$. Bell's summability method also includes the order summability introduced by Jurkat and Peyerimhoff [13, 14].

So far, Bell's summability method has been used in several papers (see, for instance, [1, 5, 6, 12, 17, 18, 19, 20]). In the present paper we use it in the approximation to derivatives of functions by a family of linear operators acting on weighted spaces.

**2. Weighted approximation theorems via summability methods.** In this section, we approximate (in the sense of summability process) a function and its derivatives by using sequences of linear operators acting on weighted spaces.

We mainly consider the following class of functions as in the paper by Èfendiev [10]. Let $r$ be a non-negative integer. As usual, the space of all functions with $r$-th continuous derivatives on $\mathbb{R}$ is denoted by $C^{(r)}(\mathbb{R})$. Let $\rho : \mathbb{R} \to [1, +\infty)$ be a weight function, that, it satisfies the following conditions: $(a)$ $\rho(0) = 1$; $(b)$ it is increasing on $(0, +\infty)$ and decreasing on $(-\infty, 0)$; $(c)$ $\lim\limits_{x \to \pm\infty} \rho(x) = +\infty$. Then, the corresponding weighted spaces are given as follows:

$$\begin{aligned}
C_{\rho}^{(r)}(\mathbb{R}) &= \left\{ f \in C^{(r)}(\mathbb{R}) : \left| f^{(r)}(x) \right| \le m_f \rho(x),\ \exists m_f > 0,\ \forall x \in \mathbb{R} \right\}, \\
\tilde{C}_{\rho}^{(r)}(\mathbb{R}) &= \left\{ f \in C_{\rho}^{(r)}(\mathbb{R}) :\ \lim_{x \to \pm\infty} f^{(r)}(x)/\rho(x) = k_f,\ \exists\ k_f \right\}, \\
\hat{C}_{\rho}^{(r)}(\mathbb{R}) &= \left\{ f \in \tilde{C}_{\rho}^{(r)}(\mathbb{R}) : \lim_{x \to \pm\infty} f^{(r)}(x)/\rho(x) = 0 \right\}, \\
B_{\rho}(\mathbb{R}) &= \left\{ g : \mathbb{R} \to \mathbb{R} :\ \ |g(x)| \le m_g \rho(x),\ \exists m_g > 0,\ \forall x \in \mathbb{R} \right\}.
\end{aligned}$$

Also, by $M^{(r)}(\mathbb{R})$ we denote the class of linear operators $L$ acting on the above weighted spaces such that $L(f) \ge 0$ whenever $f^{(r)} \ge 0$ on $\mathbb{R}$. Observe that the case of $r = 0$ coincides with the family of positive linear operators. If $r = 0$, we just write $C_{\rho}(\mathbb{R})$, $\tilde{C}_{\rho}(\mathbb{R})$, $\hat{C}_{\rho}(\mathbb{R})$, $M(\mathbb{R})$. The weighted norm on $B_{\rho}(\mathbb{R})$ is defined by $\|g\|_{\rho} = \sup\limits_{x \in \mathbb{R}} |g(x)| / \rho(x)$ for $g \in B_{\rho}(\mathbb{R})$.

We now recall the next definition.

**Definition 2.1** ([15]). *We say that the system of continuous functions*

$$\{f_0\, , f_1, \cdots , f_n\}$$

*on an interval* $[a, b]$ *is* $T$*-system provided that any polynomial* $P(x) = a_0 f_0(x) + a_1 f_1(x) + \cdots + a_n f_n(x)$ *has not more than* $n$ *zeros in this interval for which the coefficients* $a_i$ $(0 \le i \le n)$ *are not all equal to zero.*

Let $\mathcal{A} = \{[a_{nk}^{\upsilon}]\}$ $(k, n, \upsilon \in \mathbb{N})$ be a non-negative regular summability method, and let the $\{L_k\}$ be a sequence of operators acting on weighted spaces. Throughout the paper we say that, for a given function $f$ in an appropriate weighted space, the sequence $\{L_k(f)\}$ is (uniformly) $\mathcal{A}$-summable to $f$ with respect to the weighted norm if the following limit holds:

$$\lim_{n \to \infty} \left\| \sum_{k=1}^{\infty} a_{nk}^{\upsilon} L_k(f) - f \right\|_{\rho} = 0, \text{ uniformly in } \upsilon \in \mathbb{N}. \tag{2.1}$$

Throughout the paper we assume that $\mathcal{A}$ is regular and the series $\sum_{k=1}^{\infty} a_{nk}^{v} L_k(f)$ in (2.1) converges for each $n, v \in \mathbb{N}$.

It is well-known that the test functions $e_i(x) = x^i$ $(i = 0, 1, 2)$ of the classical Korovkin theorem can be replaced with the $T$-system $\{f_0, f_1, f_2\}$ on an interval $[a, b]$ (see Theorem 8 of [15]). The statistical analog of this result may be found in [9]. Furthermore, Swetits [20] improved the classical Korovkin theorem on the interval $[a, b]$ via summability methods in (2.1). The $T$-system version of Swetits' result has recently been obtained by Atlihan and Tas [7] as follows:

**Theorem 2.2.** *([7]) Let $\mathcal{A} = \{[a_{nk}^{v}]\}$ $(k, n, v \in \mathbb{N})$ be a non-negative regular summability method, and let $\{L_k\}$ be a sequence of positive linear operators from $C[a, b]$ into itself. If, for each $i = 0, 1, 2$, $\{L_k(f_i)\}$ is (uniformly) $\mathcal{A}$-summable to $f_i$ on $[a, b]$, where $\{f_0, f_1, f_2\}$ is $T$-system on $[a, b]$, then, for all $f \in C[a, b]$, the sequence $\{L_k(f)\}$ is (uniformly) $\mathcal{A}$-summable to $f$ with respect to the classical sup-norm on $C[a, b]$.*

Now, we are ready to give our approximation results on weighted spaces. We begin with the case of $r = 0$.

**Theorem 2.3.** *Let $\mathcal{A} = \{[a_{nk}^{v}]\}$ $(k, n, v \in \mathbb{N})$ be a non-negative regular summability method, and let the operators $L_k : C_\rho(\mathbb{R}) \to B_\rho(\mathbb{R})$ belong the class $M(\mathbb{R})$, that is, the class of positive linear operators. Assume that $\{f_0, f_1\}$ and $\{f_0, f_1, f_2\}$ are $T$-systems on $\mathbb{R}$. Assume further that the following conditions*

$$\lim_{x \to \pm\infty} \frac{f_i(x)}{1 + |f_2(x)|} = 0 \quad (i = 0, 1), \tag{2.2}$$

$$\lim_{x \to \pm\infty} \frac{f_2(x)}{\rho(x)} = m_{f_2} \neq 0 \tag{2.3}$$

*hold. If, for each $i = 0, 1, 2$, $\{L_k(f_i)\}$ is (uniformly) $\mathcal{A}$-summable to $f_i$ with respect to the weighted norm on $B_\rho(\mathbb{R})$, then, for all $f \in \tilde{C}_\rho(\mathbb{R})$, the sequence $\{L_k(f)\}$ is (uniformly) $\mathcal{A}$-summable to $f$ with respect to the weighted norm on $B_\rho(\mathbb{R})$.*

P r o o f. By the proof of Theorem 1 in the paper by Èfendiev [10] (see also Theorem 2.2 in [3]), for a given function $f \in \tilde{C}_\rho(\mathbb{R})$, the function $g$ on $\mathbb{R}$ defined by $g(y) := m_{f_2} f(y) - k_f f_2(y)$ belongs to $\hat{C}_\rho(\mathbb{R})$, where the constants $k_f$ and $m_{f_2}$ are the same as in the definitions of corresponding weighted spaces. Now, following Lemma 2 in [10], we get from (2.2), (2.3) and $\{f_0, f_1\}$ being $T$-system that, for a given $\varepsilon > 0$ and a fixed positive constant $s_0$, there exists a positive constant $u_0$ such that

$$|g(y)| < \frac{M}{m_a}\Phi_a(y) + s_0\varepsilon f_2(y) \quad \text{for all } y \in \mathbb{R}, \tag{2.4}$$

where $\Phi_a$ ($a$ being a fixed real number satisfying $f_i(a) \neq 0$, $i = 0, 1$) is a function such that $\Phi_a(a) = 0$ and $\Phi_a(y) > 0$ for $y < a$ and that $\Phi_a(y) = \gamma_0(a)f_0(y) + \gamma_1(a)f_1(y)$, where $|\gamma_0(a)| = |f_1(a)/f_2(a)|$ and $|\gamma_1(a)| = 1$. In (2.4), the constants $M$ and $m_a$ are given, respectively, by $M := \sup\limits_{|y|\leq u_0} |g(y)|$ and $m_a := \min\limits_{|y|\leq u_0} \Phi a(y)$. Then, using (2.4) we observe that

$$\begin{aligned}
\left|\sum_{k=1}^{\infty} a_{nk}^{v} L_k(g; x)\right| &\leq \sum_{k=1}^{\infty} a_{nk}^{v} L_k(|g(y)|; x) \\
&\leq \sum_{k=1}^{\infty} a_{nk}^{v} \left\{\frac{M}{m_a} L_k(\Phi_a; x) + \varepsilon s_0 L_k(f_2; x)\right\} \\
&= \frac{M}{m_a}\left\{\gamma_0(a)\sum_{k=1}^{\infty} a_{nk}^{v} L_k(f_0; x) \ + \gamma_1(a)\sum_{k=1}^{\infty} a_{nk}^{v} L_k(f_1; x)\right\} \\
&\quad + \varepsilon s_0 \sum_{k=1}^{\infty} a_{nk}^{v} L_k(f_2; x)
\end{aligned}$$

holds for each $n, v \in \mathbb{N}$. Using the last inequality, we get

$$\begin{aligned}
\left|\sum_{k=1}^{\infty} a_{nk}^{v} L_k(g; x)\right| \leq{}& \frac{M}{m_a}|\gamma_0(a)|\left|\sum_{k=1}^{\infty} a_{nk}^{v} L_k(f_0; x) \ - f_0(x)\right| \\
&+ \frac{M}{m_a}\left|\sum_{k=1}^{\infty} a_{nk}^{v} L_k(f_1; x) \ - f_1(x)\right| \\
&+ \varepsilon s_0\left|\sum_{k=1}^{\infty} a_{nk}^{v} L_k(f_2; x) \ - f_2(x)\right| \\
&+ \frac{M}{m_a}|\Phi_a(x)| + \varepsilon s_0 |f_2(x)|.
\end{aligned}$$

On the other hand, we have

$$\left\|\sum_{k=1}^{\infty} a_{nk}^{v} L_k(g) - g\right\|_{\rho} \leq \sup_{|x|\leq u_0} \frac{1}{\rho(x)}\left|\sum_{k=1}^{\infty} a_{nk}^{v} L_k(g; x) - g(x)\right|$$

$$+\sup_{|x|>u_0}\frac{1}{\rho(x)}\left|\sum_{k=1}^{\infty}a_{nk}^{\upsilon}L_k(g;x)\right|+\sup_{|x|>u_0}\frac{|g(x)|}{\rho(x)}.$$

Then, combining the above inequalities, we see that

$$\begin{aligned}\left\|\sum_{k=1}^{\infty}a_{nk}^{\upsilon}L_k(g)-g\right\|_{\rho} &\leq \left\|\sum_{k=1}^{\infty}a_{nk}^{\upsilon}L_k(g)-g\right\|_{[-u_0,u_0]}\\ &+\frac{M}{m_a}\left|\gamma_0(a)\right|\left\|\sum_{k=1}^{\infty}a_{nk}^{\upsilon}L_k(f_0)-f_0\right\|_{\rho}\\ &+\frac{M}{m_a}\left\|\sum_{k=1}^{\infty}a_{nk}^{\upsilon}L_k(f_1)-f_1\right\|_{\rho}\\ &+\varepsilon s_0\left\|\sum_{k=1}^{\infty}a_{nk}^{\upsilon}L_k(f_2)-f_2\right\|_{\rho}\\ &+\frac{M}{m_a}\sup_{|x|>u_0}\frac{|\Phi_a(x)|}{\rho(x)}+\varepsilon s_0\sup_{|x|>u_0}\frac{|f_2(x)|}{\rho(x)},\\ &+\sup_{|x|>u_0}\frac{|g(x)|}{\rho(x)}\end{aligned}\tag{2.5}$$

where $\|\cdot\|_{[-u_0,u_0]}$ denotes the usual sup-norm on $C[-u_0,u_0]$. Since, for $i=0,1,2$,

$$\left\|\sum_{k=1}^{\infty}a_{nk}^{\upsilon}L_k(f_i)-f_i\right\|_{[-u_0,u_0]}\leq K\left\|\sum_{k=1}^{\infty}a_{nk}^{\upsilon}L_k(f_i)-f_i\right\|_{\rho},$$

where $K:=\|\rho\|_{[-u_0,u_0]}$, the hypothesis implies that the sequence $\{L_k(f_i)\}$ $(i=0,1,2)$ is (uniformly) $\mathcal{A}$-summable to $f_i$ with respect to the sup-norm on $C[-u_0,u_0]$. Hence, from Theorem 2.2, we obtain that $\{L_k(g)\}$ $(i=0,1,2)$ is (uniformly) $\mathcal{A}$-summable to $g$ with respect to the sup-norm since $g\in C[-u_0,u_0]$. Therefore, all terms of the left hand-side of (2.5) tends to zero as $n\to\infty$, uniformly in $\upsilon\in\mathbb{N}$, which implies that

$$\lim_{n\to\infty}\left\|\sum_{k=1}^{\infty}a_{nk}^{\upsilon}L_k(g)-g\right\|_{\rho}=0,\text{ uniformly in }\upsilon\in\mathbb{N}.\tag{2.6}$$

Now, by the regularity of $\mathcal{A}$ and the linearity of $L_k$, it follows from the definition of $g$ that

$$f(y)=\frac{1}{m_{f_2}}g(y)+\frac{k_f}{m_{f_2}}f_2(y),$$

which implies, for each $n, \upsilon \in \mathbb{N}$, that

$$\left\|\sum_{k=1}^{\infty} a_{nk}^{\upsilon} L_k(f) - f\right\|_{\rho} \leq \frac{1}{m_{f_2}} \left\|\sum_{k=1}^{\infty} a_{nk}^{\upsilon} L_k(g) - g\right\|_{\rho} + \frac{k_f}{m_{f_2}} \left\|\sum_{k=1}^{\infty} a_{nk}^{\upsilon} L_k(f_2) - f_2\right\|_{\rho}.$$

Taking limit as $n \to \infty$ (uniformly in $\upsilon \in \mathbb{N}$) and also considering (2.6) and the hypothesis, the proof is completed. □

Notice that if $m_{f_2} = 0$ in (2.3), then we can approximate to functions in $\hat{C}_{\rho}(\mathbb{R})$ instead of the ones in $\tilde{C}_{\rho}(\mathbb{R})$.

Now, for $r = 1, 2, \ldots$, we approximate to the $r$-th order derivative of a function $f$ in $\tilde{C}_{\rho}^{(r)}(\mathbb{R})$.

**Theorem 2.4.** *Let $\mathcal{A} = \{[a_{nk}^{\upsilon}]\}$ $(k, n, \upsilon \in \mathbb{N})$ be a non-negative regular summability method, and let the operators $L_k : C_{\rho}^{(r)}(\mathbb{R}) \to B_{\rho}(\mathbb{R})$ belong the class $M^{(r)}(\mathbb{R})$. Assume that $f_0, f_1, f_2 \in C_{\rho}^{(r)}(\mathbb{R})$ and that $\{f_0^{(r)}, f_1^{(r)}\}$ , $\{f_0^{(r)}, f_1^{(r)}, f_2^{(r)}\}$ are $T$-systems on $\mathbb{R}$. Assume further that the following conditions*

$$\lim_{x \to \pm\infty} \frac{f_i^{(r)}(x)}{1 + \left|f_2^{(r)}(x)\right|} = 0 \quad (i = 0, 1), \tag{2.7}$$

$$\lim_{x \to \pm\infty} \frac{f_2^{(r)}(x)}{\rho(x)} = m_{f_2}^{(r)} \neq 0 \tag{2.8}$$

*hold. If, for each $i = 0, 1, 2$, $\{L_k(f_i)\}$ is (uniformly) $\mathcal{A}$-summable to $f_i^{(r)}$ with respect to the weighted norm on $B_{\rho}(\mathbb{R})$, then, for all $f \in \tilde{C}_{\rho}^{(r)}(\mathbb{R})$, the sequence $\{L_k(f)\}$ is (uniformly) $\mathcal{A}$-summable to $f^{(r)}$ with respect to the weighted norm on $B_{\rho}(\mathbb{R})$.*

Proof. As in the proof of Theorem 1 in [10] (see also [3]), if we consider the operators $L_k^* := L_k \circ D^{-r}$, where $D^{-r}$ denotes the $r$-th inverse derivative operator, then it follows from the hypothesis on $L_k$ that, for each $k \in \mathbb{N}$, $L_k^*$ is a positive linear operator mapping $C_{\rho}(\mathbb{R})$ into $B_{\rho}(\mathbb{R})$ such that $L_k^*(f^{(r)}) = L_k(D^{-r}(f^{(r)})$ for $f \in C_{\rho}^{(r)}(\mathbb{R})$. Since, for each $i = 0, 1, 2$, $\{L_k(f_i)\}$ is (uniformly) $\mathcal{A}$-summable to $f_i^{(r)}$, we may write that $\{L_k^*(\psi_i)\}$ is (uniformly) $\mathcal{A}$-summable to $\psi_i$ with respect to the weighted norm on $B_{\rho}(\mathbb{R})$, where $\psi_i := f_i^{(r)}$ $(i = 0, 1, 2)$. In this case, from (2.7), (2.8), we immediately see that conditions (2.2) and (2.3) are valid for the $T$-systems $\{\psi_0, \psi_1\}$ and $\{\psi_0, \psi_1, \psi_2\}$. Thus, Theorem 2.3 implies that, for all $\psi \in \tilde{C}_\rho(\mathbb{R})$, the sequence $\{L_k^*(\psi)\}$ is (uniformly) $\mathcal{A}$-summable to $\psi$ with respect to the weighted norm on $B_\rho(\mathbb{R})$. For a given $f \in \tilde{C}_\rho^{(r)}(\mathbb{R})$, taking $\psi = f^{(r)}$, the proof follows immediately. $\square$

In general, the concept of $A$-statistical convergence (see [11]) and $\mathcal{A}$-summability method cannot be comparable. So, our results are completely different the ones in [3, 4]. However, the next result provides an interesting connection between them.

**Corollary 2.5.** *Let, for each $\upsilon \in \mathbb{N}$, $\mathcal{A} = \{A^\upsilon\} = \{[a_{nk}]\}$ be a non-negative regular summability matrix, let $\{L_k\}$ be a uniformly bounded sequence of operators from $C_\rho^{(r)}(\mathbb{R})$ into $B_\rho(\mathbb{R})$ belonging to the class $M^{(r)}(\mathbb{R})$. Assume that $f_0^{(r)}$, $f_1^{(r)}$, $f_2^{(r)}$ satisfy the conditions of Theorem 2.4. If, for each $i = 0, 1, 2$,*

$$st_A - \lim_k \left\| L_k(f_i) - f_i^{(r)} \right\|_\rho = 0, \tag{2.9}$$

*where $st_A - \lim$ denotes the $A$-statistical limit, then, for all $f \in \tilde{C}_\rho^{(r)}(\mathbb{R})$, the sequence $\{L_k(f)\}$ is (uniformly) $\mathcal{A}$-summable to $f^{(r)}$ with respect to the weighted norm on $B_\rho(\mathbb{R})$.*

Proof. From (2.9) we may write that, for a given $\varepsilon > 0$, $\lim_{n\to\infty} \sum_{k\in K_i(\varepsilon)} a_{nk} = 0$, where $K_i(\varepsilon) := \left\{ k \in \mathbb{N} : \left\| L_k(f_i) - f_i^{(r)} \right\|_\rho \geq \varepsilon \right\}$ $(i = 0, 1, 2)$. Also, by the uniform boundedness of $\{L_k\}$ we get, for each $n \in \mathbb{N}$, that

$$\begin{aligned}
\sum_{k=1}^{\infty} a_{nk} \left\| L_k(f_i) - f_i^{(r)} \right\|_\rho &\leq \sum_{k\in K_i(\varepsilon)} a_{nk} \left\| L_k(f_i) - f_i^{(r)} \right\|_\rho \\
&\quad + \sum_{k\in \mathbb{N}\setminus K_i(\varepsilon)} a_{nk} \left\| L_k(f_i) - f_i^{(r)} \right\|_\rho \\
&\leq (MD_i + E_i) \sum_{k\in K_i(\varepsilon)} a_{nk} + \varepsilon \sum_{k=1}^{\infty} a_{nk},
\end{aligned}$$

where $M := \|L_k\|_{C_\rho(R)\to B_\rho(R)} = \sup_{\|f\|_\rho = 1} \|L_k(f)\|_\rho$, $D_i := \|f_i\|_\rho$ and $E_i := \left\| f_i^{(r)} \right\|_\rho$ $(i = 0, 1, 2)$. Then, the regularity of $A$ implies that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} \left\| L_k(f_i) - f_i^{(r)} \right\|_\rho = 0.$$

Now using the fact that

$$\left\|\sum_{k=1}^{\infty} a_{nk} L_k(f_i) - f_i^{(r)}\right\|_{\rho} \leq \sum_{k=1}^{\infty} a_{nk} \left\|L_k(f_i) - f_i^{(r)}\right\|_{\rho} + C_i \left|\sum_{k=1}^{\infty} a_{nk} - 1\right|,$$

where $C_i := \left\|f_i^{(r)}\right\|_{\rho}$, we immediately see that

$$\lim_{n\to\infty} \left\|\sum_{k=1}^{\infty} a_{nk} L_k(f_i) - f_i^{(r)}\right\|_{\rho} = 0 \quad \text{for each } i = 0, 1, 2,$$

which means the sequence $\{L_k(f_i)\}$ is (uniformly) $A$-summable to $f_i$ for each $i = 0, 1, 2$ with respect to the weighted norm on $B_\rho(\mathbb{R})$. Hence, the proof follows from Theorem 2.4. □

In order to get an approximation on the space $C_\rho^{(r)}(\mathbb{R})$ we need a new weight function as follows:

**Theorem 2.6.** *Let $\mathcal{A} = \{[a_{nk}^{v}]\}$ $(k, n, v \in \mathbb{N})$ be a non-negative regular summability method, and let the operators $L_k : C_\rho^{(r)}(\mathbb{R}) \to B_{\rho_1}(\mathbb{R})$ belong the class $M^{(r)}(\mathbb{R})$. Assume that $f_0^{(r)}$, $f_1^{(r)}$, $f_2^{(r)}$ satisfy the conditions of Theorem 2.4. Let $\rho_1 : \mathbb{R} \to [1, \infty)$ be a weight function. Assume further that*

$$\lim_{x\to\pm\infty} \frac{\rho(x)}{\rho_1(x)} = 0 \tag{2.10}$$

*and*

$$\lim_{x\to\pm\infty} \frac{f_2^{(r)}}{\rho_1(x)} = m_{f_2}^{(r)} \neq 0 \tag{2.11}$$

*holds. If, for each $i = 0, 1, 2$, $\{L_k(f_i)\}$ is (uniformly) $\mathcal{A}$-summable to $f_i^{(r)}$ with respect to the weighted norm on $B_\rho(\mathbb{R})$, then, for all $f \in C_\rho^{(r)}(\mathbb{R})$, the sequence $\{L_k(f)\}$ is (uniformly) $\mathcal{A}$-summable to $f^{(r)}$ with respect to the weighted norm on $B_{\rho_1}(\mathbb{R})$.*

Proof. It is easy to check that $C_\rho^{(r)}(\mathbb{R}) \subset \hat{C}_{\rho_1}^{(r)}(\mathbb{R}) \subset \tilde{C}_{\rho_1}^{(r)}(\mathbb{R})$. Also, we get from (2.10) and (2.11) that all conditions of Theorem 2.4 are valid for the weight function $\rho_1$. Thus, the proof is a direct consequence of Theorem 2.4. □

Specializing Theorem 2.6 (consider the case of $r = 0$), it is also possible to obtain a modification of Theorem 3 presented by Atlihan and Orhan [6].

Now, we can summarize some important conclusions of our weighted approximation results via summability process.

- Theorems 2.3 and 2.4 improve Èfendiev's results in [10]. Indeed, for each $\upsilon \in \mathbb{N}$, it is enough to take $\mathcal{A} = \{A^{\upsilon}\} = \{I\}$, the identity matrix.
- If we take $\mathcal{A} = \mathcal{F}$, the almost convergence method, then we get an almost approximation result in Lorentz's sense [16].
- If we take $\mathcal{A} = \{C_1\}$, the Cesàro method, we obtain an arithmetic mean approximation in Cesàro's sense. In order to verify it we give the following application.

Define the functions $u_k : [0, +\infty) \to \mathbb{R}$ $(k \in \mathbb{N})$ by

$$u_k(x) = \begin{cases} 1 + \sin x, & \text{if } k \text{ is odd,} \\ 1 - \sin x, & \text{if } k \text{ is even.} \end{cases} \tag{2.12}$$

Then, taking $\rho(x) = 1 + x^2$ and also considering the test functions

$$f_i(x) = \frac{x^{i+1}\rho(x)}{(i+1)\,(1+x^2)} = \frac{x^{i+1}}{(i+1)} \ (i = 0, 1, 2)$$

the following positive linear operators on $C_\rho[0, +\infty)$

$$L_k(f; x) = u_k(x) e^{-kx} \sum_{j=0}^{\infty} f\left(\frac{j}{k}\right) \frac{n^j}{j!} \left(jx^{j-1} - kx^j\right), \tag{2.13}$$

satisfy all conditions of Theorem 2.4 for $r = 1$. In this case, observe that

$$L_k(f; x) = u_k(x) S_k'(f; x), \tag{2.14}$$

where, for each $k \in \mathbb{N}$, $S_k(f; x)$ denotes the classical Szász-Mirakjan operator [21] (see also [2]) and $S_k'(f; x)$ denotes its derivative with respect to $x$. If we use the Cesàro matrix $\mathcal{A} = \{C_1\} = \{[c_{nk}]\}$ $(k, n \in \mathbb{N})$, then it follows from (2.12) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} u_k(x) = 1, \text{ uniformly on } x \in [0, b] \ (b > 0). \tag{2.15}$$

We also know that

$$\lim_{k\to\infty} S_k'(f; x) = f'(x), \text{ uniformly on } x \in [0, b], \tag{2.16}$$

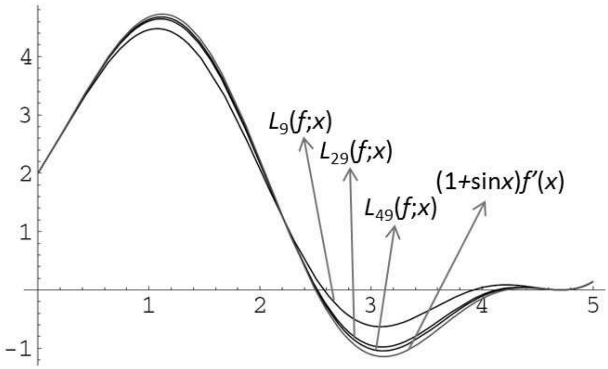


Fig. 1. $L_{2k+1}(f;x)$ in (2.13) approaches to $(1+\sin(x))f'(x)$ on $[0,5]$ for sufficiently large $k$, where $f(x)=2x+\cos x+x\sin x$

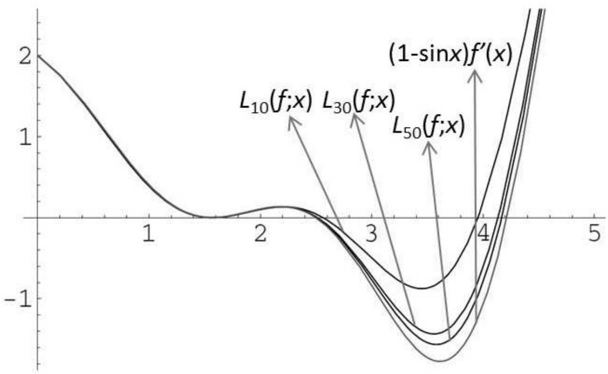


Fig. 2. $L_{2k}(f;x)$ in (2.13) approaches to $(1-\sin(x))f'(x)$ on $[0,5]$ for sufficiently large $k$, where $f(x)=2x+\cos x+x\sin x$

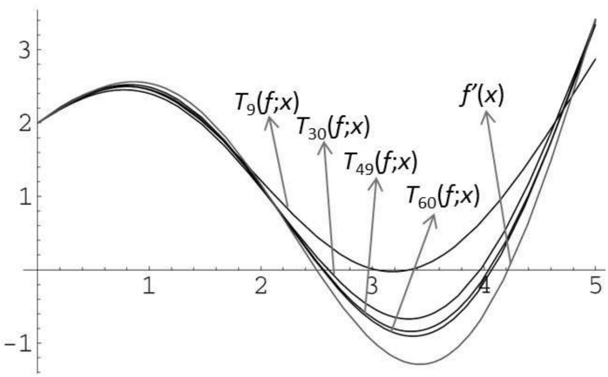


Fig. 3. $T_n(f;x)$ in (2.20), which is the arithmetic mean of $L_k(f;x)$, approaches to $f'(x)$ on $[0,5]$ for sufficiently large $n$

for all $f \in C_{\rho}^{(1)}[0,+\infty)$. By the regularity of the Cesàro method, (2.16) guarantees that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| S_k'(f) - f'(x) \right| = 0, \text{ uniformly on } x \in [0,b]. \tag{2.17}$$

On the other hand, one can write from (2.14) that

$$\begin{aligned} L_k(f;x) - f'(x) = {} & (u_k(x)-1)\left(S_k'(f;x) - f'(x)\right) + f'(x)\left((u_k(x)-1)\right) \\ & + S_k'(f;x) - f'(x), \end{aligned}$$

which implies

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} L_k(f;x) - f'(x) = {} & \frac{1}{n}\sum_{k=1}^{n} (u_k(x)-1)\left(S_k'(f;x) - f'(x)\right) \\ & + f'(x)\left(\frac{1}{n}\sum_{k=1}^{n} u_k(x) - 1\right) + \frac{1}{n}\sum_{k=1}^{n}\left(S_k'(f;x) - f'(x)\right). \end{aligned}$$

Since $|u_k(x) - 1| \le 1$ for all $k \in \mathbb{N}$ and $x \ge 0$, we obtain that

$$\left| \frac{1}{n}\sum_{k=1}^{n} L_k(f;x) - f'(x) \right| \le \frac{2}{n}\sum_{k=1}^{n} \left| S_n'(f;x) - f'(x) \right| + \left| f'(x) \right| \left| \frac{1}{n}\sum_{k=1}^{n} u_k(x) - 1 \right|. \tag{2.18}$$

Now using (2.15), (2.17) and (2.18), we conclude that

$$\lim_{n\to\infty} T_n(f;x) = f'(x), \text{ uniformly on } x \in [0,b], \tag{2.19}$$

where

$$T_n(f;x) := \frac{L_1(f;x) + L_2(f;x) + \ldots + L_n(f;x)}{n}. \tag{2.20}$$

Thus, this arithmetic mean approximation in (2.19) by the operators $L_k$ verifies Theorem 2.4 on the compact intervals of $[0,+\infty)$. However, it is impossible to approximate $f'$ by the operators $L_k(f)$ in (2.13) except for the constant functions since the function sequence $(u_k(x))$ given by (2.12) is non-convergent (pointwisely) on $[0,+\infty)\backslash\{(2k-1)\pi/2 : k = 1,2,\ldots\}$. Furthermore, observe that, for any non-negative regular matrix $A$, the $A$-statistical limit of $(u_k(x))$ does not exist. Therefore, it is also impossible to approximate statistically to $f'$ by the operators $L_k(f)$.

Now we use the function $f(x) = 2x + \cos x + x \sin x$ and its derivative $f'(x) = 2 + x \cos x$. Then, we see from Figure 1 that $L_{2k+1}(f; x)$ approaches to $(1 + \sin x)f'(x)$ on the compact interval $[0, 5]$ while, according to Figure 2, $L_{2k}(f; x)$ goes to the function $(1 - \sin x)f'(x)$ on $[0, 5]$ for sufficiently large $k$. This means that it is impossible to approximate $f'$ by $L_k(f)$ on $[0, 5]$. However, Figure 3 indicates that $T_n(f)$, which is the arithmetic mean of $L_k$, approaches uniformly to $f'$ on $[0, 5]$ for sufficiently large $n$.

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