Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **41** (2015), 431–456

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

NOTES ON OPTIMALITY CONDITIONS USING NEWTON DIAGRAMS AND SUMS OF SQUARES

Yoshiyuki Sekiguchi

Communicated by A. L. Dontchev

Abstract. We consider relationships between optimality conditions using Newton diagrams and sums of squares of polynomials and power series.

1. Introduction. We consider the set of sums of squares of real polynomials $\mathbb{R}[x]$ denoted by $\sum \mathbb{R}[x]^2$ and the quadratic module $M(g_1, \ldots, g_l) = \left\{ \sum_i \sigma_i g_i \mid \sigma_i \in \sum \mathbb{R}[x]^2 \right\}$ generated by $g_i \in \mathbb{R}[x], i = 1, \ldots, l$. In addition, let sums of squares of power series $\mathbb{R}[[x]]$ be denoted by $\sum \mathbb{R}[[x]]^2$ and $\widetilde{M}(g_1, \ldots, g_l) = \left\{ \sum_i \tau_i g_i \mid \tau_i \in \sum \mathbb{R}[[x]]^2 \right\}$. It is well known that these play important roles in

2010 *Mathematics Subject Classification:* 90C46, 13J30, 14M25.
Key words: Polynomial optimization, Newton diagram, optimality conditions, sums of squares.

polynomial optimization problems; see [7] and references therein. On the other hand, optimality conditions in optimization theory can be used to give sufficient conditions for a function to belong to quadratic modules generated by constraint functions (sos-representability).

A polynomial optimization problem is the following:

$$\begin{array}{lll}\text{(POP)} & \min & f(x) \\ & \text{s.t.} & g_i(x) \geq 0, i = 1, \dots l, \\ & & h_j(x) = 0, j = 1, \dots, m,\end{array}$$

where $f, g_i, h_j \in \mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$. We say the second order condition holds at z if z is a minimizer and there exist $\lambda_i \geq 0, \mu_i \in \mathbb{R}$ such that $\nabla f(z) = \sum_i \lambda_i \nabla g_i(z) + \sum_j \mu_j \nabla h_j(z)$, $\lambda_i g_i(z) = 0$ and

$$\nabla^2 \left(f - \sum_i \lambda_i g_i - \sum_j \mu_j h_j \right)(z)$$

is positive definite on the subspace $\{x \in \mathbb{R}^n \mid \lambda_i \nabla g_i(z)x = 0, \nabla h_j(z)x = 0\}$. Then [1], [8] showed that if the second order condition and some constraint qualification conditions hold at each global minimizer, then $f - f_{\min}$ is contained in the quadratic module $M(g_1, \dots, g_l) + \langle h_1, \dots, h_m \rangle$, where $f_{\min}$ is the global minimum.

We are interested in relationships between other optimality conditions and sos-representability. In this notes, we investigate an optimality condition using Newton diagrams given in [10].

2. Preliminaries. For a polyhedral convex set $P \subset \mathbb{R}^n$, $F \subset P$ is called a *face* of P,

For $f \in \mathbb{R}[x]$, the *support* of f is the set of all exponents of monomials of f and be denoted by $\operatorname{supp} f$. For $\alpha \in \mathbb{Z}^n_+$, $|\alpha| = a_1 + \cdots + a_n$ and α is said to be *even* if all coordinates are even. Let

$$\Delta(f) = \bigcup \{\alpha + \mathbb{R}^n_+ \mid \alpha \in \operatorname{supp} f\},$$
$$\Delta_E(f) = \bigcup \{\alpha + \mathbb{R}^n_+ \mid \alpha \in \operatorname{supp} f \cap (2\mathbb{Z})^n\}.$$

The convex hull $\operatorname{conv} \Delta(f)$ of $\Delta(f)$ is called the *Newton polyhedron* of f. The *Newton diagram* $\Gamma(f)$ is the union of the compact faces of $\operatorname{conv} \Delta(f)$. For $\gamma \subset \mathbb{R}^n_+$, define $f_\gamma = \sum \{f_\alpha x^\alpha \mid \alpha \in \gamma \cap \operatorname{supp} f\}$ and $\mathbb{R}[x]_\gamma$ as the set of polynomials whose supports are included in $\gamma \cap \mathbb{Z}^n$. The polynomial $p_\gamma = \sum_{\alpha \in \gamma \cap (2\mathbb{Z})^n} x^\alpha$ is called the *principal polynomial* of γ.

We consider the *finest locally convex topology* on $\mathbb{R}[x]$; see [3], [9]. This topology is Hausdorff and each finite dimensional subspaces of $\mathbb{R}[x]$ inherits the Euclidean topology, and every converging sequence in $\mathbb{R}[x]$ is contained in a finite dimensional subspace. For a subset C of a finite dimensional subspace of $\mathbb{R}[x]$, the *relative interior* $\operatorname{rint} C$ is defined as the interior of C with respect to the minimal finite dimensional subspace which includes C.

3. Necessary condition. Vasil'ev showed a necessary condition for locally isolated minimality using Newton diagrams [10, Theorem 1.5 (1)].

Theorem 3.1 (Vasil'ev). *Let $f \in \mathbb{R}[x]$ with $f(0) = 0$ have an isolated minimum at* 0. *Then*

(1) $\Gamma(f)$ *meets all coordinate axes;*

(2) *Every vertex of* $\Gamma(f)$ *is even;*

(3) *For each vertex α of* $\Gamma(f)$, $f_\alpha > 0$*;*

(4) *For each face γ of* $\Gamma(f)$, $f_\gamma(x) \geq 0,\ \forall x \in \mathbb{R}^n$.

The following theorem gives necessary conditions using Newton diagrams for sos-representability.

Theorem 3.2. *Let $f \in \mathbb{R}[x]$ with $f(0) = 0$ be a sum of square polynomials. Then*

(1) *Every vertex of* $\Gamma(f)$ *is even.*

(2) *For each vertex α of* $\Gamma(f)$, $f_\alpha > 0$.

(3) *For each face γ of* $\Gamma(f)$, $f_\gamma(x) \in \sum \mathbb{R}[x]^2$.

P r o o f. As a easy consequence of the proof of [10, Proposition 1.2], we have that nonnegativity of f implies the properties (1) and (2)

We will show the properties (3).

For each face $\gamma \subset \Gamma(f)$, let $\Delta = \{\alpha \in \mathbb{Z}_+^n \mid A_1\alpha_1 + A_2\alpha_2 + \cdots + A_n\alpha_n = v\}$ be the supporting hyperplane including the face γ but not $\Gamma(f) \setminus \gamma$. Here we may assume $A = (A_1, \ldots, A_n) \in \mathbb{Z}_+^n \setminus \{0\}^n$ and hence $v = \min\{A \cdot \alpha \mid \alpha \in \operatorname{supp} f\}$, where the dot product is defined by $A \cdot \alpha = \sum_i A_i\alpha_i$. We can write

$f = f_v + f_{v+1} + \cdots$, where $f_{v'}$ is a polynomial each of whose exponents α satisfy $A \cdot \alpha = v'$. Then we have $f_v = f_\gamma$.

Next, let $f = \sum_i^s g_i^2$. We define $w_i = \min\{A \cdot \alpha \mid \alpha \in \operatorname{supp} g_i\}$, $w = \min\{w_1, \ldots w_s\}$. For $i = 1, \ldots, s$, g_i is decomposed as $g_i = g_{i,w} + g_{i,w+1} + \cdots + g_{i,w+t}$ for some $t_i \in \mathbb{N}$. Then we write

$$f = \sum_i^s (g_{i,w} + g_{i,w+1} + \ldots + g_{i,w+t_i})^2 = \sum_i^s g_{i,w}^2 + \tilde{f},$$

where all exponents α of $\tilde{f}$ satisfies $A \cdot \alpha > 2w$. Since $\sum_{i=1}^s g_{i,w}^2 \neq 0$, we have $v \leq 2w$. If $v < 2w$, there exists $\beta \in \gamma$ such that $A \cdot \beta = v < 2w$ and x^β is a monomial of $f - \sum_i^s g_{i,w}^2 = \tilde{f}$. This is a contradiction and we have $v = 2w$. Therefore $f_\gamma = \sum_i^s g_{i,w}^2$. $\square$

4. Sufficient condition. We investigate sufficient conditions for a polynomial to be a sum of squares of power series $\mathbb{R}[[x]]$ using Newton diagrams. We present a sufficient condition for locally isolated minimality by Vasil'ev [10, Theorem 1.5 (2)].

Theorem 4.1 (Vasil'ev)**.** *Let $f \in \mathbb{R}[x]$ with $f(0) = 0$.*

(1) *$\Gamma(f)$ meets all coordinate axes.*

(2) *Every vertex of $\Gamma(f)$ is even.*

(3) *For each vertex α of $\Gamma(f)$, $f_\alpha > 0$.*

(4) *For each face γ of $\Gamma(f)$, $f_\gamma(x) > 0, \forall x$ with $x_1 \cdots x_n \neq 0$.*

Then f has an isolated minimum at 0.

Example 4.2.

$$f(x, y) = x^6 + x^4 y + x^3 y^3 + x^2 y^2 + y^4$$

The vertices of the Newton diagram of f are $(0,4)$, $(2,2)$ and $(6,0)$. The compact faces consist of $\gamma_1 = \{t(0,4)+(1-t)(2,2) \mid 0 \le t \le 1\}$, $\gamma_2 = \{t(2,2)+(1-t)(6,0) \mid 0 \le t \le 1\}$ and the vertices. Here we have for x, y with $xy \neq 0$,

$$f_{\gamma_1} = x^2y^2 + y^4 > 0$$
$$f_{\gamma_2} = x^6 + x^4y + x^2y^2 = x^2 \left\{ \left(x^2 + \frac{1}{2}y \right)^2 + \frac{3}{4}y^2 \right\} > 0.$$

Therefore $(0,0)$ is an isolated minimum of f.

4.1. Simple Newton diagrams. We seek conditions which are analogous to the one by Vasil'ev. We consider the following well-known sufficient condition from the point of view of Newton diagrams; see e.g. [7, Lemma 9.5.1].

Lemma 4.3. *Let $f \in \mathbb{R}[x]$. Suppose $f = \sum_k f_k$ be the expansion of its homogeneous components where $\deg f_k = k$. If $f_0 = f_1 = 0$ and f_2 is a positive definite form, then $f \in \sum \mathbb{R}[[x]]^2$.*

Here we note that if f_2 is positive definite, then $f_2 \in \operatorname{rint}\left(\sum \mathbb{R}[x]_1^2\right)$ [5, Corollary 2.5, Remark 2.6]. Thus the lemma tells us that $f \in \sum \mathbb{R}[[x]]^2$ if the Newton diagram $\Gamma := \Gamma(f)$ is contained in the plane $|\alpha| = 2$ and f_Γ is contained in $\operatorname{rint}\left(\sum \mathbb{R}[x]_1^2\right)$. From this observation, we first obtain an extension of the lemma in the case that the Newton diagram is contained in a plane which is parallel to $|\alpha| = 2$.

Theorem 4.4. *Let f_{2m} be the lowest homogeneous part of $f \in \mathbb{R}[x]$. If $f_{2m} \in \operatorname{rint}\left(\sum \mathbb{R}[x]_m^2\right)$, then $f \in \sum \mathbb{R}[[x]]^2$.*

To show this, we need the following lemmas. In addition, we will use the well-known fact that for any $u \in \mathbb{R}[x]$ with $u(0) = 0$,

$$1 + u,\ \frac{1}{1+u} \in \sum \mathbb{R}[[x]]^2,$$

see e.g. [7, Section 1.6].

Lemma 4.5. *Suppose $f \in \mathbb{R}[x]$ is a homogeneous polynomial of degree $2d$ and $\{e_i\}$ is the canonical basis of $\mathbb{Z}^n$. Then there exists $\widetilde{M} > 0$ such that $f + \sum_{i=1}^n Mx^{2de_i} \in \sum \mathbb{R}[x]^2$ for $M > \widetilde{M}$.*

Proof. This is easily implied by Ghasemi-Marshall [5, Theorem 2.1]. □

Lemma 4.6. *Let $f \in \mathbb{R}[x]$ and γ be a face of $\Gamma(f)$. Then we have the following:*

(1) *The principal polynomial p_γ of γ lies in* $\operatorname{rint}\left(\sum \mathbb{R}[x]^2_{\frac{1}{2}\gamma}\right)$.

(2) $f_\gamma \in \operatorname{rint}\left(\sum \mathbb{R}[x]^2_{\frac{1}{2}\gamma}\right)$ *if and only if* $f_\gamma - \varepsilon p_\gamma \in \sum \mathbb{R}[x]^2_{\frac{1}{2}\gamma}$ *for sufficiently small* $\varepsilon > 0$.

Proof. The proof of (1) is almost identical to the one given in [2, Proposition 5.5]. Let $g \in \sum \mathbb{R}[x]^2_{\frac{1}{2}\gamma}$. Then $g = \sum_t h_t^2$ for some $h_t \in \mathbb{R}[x]_{\frac{1}{2}\gamma}$. For each $a \in \operatorname{supp} g$, there exist $b_1, b_2 \in \bigcup_t \operatorname{supp} h_t$ such that $a = b_1 + b_2$. Since we have $b_1, b_2 \in \frac{1}{2}\gamma$, there exist $\alpha_1, \alpha_2 \in \gamma \cap (2\mathbb{Z})^n$ such that $a = \frac{1}{2}(\alpha_1 + \alpha_2)$. Since

$$x^{\alpha_1} + x^{\alpha_2} \pm 2x^a = \left(x^{\frac{1}{2}\alpha_1} \pm x^{\frac{1}{2}\alpha_2}\right)^2,$$

we conclude that $p_\gamma \pm 2x^a \in \sum \mathbb{R}[x]^2_{\frac{1}{2}\gamma}$ and hence that $p_\alpha - \varepsilon g \in \sum \mathbb{R}[x]^2_{\frac{1}{2}\gamma}$ for sufficiently small $\varepsilon > 0$.

For (2), consider V as the affine hull of $\sum \mathbb{R}[x]^2_{\frac{1}{2}\gamma}$ in the proof of [3, Proposition 1.4]. □

Proof of Theorem 4.4. Let $\Gamma = \Gamma(f_{2m})$. Since $f_{2m} \in \operatorname{rint}\left(\sum \mathbb{R}[x]^2_m\right)$, Γ meets all coordinate axes and then $\Gamma = \Gamma(f)$. By (2) of Lemma 4.6, there exists $\varepsilon > 0$ such that $f_{2m} - 2\varepsilon p_\Gamma \in \sum \mathbb{R}[x]^2$. Let $t = \lceil \frac{1}{2} \deg f \rceil$ and $\{e_i\}$ be the canonical basis of $\mathbb{Z}^n$. Then we write

$$f_{2m} = f_{2m} - 2\varepsilon p_\Gamma + f^{(1)} + f^{(2)},$$

where for $M_k > 0$,

$$f^{(1)} = \varepsilon p_\Gamma - \sum_{k=m+1}^{t} \sum_{i=1}^{n} M_{2k} x^{2ke_i},$$

$$f^{(2)} = \varepsilon p_\Gamma + \sum_{k=m+1}^{t} \sum_{i=1}^{n} M_{2k} x^{2ke_i} + \sum_{|\alpha| \geq 2m+1} f_\alpha x^\alpha.$$

Since Γ meets all coordinate axes, p_Γ contains x^{2me_i} for all i. Then we have

$$\varepsilon x^{2me_i} - \sum_{k=m+1}^{t} M_{2k} x^{2ke_i} = x^{2me_i}\left(\varepsilon - \sum_{k=m+1}^{t} M_{2k} x^{(2k-2m)e_i}\right) \in \sum \mathbb{R}[[x]]^2$$

and hence $f^{(1)} \in \sum \mathbb{R}[[x]]^2$ for any $M_k > 0$.

Next we will show $f^{(2)} \in \sum \mathbb{R}[x]^2$. We claim that for arbitrary $C_\alpha > 0$, there exists $D > 0$ such that

$$T_k := \sum_{\substack{\alpha:\text{even} \\ |\alpha|=2k}} C_\alpha x^\alpha + \sum_{|\alpha|=2k+1} f_\alpha x^\alpha + \sum_{i=1}^{n} D x^{(2k+2)e_i}$$

is contained in $\sum \mathbb{R}[x]^2$. Let $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = 2k+1$. For the index s such that $\sum_i^s 2\alpha_i \le 2k < \sum_i^{s+1} 2\alpha_i$, we define $\beta(\alpha), \beta'(\alpha) \in \mathbb{Z}_+^n$ as

$$\beta(\alpha)_i = \begin{cases} 2\alpha_i, & i = 1, \dots, s \\ 2k - \sum_i^s 2\alpha_i, & i = s+1 \\ 0, & \text{otherwise} \end{cases}$$

and $\beta'(\alpha) = 2\alpha - \beta(\alpha)$. Then $\beta(\alpha), \beta'(\alpha)$ are even, $|\beta(\alpha)| = 2k, |\beta'(\alpha)| = 2k+2$ and $2\alpha = \beta(\alpha) + \beta'(\alpha)$. Thus

$$C_{\beta(\alpha)} x^{\beta(\alpha)} + f_\alpha x^\alpha = C_{\beta(\alpha)}\left(x^{\frac{\beta(\alpha)}{2}} + \frac{f_\alpha}{2C_{\beta(\alpha)}} x^{\frac{\beta'(\alpha)}{2}}\right)^2 - \frac{f_\alpha^2}{4C_{\beta(\alpha)}} x^{\beta'(\alpha)}.$$

Let

$$\begin{aligned} S(k) &= \{\alpha \in \operatorname{supp} f \mid |\alpha| = k\}, \\ I &= \{\beta \in \mathbb{Z}_+^n \mid \beta \text{ is even}, |\beta| = 2k\} \setminus \{\beta(\alpha) \mid S(2k+1)\}. \end{aligned}$$

Then we have

$$T_k = \sum_{\beta \in I} C_\alpha x^\alpha + \sum_{\alpha \in S(2k+1)} \left(C_{\beta(\alpha)} x^{\beta(\alpha)} + f_\alpha x^\alpha\right) + \sum_{i=1}^{n} D x^{(2k+2)e_i}$$

$$= \sum_{\beta \in I} C_\alpha x^\alpha + \sum_{\alpha \in S(2k+1))} C_{\beta(\alpha)} \left(x^{\frac{\beta(\alpha)}{2}} + \frac{f_\alpha}{2C_{\beta(\alpha)}} x^{\frac{\beta'(\alpha)}{2}} \right)^2 + \left(\sum_{i=1}^n D x^{(2k+2)e_i} - \sum_{\substack{|\alpha|=2k+1 \\ \alpha \in \operatorname{supp} f}} \frac{f_\alpha^2}{4C_{\beta(\alpha)}} x^{\beta'(\alpha)} \right).$$

Here the last parenthesis is a homogeneous polynomial of degree $2k+2$ and Lemma 4.5 implies that it is a sum of square polynomials for sufficiently large D_i. Thus the claim is proved.

Now we have

$$\begin{aligned} f^{(2)} &= \varepsilon p_\Gamma + \sum_{k=m+1}^t \sum_{i=1}^n M_{2k} x^{2ke_i} + \sum_{k=m+1}^t \left(\sum_{|\alpha|=2k-1} f_\alpha x^\alpha + \sum_{|\alpha|=2k} f_\alpha x^\alpha \right) \\ &= g^{(1)} + g^{(2)} + g^{(3)}, \end{aligned}$$

where

$$\begin{aligned} g^{(1)} &= \varepsilon p_\Gamma + \sum_{|\alpha|=2m+1} f_\alpha x^\alpha + \sum_{i=1}^n \frac{M_{2m+2}}{4} x^{(2m+2)e_i} \\ g^{(2)} &= \sum_{k=m+2}^t \left(\sum_{\alpha \in S(2k-1)} x^{\beta(\alpha)} + \sum_{|\alpha|=2k-1} f_\alpha x^\alpha + \sum_{i=1}^n \frac{M_{2k}}{4} x^{2ke_i} \right) \\ g^{(3)} &= \sum_{k=m+2}^t \left(\sum_{i=1}^n \frac{M_{2k-2}}{4} x^{(2k-2)e_i} - \sum_{\alpha \in S(2k-1)} x^{\beta(\alpha)} \right) \\ g^{(4)} &= \sum_{k=m+1}^t \left(\sum_{|\alpha|=2k} f_\alpha x^\alpha + \sum_{i=1}^n \frac{M_{2k}}{2} x^{2ke_i} \right) + \sum_{i=1}^n \frac{M_{2t}}{4} x^{2te_i}. \end{aligned}$$

Note that $\sum_{\alpha \in S(2k-1)} x^{\beta(\alpha)}$ is a homogeneous polynomials of degree $2k-2$. Again by Lemma 4.5, there exist $\widetilde{M}$ such that $g^{(3)}, g^{(4)} \in \sum \mathbb{R}[x]^2$ for $M_{2k} > \widetilde{M}$. The claim above implies that there exist $M_{2m+2} > \widetilde{M}$ such that $g^{(1)} \in \sum \mathbb{R}[x]^2$. Similarly for $k = m+2, \dots t$, there exist $M_{2k} > \widetilde{M}$ such that $g^{(2)} \in \sum \mathbb{R}[x]^2$. Therefore $f^{(2)} \in \sum \mathbb{R}[x]^2$ and hence $f \in \sum \mathbb{R}[[x]]^2$. $\square$

Example 4.7. Consider

$$f(x,y,z) = 2x^6 + 2y^6 + 2z^6 + xy^3z^3 + x^2y^4z^3.$$

The lowest homogeneous part is $2x^6 + 2y^6 + 2z^6$, which is contained in $\operatorname{rint}\left(\sum \mathbb{R}[x]_2^2\right)$. The monomials xy^3z^3 and $x^2y^4z^3$ are not even and their exponent vectors are $(1,3,3)$ and $(2,4,3)$ respectively. Now we have

$$\begin{aligned} 2(1,3,3) &= (2,6,6) = (2,4,0) + (0,2,6) \\ 2(2,4,3) &= (4,8,6) = (4,4,0) + (0,4,6) \end{aligned}$$

and then

$$\begin{aligned} x^2y^4 + xy^3z^3 &= \left(xy^2 + \frac{1}{2}yz^3\right)^2 - \frac{1}{4}y^2z^6 \\ x^4y^4 + x^2y^4z^3 &= \left(x^2y^2 + \frac{1}{2}y^2z^3\right)^2 - \frac{1}{4}y^4z^6. \end{aligned}$$

Now we have

$$\begin{aligned} f &= x^6 + y^6 + z^6 - 2a(x^8 + y^8 + z^8) - b(x^{10} + y^{10} + z^{10}) \\ &\quad + (x^6 + y^6 + z^6 - \varepsilon x^2y^4) \\ &\quad + [\varepsilon x^2y^4 + xy^3z^3 + a(x^8 + y^8 + z^8)] \\ &\quad + [x^4y^4 + x^2y^4z^3 + b(x^{10} + y^{10} + z^{10})] \\ &\quad + [a(x^8 + y^8 + z^8) - x^4y^4] \\ &= x^6(1 - 2ax^2 - bx^4) + y^6(1 - 2ay^2 - by^4) + z^6(1 - 2az^2 - bz^4) \\ &\quad + (x^6 + y^6 + z^6 - \varepsilon x^2y^4) \\ &\quad + \left[\varepsilon\left(xy^2 + \frac{1}{2\varepsilon}yz^3\right)^2 - \frac{1}{4\varepsilon}y^2z^6 + a(x^8 + y^8 + z^8)\right] \\ &\quad + \left[\left(x^2y^2 + \frac{1}{2}y^2z^3\right)^2 - \frac{1}{4}y^4z^6 + b(x^{10} + y^{10} + z^{10})\right] \\ &\quad + \left[a(x^8 + y^8 + z^8) - x^4y^4\right] \end{aligned}$$

By Lemma 4.6, there exists $\varepsilon > 0$ such that $x^6 + y^6 + z^6 - \varepsilon x^2y^4 \in \sum \mathbb{R}[x]^2$. Then by Lemma 4.5, we can choose $a, b > 0$ large enough so that the last three brackets are contained in $\sum \mathbb{R}[x]^2$. Therefore $f \in \sum \mathbb{R}[[x]]^2$.

4.2. General Newton diagrams. Next, we consider the case that the Newton diagram has several faces which are contained in different planes. For this general case, we need an assumption on the distributions of exponent vectors of polynomials in addition to conditions corresponding to those of Theorem 4.1.

For $\alpha^1, \dots, \alpha^t \in (2\mathbb{Z}_+)^n$, a *binary convex combination* of these points is $\alpha \in \mathbb{Z}_+^n$ which can be written as

$$\alpha = \lambda_1 \alpha^1 + \cdots + \lambda_t \alpha^t,$$

for some $\lambda_s > 0, \sum_{s=1}^{t} \lambda_s = 1$ such that 2-adic expansions of $\lambda_1, \dots, \lambda_t$ have finite digits. We also say that a binary convex combination has *full digits* if there exists $N \in \mathbb{N}$ such that

(1) $\lambda_s = \sum_{k=1}^{N} \delta_{sk} 2^{-k}$ for $\delta_{sk} \in \{0, 1\}$, $s = 1, \dots, t$;

(2) for each k, there exists s with $\delta_{sk} = 1$.

For $\Delta_E \subset \mathbb{Z}_+^n$, the set of all binary convex combinations of points in $\Delta_E \cap (2\mathbb{Z}_+)^n$ which have full digits and are contained in $\mathbb{Z}^n$ is called the *bisectional convex hull* of Δ_E and denoted by $\operatorname{bconv} \Delta_E$. Note that we have

$$\Delta_E \cap Z^n \subset \operatorname{bconv} \Delta_E \subset \operatorname{conv} \Delta_E \cap Z^n.$$

Example 4.8. Let $\Delta_E = \left\{(16, 0) + \mathbb{Z}_+^2\right\} \cup \left\{(0, 10) + \mathbb{Z}_+^2\right\}$. Then $(11, 7) \in \operatorname{bconv} \Delta_E$. In fact, we have

$$(11, 7) = \left(\frac{1}{2} + \frac{1}{2^3}\right)(16, 0) + \frac{1}{2^2}(4, 22) + \frac{1}{2^3}(0, 12),$$

$(4, 22), (0, 12) \in \Delta_E \cap (2\mathbb{Z}_+)^n$ and it has full digits.

Proposition 4.9. *Let $\Delta_E \subset \mathbb{Z}_+^n$. Then we have*

$$\operatorname{bconv} \Delta_E = \mathbb{Z}^n \cap$$
$$\left\{ \sum_{k=1}^{N} 2^{-k} \beta^k + 2^{-N} \beta^{N+1} \mid \beta^k \in \Delta_E \cap (2\mathbb{Z}_+)^n, k = 1, \dots, N+1 \text{ for some } N \in \mathbb{N} \right\}.$$

P r o o f. Let $\alpha \in \operatorname{bconv} \Delta_E$. Then there exist $\alpha^1, \dots, \alpha^t \in \Delta_E \cap (2\mathbb{Z}_+)^n, \lambda_1, \dots, \lambda_t > 0, \sum_{s=1}^{t} \lambda_s = 1$ such that

$$\alpha = \lambda_1 \alpha^1 + \cdots + \lambda_t \alpha^t$$

and it has full digits. Suppose that $\lambda_s = \sum_{k=1}^{N+1} \delta_{sk} 2^{-k}$ for $s = 1, \dots, t$. Since $\sum_{s=1}^{t} \lambda_s = 1$ and $\{\delta_{sN+1}\}_s$ corresponds to the $N+1$ st digits which are the last ones, the number of nonzero $\{\delta_{sN+1}\}_s$ is even. Thus there exist at least two nonzero $\delta_{s'N+1}, \delta_{s''N+1}$.

Since α has full digits, for each $k = 1, \dots, N$, there exists $\tau \in \{1, \dots, t\}$ such that $\delta_{\tau k} = 1$ and then let $\tau(k)$ be the least such index. Then we have

$$\begin{aligned} 1 = \sum_{s=1}^{t} \lambda_s \geq \sum_{k=1}^{N} \delta_{\tau(k)k} 2^{-k} + \delta_{s'N+1} 2^{-N-1} + \delta_{s''N+1} 2^{-N-1} \\ = \sum_{k=1}^{N} 2^{-k} + 2^{-N-1} + 2^{-N-1} = 1. \end{aligned}$$

Therefore there is only one s with $\delta_{sk} = 1$ for each $k = 1, \dots, N$. It gives the desired representation. □

Proposition 4.10. *For $\beta^k \in (2\mathbb{Z}_+)^n, k = 1, \dots N+1$, let*

$$\alpha = \sum_{k=1}^{N} \frac{1}{2^k} \beta^k + \frac{1}{2^N} \beta^{N+1}.$$

be contained in $\mathbb{Z}^n$. Then we have

$$\sum_{k=N'}^{N} \frac{1}{2^{k-N'+2}} \beta^k + \frac{1}{2^{N-N'+2}} \beta^{N+1}.$$

is contained in $\mathbb{Z}_+^n$ for $N' = 2, \dots, N+1$ with the convention $\sum_{k=N+1}^{N} a_k = 0$.

Proof. Since $\beta^k \in (2\mathbb{Z}_+)^n$, the left hand side of

$$2^{N'-2}\left(\alpha - \sum_{k=1}^{N'-1} \frac{1}{2^k}\beta^k\right) = \sum_{k=N'}^{N} \frac{1}{2^{k-N'+2}}\beta^k + \frac{1}{2^{N-N'+2}}\beta^{N+1}.$$

is contained in $\mathbb{Z}_+^n$ and so is the right hand side. □

Example 4.11. By Example 4.8,

$$(11,7) = \frac{1}{2}(16,0) + \frac{1}{2^2}(4,22) + \frac{1}{2^3}(16,0) + \frac{1}{2^3}(0,12)$$

and $(4,22) \in \Delta_E$. In addition, we have all of right hand sides of

$$\begin{aligned}
(11,7) - \frac{1}{2}(16,0) &= \frac{1}{2^2}(4,22) + \frac{1}{2^3}(16,0) + \frac{1}{2^3}(0,12),\\
2\left((11,7) - \frac{1}{2}(16,0) - \frac{1}{2^2}(4,22)\right) &= \frac{1}{2^2}(16,0) + \frac{1}{2^2}(0,12)\\
2^2\left((11,7) - \frac{1}{2}(16,0) - \frac{1}{2^2}(4,22) - \frac{1}{2^3}(16,0)\right) &= \frac{1}{2}(0,12)
\end{aligned}$$

are contained in $\mathbb{Z}_+^2$.

Now we present sufficient conditions.

Theorem 4.12. *Let $f \in \mathbb{R}[x]$ with $f(0) = 0$. Suppose that*

(1) *Every vertex of $\Gamma(f)$ is even.*

(2) *For each vertex α of $\Gamma(f)$, $f_\alpha > 0$.*

(3) *For each maximal face γ of $\Gamma(f)$, $f_\gamma(x) \in \operatorname{rint}\left(\sum \mathbb{R}[x]^2_{\frac{1}{2}\gamma}\right)$.*

(4) *If for each maximal face γ of $\Gamma(f)$,*

$$\{\alpha \in \operatorname{supp} f \cap \operatorname{conv} \Delta(f_\gamma) \setminus \gamma \mid \alpha \textit{ is odd or } f_\alpha < 0\} \subset \operatorname{bconv} \Delta_E(f_\gamma).$$

Then $f \in \sum \mathbb{R}[[x]]$.

We note that by Theorem 3.2, Condition (3) of Theorem 4.12 implies the corresponding interiority condition for each face of $\Gamma(f)$. To show the theorem, we need the following lemmas.

Lemma 4.13. *Let*

$$\alpha = \sum_{k=1}^{N} \frac{1}{2^k}\beta^k + \frac{1}{2^N}\beta^{N+1},$$

where $\beta^k \in (2\mathbb{Z}_+)^n, N \in \mathbb{N}$. *For any* $\varepsilon > 0, a \in \mathbb{R}, t \in \{1, \dots, N+1\}$ *there exists* $M > 0$ *such that*

$$\sum_{k=1}^{N+1} \varepsilon x^{\beta^k} - a x^{\alpha} + M x^{\beta^t} \in \sum \mathbb{R}[x]^2.$$

Proof. Case $t = N+1$.

$$\begin{aligned}
&\sum_{k=1}^{N+1} \varepsilon x^{\beta^k} - a x^{\alpha} \\
&= \sum_{k=2}^{N+1} \varepsilon x^{\beta^k} + \varepsilon \left(x^{2^{-1}\beta^1} - \frac{a}{2\varepsilon} x^{\sum_{k=2}^{N} 2^{-k}\beta^k + 2^{-N}\beta^{N+1}} \right)^2 \\
&\quad - \varepsilon \left(\frac{a}{2\varepsilon} \right)^2 x^{\sum_{k=2}^{N} 2^{-k+1}\beta^k + 2^{-N+1}\beta^{N+1}} \\
&= \sum_{k=3}^{N+1} \varepsilon x^{\beta^k} + \varepsilon \left(x^{2^{-1}\beta^1} - \frac{a}{2\varepsilon} x^{\sum_{k=2}^{N} 2^{-k}\beta^k + 2^{-N}\beta^{N+1}} \right)^2 \\
&\quad + \varepsilon \left(x^{2^{-1}\beta^2} - \frac{1}{2} \left(\frac{a}{2\varepsilon} \right)^2 x^{\sum_{k=3}^{N} 2^{-k+1}\beta^k + 2^{-N+1}\beta^{N+1}} \right)^2 \\
&\quad - \varepsilon \left(\frac{1}{2} \left(\frac{a}{2\varepsilon} \right)^2 \right)^2 x^{\sum_{k=3}^{N} 2^{-k+2}\beta^k + 2^{-N+2}\beta^{N+1}} \\
&\quad \vdots \\
&= \varepsilon x^{\beta^{N+1}} + \sum_{j=1}^{N-1} \varepsilon \left(x^{2^{-1}\beta^j} - C_j x^{\sum_{k=j+1}^{N} 2^{-k+j-1}\beta^k + 2^{-N+j-1}\beta^{N+1}} \right)^2 \\
&\quad + \varepsilon \left(x^{2^{-1}\beta^N} - C_N x^{2^{-1}\beta^{N+1}} \right)^2 - \varepsilon C_N^2 x^{\beta^{N+1}}
\end{aligned}$$

where

$$C_1 = \frac{a}{2\varepsilon},\ C_j = 2^{-1} C_{j-1}^2,\ j = 1, 2, \dots, N.$$

Thus we have

$$C_j = \frac{a^{2^{j-1}}}{2^{2^j - 1} \varepsilon^{2^{j-1}}},\ j = 1, 2, \dots, N.$$

By Proposition 4.10, we have $\sum_{k=j+1}^{N} 2^{-k+j-1}\beta^k + 2^{-N+j-1}\beta^{N+1}$ is contained in $\mathbb{Z}_+^n$ for each j. Therefore $\sum_{k=1}^{N+1} \varepsilon x^{\beta^k} - ax^\alpha + \varepsilon C_N^2 \in \sum \mathbb{R}[x]^2$.

Case $t = \{2, \ldots, N\}$.

$$
\begin{aligned}
&\sum_{k=1}^{N+1} \varepsilon x^{\beta^k} - ax^\alpha \\
&= \sum_{j=t}^{N+1} \varepsilon x^{\beta^j} + \sum_{j=1}^{t-1} \varepsilon \left(x^{2^{-1}\beta^j} + C_j x^{\sum_{k=j+1}^{N} 2^{-k+j-1}\beta^k + 2^{-N+j-1}\beta^{N+1}} \right)^2 \\
&\quad - \varepsilon C_{t-1}^2 x^{\sum_{k=t}^{N} 2^{-k+t-1}\beta^k + 2^{-N+j-1}\beta^{N+1}} \\
&= \sum_{j=t}^{N+1} \varepsilon x^{\beta^j} + \sum_{j=1}^{t-1} \varepsilon \left(x^{2^{-1}\beta^j} + C_j x^{\sum_{k=j+1}^{N} 2^{-k+j-1}\beta^k + 2^{-N+j-1}\beta^{N+1}} \right)^2 - Lx^{\beta^t} \\
&\quad + Lx^{\beta^t} - \varepsilon C_{t-1}^2 x^{\sum_{k=t}^{N} 2^{-k+t-1}\beta^k + 2^{-N+t-1}\beta^{N+1}} \\
&= \sum_{j=t+1}^{N+1} \varepsilon x^{\beta^j} + \sum_{j=1}^{t-1} \varepsilon \left(x^{2^{-1}\beta^j} + C_j x^{\sum_{k=j+1}^{N} 2^{-k+j-1}\beta^k + 2^{-N+j-1}\beta^{N}} \right)^2 - (L+\varepsilon)x^{\beta^t} \\
&\quad + L\left(x^{2^{-1}\beta^t} - \frac{C_t}{2L} x^{\sum_{k=t+1}^{N} 2^{-k+t-1}\beta^k + 2^{-N+t-1}\beta^{N+1}} \right)^2 \\
&\quad - \frac{C_t^2}{2^2 L} x^{\sum_{k=t+1}^{N} 2^{-k+t}\beta^k + 2^{-N+t}\beta^{N+1}} \\
&= \sum_{j=t+2}^{N+1} \varepsilon x^{\beta^j} - (L+\varepsilon)x^{\beta^t} + \sum_{j=1}^{t-1} \varepsilon \left(x^{2^{-1}\beta^j} + C_j x^{\sum_{k=j+1}^{N} 2^{-k+j-1}\beta^k + 2^{-N+j-1}\beta^{N}} \right)^2 \\
&\quad + L\left(x^{2^{-1}\beta^t} - \frac{C_t}{2L} x^{\sum_{k=t+1}^{N} 2^{-k+t-1}\beta^k + 2^{-N+t-1}\beta^{N+1}} \right)^2 \\
&\quad + \varepsilon \left(x^{2^{-1}\beta^{t+1}} - \frac{1}{2\varepsilon}\frac{C_t^2}{2^2 L} x^{\sum_{k=t+2}^{N} 2^{-k+t}\beta^k + 2^{-N+t}\beta^{N+1}} \right)^2 \\
&\quad - \frac{1}{2^2\varepsilon}\left(\frac{C_t^2}{2^2 L} \right)^2 x^{\sum_{k=t+2}^{N} 2^{-k+t+1}\beta^k + 2^{-N+t+1}\beta^{N+1}} \\
&= \varepsilon x^{\beta^{N+1}} - (L+\varepsilon)x^{\beta^t}
\end{aligned}
$$

$$+\sum_{j=1}^{t-1}\varepsilon\left(x^{2^{-1}\beta^j}+C_jx^{\sum_{k=j+1}^{N}2^{-k+j-1}\beta^k+2^{-N+j-1}\beta^{N+1}}\right)^2$$
$$+L\left(x^{2^{-1}\beta^t}-\frac{C_t}{2L}x^{\sum_{k=t+1}^{N}2^{-k+t-1}\beta^k+2^{-N+t-1}\beta^{N+1}}\right)^2$$
$$+\sum_{j=t+1}^{N}\varepsilon\left(x^{2^{-1}\beta^j}-D_jx^{\sum_{k=j+1}^{N}2^{-k+j-1}\beta^k+2^{-N+j-1}\beta^{N+1}}\right)^2-\varepsilon D_N^2x^{\beta^{N+1}},$$

where

$$D_{t+1}=\frac{C_t^2}{2^3\varepsilon L},\quad D_j=2^{-1}D_{j-1}^2,\ j=t+2,\dots,N,$$

and hence we have

$$D_j=\frac{1}{2^{2^{j-t-1}-1}}\left(\frac{a^{2^t}}{2^{2^{t+1}+1}\varepsilon^{2^t+1}L}\right)^{2^{j-t-1}},\ j=t+2,\dots,N.$$

Then by taking L large so that $D_N<1$, we obtain $\sum_{k=1}^{N+1}\varepsilon x^{\beta^k}-ax^\alpha+(L+\varepsilon)x^{\beta^t}\in\sum\mathbb{R}[x]^2$. Case $t=1$ is identical to the case $t=N+1$. □

Lemma 4.14. *For $f\in\mathbb{R}[x]$ with $f(0)=0$, let γ be a face of $\Gamma(f)$. Suppose that $f_\gamma\in\operatorname{rint}\left(\sum\mathbb{R}[x]^2_{\frac{1}{2}\gamma}\right)$. Then for any $a>0$, $\alpha\in\operatorname{bconv}\Delta_E(f_\gamma)\setminus\gamma$,*

$$f_\gamma\pm ax^\alpha\in\sum\mathbb{R}[[x]]^2.$$

Proof. Since $f_\gamma\in\operatorname{rint}\left(\sum\mathbb{R}[x]^2_{\frac{1}{2}\gamma}\right)$, there exists $\varepsilon>0$ such that $f_\gamma-\varepsilon p_\gamma\in\sum\mathbb{R}[x]^2$. Let arbitrary $\alpha\in\operatorname{bconv}\Delta_E(f_\gamma)\setminus\gamma$ be fixed. Then there exist $\{\beta^k\}\in\Delta_E(f_\gamma)\cap(2\mathbb{Z}_+)^n$ such that $\alpha=\sum_{k=1}^{N}2^{-k}\beta^k+2^{-N}\beta^{N+1}$. Since γ is a face, there exist $A=(A_1,\dots,A_n)\in\mathbb{Z}_+^n\setminus\{0\}^n$ and $v>0$ such that $\{\alpha'\in\mathbb{Z}_+^n\mid A\cdot\alpha'=v\}$ contains γ. By taking the dot product of A and α, we have

$$A\cdot\alpha=\sum_{k=1}^{N}\frac{1}{2^k}A\cdot\beta^k+\frac{1}{2^N}A\cdot\beta^{N+1}.$$

Since $\alpha \notin \gamma$, we have $A \cdot \alpha > v$. In addition, since $\sum_{k=1}^{N} 2^{-k} + 2^{-N-1} = 1$, there exists $t \in \{1, \ldots, N+1\}$ such that $A \cdot \beta^t > v$ and thus $\beta^t \notin \gamma$.

Now, for $M > 0$ we have

$$\begin{aligned} & f_\gamma \pm a x^\alpha \\ & = f_\gamma - \varepsilon p_\gamma + \varepsilon p_\gamma - \frac{\varepsilon}{N+2} \sum_{k=1}^{N+1} x^{\beta^k} + \frac{\varepsilon}{N+2} \sum_{k=1}^{N+1} x^{\beta^k} \pm a x^\alpha - M x^{\beta^t} + M x^{\beta^t} \\ & = (f_\gamma - \varepsilon p_\gamma) + \left(\varepsilon p_\gamma - \frac{\varepsilon}{N+2} \sum_{k=1}^{N+1} x^{\beta^k} - M x^{\beta^t} \right) \\ & + \left(\frac{\varepsilon}{N+2} \sum_{k=1}^{N+1} x^{\beta^k} \pm a x^\alpha + M x^{\beta^t} \right). \end{aligned}$$

By Lemma 4.13, there exists $M > 0$ such that the last parenthesis is contained in $\sum \mathbb{R}[x]^2$. Since $\beta^t \in \Delta_E(f_\gamma) \setminus \gamma \cap \mathbb{Z}^n$, there exist $\tilde{\beta^t} \in \gamma \cap (2\mathbb{Z}_+)^n$ and $\omega \in (2\mathbb{Z}_+)^n \setminus \{0\}^n$ such that $\beta^t = \tilde{\beta^t} + \omega$. In addition, let $r = \#\{\beta^k \mid \beta^k = \beta^t, k = 1, \ldots, N+1\}$ and $\tilde{r} = \#\{\beta^k \mid \beta^k = \tilde{\beta^t}, k = 1, \ldots, N+1\}$. Then $0 \leq r, \tilde{r} \leq N+1$ and

$$\begin{aligned} & \varepsilon p_\gamma - \frac{\varepsilon}{N+2} \sum_{k=1}^{N+1} x^{\beta^k} - M x^{\beta^t} \\ & = \varepsilon \sum_{\substack{\alpha' \in \gamma \cap (2\mathbb{Z}_+)^n \\ \alpha' \neq \tilde{\beta^t}}} x^{\alpha'} + \varepsilon x^{\tilde{\beta^t}} - \frac{\tilde{r}\varepsilon}{N+2} x^{\tilde{\beta^k}} - \frac{\varepsilon}{N+2} \sum_{\substack{k \neq t \\ \beta^k \neq \beta^t, \tilde{\beta^k}}} x^{\beta^k} \\ & \quad - (r\varepsilon(N+2)^{-1} + M) x^{\beta^t} \\ & = \varepsilon \sum_{\substack{\alpha' \in \gamma \cap (2\mathbb{Z}_+)^n \\ \alpha' \neq \tilde{\beta^t}}} x^{\alpha'} - \frac{\varepsilon}{N+2} \sum_{\substack{k \neq t \\ \beta^k \neq \beta^t, \tilde{\beta^k}}} x^{\beta^k} \\ & \quad + \varepsilon x^{\tilde{\beta^t}} \left(1 - \tilde{r}(N+2)^{-1} - \varepsilon^{-1}(r\varepsilon(N+2)^{-1} + M) x^\omega \right) \end{aligned}$$

is contained in $\sum \mathbb{R}[[x]]^2$. Therefore, we have $f_\gamma \pm a x^\alpha \in \sum \mathbb{R}[[x]]^2$. □

Example 4.15. Let $f(x, y) = x^{16} + y^{10} - x^{13} y^2$ and $\Gamma = \Gamma(f)$. Then $\Gamma = \{\lambda(16, 0) + (1 - \lambda)(0, 10) \mid 0 \leq \lambda \leq 1\}$ and $\Delta_E(f_\Gamma) = \{(16, 0) + (2\mathbb{Z}_+)^2\} \cup$

$\{(0,10)+(2\mathbb{Z}_+)^2\}$. We have $(13,2)\in \operatorname{bconv}\Delta_E$. In fact,

$$(13,2)=\frac{1}{2}(16,0)+\frac{1}{2^2}(16,0)+\frac{1}{2^3}(0,10)+\frac{1}{2^4}(16,0)+\frac{1}{2^4}(0,12)$$

and $(0,12)=(0,10)+(0,2)\in\Delta_E(f_\Gamma)\setminus\Gamma$. Now we have

$$\begin{aligned}\frac{1}{2^2}(16,0)+\frac{1}{2^3}(0,10)+\frac{1}{2^4}(16,0)+\frac{1}{2^4}(0,12)&=(5,2)\\ \frac{1}{2^2}(0,10)+\frac{1}{2^3}(16,0)+\frac{1}{2^3}(0,12)&=(2,4)\\ \frac{1}{2^2}(16,0)+\frac{1}{2^2}(0,12)&=(4,3)\\ \frac{1}{2}(0,12)&=(0,6).\end{aligned}$$

Thus we obtain that for any $\varepsilon_0>0$ there exists $M>0$

$$\begin{aligned}&\varepsilon_0\left(3x^{16}+y^{10}+y^{12}\right)-x^{13}y^2+My^{12}\\ &=\varepsilon_0(x^8-(2^{-1}\varepsilon_0^{-1})x^5y^2)^2+\varepsilon_0(x^8-(2^{-3}\varepsilon_0^{-2})x^2y^4)^2+\varepsilon_0(y^5-(2^{-7}\varepsilon_0^{-4})x^4y^3)^2\\ &\qquad+\varepsilon_0(x^8-(2^{-15}\varepsilon_0^{-8})y^6)^2+(\varepsilon_0+M-2^{-30}\varepsilon_0^{-15})y^{12}.\end{aligned}$$

is contained in $\sum\mathbb{R}[x]^2$. Therefore

$$\begin{aligned}f(x,y)&=x^{16}+y^{10}-x^{13}y^2-\varepsilon(x^{16}+y^{16})+\varepsilon(x^{16}+y^{10})\\ &\qquad-6^{-1}\varepsilon\left(3x^{16}+y^{10}+y^{12}\right)+6^{-1}\varepsilon\left(3x^{16}+y^{10}+y^{12}\right)-My^{12}+My^{12}\\ &=(1-\varepsilon)x^{16}+(1-\varepsilon)y^{10}+\varepsilon(1-2^{-1})x^{16}\\ &\qquad+\varepsilon y^{10}\left(1-6^{-1}-\varepsilon^{-1}(6^{-1}\varepsilon+M)y^2\right)\\ &\qquad+6^{-1}\varepsilon\left(3x^{16}+y^{10}+y^{12}\right)-x^{13}y^2+My^{12}\end{aligned}$$

is contained in $\sum\mathbb{R}[[x]]^2$.

Proof of Theorem, 4.12. For a maximal face γ of $\Gamma:=\Gamma(f)$, let s be the number of elements of $\operatorname{supp} f\cap(\operatorname{conv}\Delta(f_\gamma)\setminus\gamma)$. For arbitrary small $\varepsilon>0$ and each $\alpha\in\operatorname{supp} f\cap(\operatorname{conv}\Delta(f_\gamma)\setminus\gamma)$, Lemma 4.14 ensures that

$$\frac{\varepsilon}{s}f_\gamma+f_\alpha x^\alpha\in\sum\mathbb{R}[[x]]^2.$$

Therefore

$$\varepsilon f_\gamma+\sum_\alpha\{f_\alpha x^\alpha\mid \alpha\in\operatorname{supp} f\cap(\Delta(f_\gamma)\setminus\gamma)\}$$

is contained in $\sum \mathbb{R}[[x]]^2$. Since $\varepsilon > 0$ is an arbitrary small constant and $f_{\gamma'} \in \operatorname{rint}\left(\sum \mathbb{R}[x]^2_{\gamma}\right)$ for any face γ' of Γ, we conclude that

$$f = f_\Gamma - \varepsilon \sum_{\gamma} f_\gamma + \varepsilon \sum_{\gamma} f_\gamma + \sum_{\alpha} \{ f_\alpha x^\alpha \mid \alpha \in \operatorname{supp} f \setminus \Gamma \}$$

belongs to $\sum \mathbb{R}[[x]]^2$, where the first and second summations are taken with respect to every maximal face γ of Γ. □

4.3. Regularity of Newton polyhedra. In Theorem 4.12, Condition (4) is hard to check. However there are some kinds of Newton diagrams which the condition is automatically satisfied. In addition, it will be shown that when we use Theorem 4.12, we need to check the condition for only lower degree parts of polynomials. First we define a regularity property of Newton polyhedra.

Definition 4.16. *Let $f \in \mathbb{R}[x]$ with $f(0) = 0$. We say that f has a regular Newton polyhedron, if f satisfies that*

(1) *Every vertex of $\Gamma(f)$ is even;*

(2) *For each vertex α of $\Gamma(f)$, $f_\alpha > 0$;*

(3) *If for each maximal face γ of $\Gamma(f)$,*

$$\{\alpha \in \operatorname{supp} f \cap \operatorname{conv} \Delta(f_\gamma) \setminus \gamma \mid \alpha \textit{ is odd or } f_\alpha < 0\} \subset \operatorname{bconv} \Delta_E(f_\gamma).$$

With this regularity, Theorem 4.12 can be restated as follows:

Theorem 4.17. *Let $f \in \mathbb{R}[x]$ with $f(0) = 0$. Suppose that f has a regular Newton polyhedron. If we have $f_\gamma(x) \in \operatorname{rint}\left(\sum \mathbb{R}[x]^2_{\frac{1}{2}\gamma}\right)$ for each maximal face γ of $\Gamma(f)$, then $f \in \sum \mathbb{R}[[x]]^2$.*

The following proposition explains a different aspect of Lemma 4.3 that if a Newton diagram is included in the plane $|\alpha| = 2$ and meets all coordinate axes, its Newton polyhedron is regular.

Proposition 4.18. *Let $f \in \mathbb{R}[x]$. Suppose that*

$$\Gamma := \Gamma(f) = \left\{ \alpha \in \mathbb{Z}^n_+ \mid \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n = 2 \right\}.$$

If f_Γ is positive definite, then $\operatorname{conv} \Delta(f_\Gamma) \cap Z^n \subset \operatorname{bconv} \Delta_E(f_\Gamma)$ *and thus f has a regular Newton polyhedron.*

Proof. Let $f = \sum_k f_k$ be the expansion of its homogeneous components where $\deg f_k = k$. We note that the assumption is equivalent to that $f_0 = f_1 = 0$ and f_2 is positive definite. We show the conclusion by induction on the number of variables.

If $n = 1$, we can write $f = f_2 x^2 + \sum_{k=3}^{d} f_k x^k$ where $d = \deg f$. Then $f_2 > 0$ and $\operatorname{supp} f \subset \{2\} + \mathbb{Z}_+$. Thus $\operatorname{conv} \Delta(f_\Gamma) \cap \mathbb{Z} \subset \Delta_E(f_\Gamma)$.

Suppose that the conclusion holds for n. Let $f \in \mathbb{R}[x_1, \dots, x_{n+1}]$ be such that

$$\Gamma = \Gamma(f) = \{\alpha \in \mathbb{Z}_+^{n+1} \mid \alpha_1 + \cdots + \alpha_{n+1} = 2\}$$

and f_Γ is positive definite. Then for the canonical basis $\{e_i\}$ of $\mathbb{Z}^{n+1}$, we have $2e_i \in \Gamma \cap \operatorname{supp} f_2$ for $i = 1, \dots, n+1$. Clearly, f satisfies the condition (1) and (2) of Definition 4.16. Suppose $\alpha \in \operatorname{conv} \Delta(f_\Gamma) \cap \mathbb{Z}^{n+1}$.

Case $\alpha_{n+1} \geq 2$. Then

$$\alpha \in \{2e_{n+1}\} + \mathbb{Z}_+^{n+1} \subset \operatorname{supp} f_\Gamma \cap (2\mathbb{Z})^{n+1} + \mathbb{Z}_+^{n+1} \subset \Delta_E(f_\Gamma) \cap \mathbb{Z}_+^{n+1}.$$

Since $\Delta_E(f_\Gamma) \cap \mathbb{Z}_+^{n+1} \subset \operatorname{bconv} \Delta_E(f_\Gamma)$, we have $\alpha \in \operatorname{bconv} \Delta_E(f_\Gamma)$.

Case $\alpha_{n+1} = 1$. Then $\alpha = e_{n+1} + (\beta, 0)$ for some $\beta \in \mathbb{Z}_+^n$. Now we have

$$\alpha = \frac{1}{2}\{2e_{n+1} + (2\beta, 0)\}.$$

Since at least one component of 2β is greater than or equal to 2, the same arguments in the previous case implies that $(2\beta, 0) \in \Delta_E(f_\Gamma)$. In addition $2e_{n+1} \in \Delta_E(f_\Gamma)$ and thus $\alpha \in \operatorname{bconv} \Delta_E(f_\Gamma)$.

Case $\alpha_{n+1} = 0$. Then $\alpha = (\tilde{\alpha}, 0)$ for some $\tilde{\alpha} \in Z_+^n$. Define $\tilde{f} = f(x_1, \dots, x_n, 0)$. Then $\tilde{f} \in \mathbb{R}[x_1, \dots, x_n]$, $\tilde{f}_0 = \tilde{f}_1 = 0$ and $\tilde{f}_2$ is positive definite. Since $\{\alpha \in \operatorname{supp} f_2 \mid \alpha_{n+1} = 0\} = \operatorname{supp} \tilde{f}_2 \times \{0\}$, we have

$$\alpha \in (\operatorname{conv} \Delta(\tilde{f}_{\tilde{\Gamma}}) \cap \mathbb{Z}^n) \times \{0\} \subset \operatorname{bconv} \Delta_E(\tilde{f}_{\tilde{\Gamma}}) \times \{0\},$$

where the inclusion is implied by the induction hypothesis. Now we claim that $\Delta_E(\tilde{f}_{\tilde{\Gamma}}) \times \{0\} \subset \Delta_E(f_\Gamma)$. Let $\alpha' \in \Delta_E(\tilde{f}_{\tilde{\Gamma}}) \times \{0\}$. Then $\alpha' = (\beta + r, 0)$ for some $\beta \in \operatorname{supp} \tilde{f}_{\tilde{\Gamma}} \cap (2\mathbb{Z})^n$, $r \in \mathbb{R}_+^n$. Since $(\beta, 0) \in \operatorname{supp} f_\Gamma \cap (2\mathbb{Z})^{n+1}$, we have $(\beta + r, 0) = (\beta, 0) + (r, 0) \in \Delta_E(f_\Gamma)$. Thus

$$\alpha \in \operatorname{bconv} \Delta_E(\tilde{f}_{\tilde{\Gamma}}) \times \{0\} = \operatorname{bconv}(\Delta_E(\tilde{f}_{\tilde{\Gamma}}) \times \{0\}) \subset \operatorname{bconv} \Delta_E(f_\Gamma).$$

Therefore $\operatorname{conv}\Delta(f_\Gamma)\cap \subset \operatorname{bconv}\Delta_E(f_\Gamma)$. Since Γ is the unique maximal face of f, f has a regular Newton polyhedron. $\square$

In the case that a Newton diagram is contained in a plane, we can slightly relax a condition of Theorem 4.4 which means that it has to be parallel to the plane $|\alpha| = 2$.

Theorem 4.19. *Let $f \in \mathbb{R}[x]$. Suppose that*

$$\Gamma := \Gamma(f) = \{\alpha \in \mathbb{Z}_+^n \mid k\alpha_1 + \cdots + k\alpha_{n-1} + \alpha_n = 2k\}$$

for some $k \in \mathbb{Z}_+$. If $f_\Gamma \in \operatorname{rint}(\sum \mathbb{R}[x]^2_{\frac{1}{2}\Gamma})$, then f has a regular Newton polyhedron.

Proof. Suppose $\alpha = (\tilde{\alpha}, \alpha_n) \in (\operatorname{conv}\Delta(f_\Gamma)\cap\mathbb{Z}_+^n)\setminus\Gamma$. Then $k|\tilde{\alpha}| + \alpha_n > 2k$.

Case $|\tilde{\alpha}| \geq 2$. Let $\gamma = \Gamma \cap \{\alpha \in \mathbb{Z}_+^n \mid \alpha_n = 0\}$ Then $\gamma = \{(\alpha', 0) \in \mathbb{Z}_+^n \mid \alpha'_1 + \cdots + \alpha'_{n-1} = 2\}$ and γ is a face of Γ. In addition $\gamma = \Gamma(\tilde{f}) \times \{0\}$ where $\tilde{f}(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, 0)$. Let $\tilde{\gamma} = \Gamma(\tilde{f})$. Then $f_\gamma = \tilde{f}_{\tilde{\gamma}}$ and $\tilde{\alpha} \in \operatorname{conv}\Delta(\tilde{f}_{\tilde{\gamma}}) \cap \mathbb{Z}^{n-1}$. By Lemma 4.6, $f_\Gamma - \varepsilon p_\Gamma$ belongs to $\sum \mathbb{R}[x]^2$ for a sufficiently small $\varepsilon > 0$. Applying Theorem 3.2 to the face γ of Γ, we also have $f_\gamma - \varepsilon p_\gamma \in \sum \mathbb{R}[x]^2$. Then we have $\tilde{f}_{\tilde{\gamma}} = f_\gamma = (f_\gamma - \varepsilon p_\gamma) + \varepsilon p_\gamma$ is a positive definite quadratic form in $x_1, \ldots, x_{n-1}$. Thus Proposition 4.18 implies that

$$\begin{aligned}\alpha = (\tilde{\alpha}, 0) + (0, \ldots, 0, \alpha_n) &\in \operatorname{conv}\Delta(\tilde{f}_{\tilde{\gamma}}) \cap \mathbb{Z}^{n-1} \times \{0\} + \mathbb{Z}_+^n \\ &\subset \operatorname{conv}\Delta(\tilde{f}_{\tilde{\gamma}}) \cap \mathbb{Z}^{n-1} \times \mathbb{Z}_+ \subset \operatorname{bconv}\Delta_E(\tilde{f}_{\tilde{\gamma}}) \times \mathbb{Z}_+.\end{aligned}$$

Since $\operatorname{supp}\tilde{f}_{\tilde{\gamma}}\cap(2\mathbb{Z})^{n-1}\times\{0\} \subset \operatorname{supp} f_\gamma \cap (2\mathbb{Z})^n$, we have $\Delta_E(\tilde{f}_{\tilde{\gamma}})\times\mathbb{Z}_+ \subset \Delta_E(f_\gamma)$. Thus

$$\alpha \in \operatorname{bconv}\Delta_E(\tilde{f}_{\tilde{\gamma}}) \times \mathbb{Z}_+ \subset \operatorname{bconv}\Delta_E(f_\gamma) \subset \operatorname{bconv}\Delta_E(f_\Gamma).$$

Case $|\tilde{\alpha}| = 1$. Notice that $\alpha_n \geq k$ and there exists an unique index t such that $\alpha_t = 1$ and $\alpha_s = 0$ for $s \neq t$. Suppose that $t = 1$. Then we have

$$\alpha = (1, 0, \cdots, 0, \alpha_n) = \frac{1}{2}\{(2, 0, \cdots, 0) + (0, \cdots, 0, 2\alpha_n)\} \in \operatorname{bconv}\Delta_E(f_\Gamma).$$

The same argument gives the inclusion for the case $t = 2, \ldots, n$.

The case $|\tilde{\alpha}| = 0$ is obvious. $\square$

Example 4.20. Let $f(x, y, z) = x^2 + y^2 + xyz + yz^6 + z^{10}$. Then $\Gamma(f) = \{\alpha \in \mathbb{R}_+^3 \mid 5\alpha_1 + 5\alpha_2 + \alpha = 10\}$. Here the lowest form $g(x, y, z) = x^2 + y^2$ is only a positive semidefinite form and thus Lemma 4.3 can not be applied. However $f \in \sum \mathbb{R}[[x]]^2$ by Theorem 4.19 and Theorem 4.17. In fact, we can see it directly by

$$f = y^2\left(1 - \frac{3}{4}z^2\right) + \left(x + \frac{1}{2}yz\right)^2 + \frac{1}{2}z^{10} + \frac{1}{2}\left(yz + z^5\right)^2.$$

The following proposition ensures that the regularity of lower degree parts is enough for a polynomials to belong $\sum \mathbb{R}[[x]]^2$.

Theorem 4.21. *Suppose that $f \in \mathbb{R}[x]$ satisfies the following;*

(1) *$\Gamma(f)$ meets all coordinate axes;*

(2) *for each maximal face γ of $\Gamma(f)$, $f_\gamma(x) \in \operatorname{rint}\left(\sum \mathbb{R}[x]^2_{\frac{1}{2}\gamma}\right)$.*

If $\sum\{f_\alpha x^\alpha : |\alpha| \leq \deg(f_\Gamma) + 1\}$ has a regular Newton polyhedron, then we have $f \in \sum \mathbb{R}[[x]]^2$.

Proof. Let $d = \deg(f_\Gamma)$, $f_0 = \sum\{f_\alpha x^\alpha : |\alpha| \leq d + 1\}$. Then there exists $\varepsilon >$ such that $f_0 - \varepsilon p_\gamma \in \operatorname{rint}\left(\sum \mathbb{R}[x]^2_{\frac{1}{2}\gamma}\right)$ for each maximal face γ of $\Gamma(f)$. By Theorem 4.12, we have $f_0 - \varepsilon p_\gamma \in \sum \mathbb{R}[[x]]^2$. Thus we also have $f_0 - \varepsilon p_\Gamma \in \sum \mathbb{R}[[x]]^2$, by taking $\varepsilon > 0$ smaller if necessary.

Since d is even, Lemma 4.5 ensures that for any $K > 0$ there exists $M > 0$ such that

$$M \sum_i x_i^{d+2} + \sum_{|\alpha|=d+2} f_\alpha x^\alpha \in \operatorname{rint} \sum \mathbb{R}[x]^2_{\frac{d}{2}+1}.$$

$$f = (f_0 - \varepsilon p_\Gamma) + \left(\varepsilon p_\Gamma - M \sum_i x_i^{d+2}\right) + \left(M \sum_i x_i^{d+2} + \sum_{|\alpha|=d+2} f_\alpha x^\alpha + \sum_{|\alpha|>d+2} f_\alpha x^\alpha\right)$$

Since $\Gamma(f)$ meets all coordinate axes, the second parenthesis is contained in $\sum \mathbb{R}[[x]]^2$. By Theorem 4.4, the last parenthesis is contained in $\sum \mathbb{R}[x]^2$. □

As an easy consequence of Theorem 4.21, if the Newton diagram stays away from other exponents, regularity is not necessary to ensure $f \in \sum \mathbb{R}[[x]]^2$.

Corollary 4.22. *Suppose that* $f \in \mathbb{R}[x]$ *satisfies*

(1) $\Gamma(f)$ *meets all coordinate axes;*

(2) *for each maximal face* γ *of* $\Gamma(f)$, $f_\gamma(x) \in \operatorname{rint}\left(\sum \mathbb{R}[x]^2_{\frac{1}{2}\gamma}\right)$.

If the degree of each monomial in $f - f_\Gamma$ *is greater than* $\deg(f_\Gamma) + 1$, *then we have* $f \in \sum \mathbb{R}[[x]]^2$.

5. Constrained case. In this section, we seek a sufficient condition for $f \in \mathbb{R}[x]$ to belong to a quadratic module generated by several polynomials. Here we consider a *local order* on monomials in $\mathbb{R}[x]$. For example, the *anti-graded rex order* on $\mathbb{R}[x, y]$ is a local order satisfying that

$$1 > x > y > x^2 > xy > y^2.$$

For the detailed definition and discussion, see [4, Section 4.3]. For a given ordering, the *leading term* $\mathrm{LT}(f)$ of f be the maximal monomial appearing in f. The following theorem is well-known [4, Cor. 3.13 in Chap.4].

Theorem 5.1 (Mora's division)**.** *For* $f, g_i \in \mathbb{R}[x], i = 1, \dots, l$ *and a local order* $>$, *there exist* $u, q_i, r \in \mathbb{R}[x]$ *such that*

(1) $(1 + u)f = \sum_i q_i g_i + r$,

(2) $u(0) = 0$,

(3) $\mathrm{LT}(f) \geq \mathrm{LT}(q_i g_i)$ *for all* i,

(4) $\mathrm{LT}(r)$ *can not be divided by* $\mathrm{LT}(g_i)$ *for all* i.

Here we consider slightly modified version of the division.

Definition 5.2 (Modified Mora's division)**.** *After applying the Mora's division*

$$(1 + u)f = \sum_i q_i g_i + r,$$

let r_0 *be the polynomial obtained by eliminating all terms of* r *included in the ideal generated by the leading monomials of linear parts of* g_i, h_j. *For* $\Gamma := \Gamma(r_0)$, *let* $d = \deg(r_{0,\Gamma})$.

(1) *Divide further as*

$$(1+u')f = \sum_i q_i' g_i + r',$$

where any monomials of r' *with the degree* $\leq d+1$ *can not be divided by* $\mathrm{LT}(g_i)$ *for all* i.

(2) *Let* $\tilde{r}$ *be a polynomial obtained by eliminating all monomials of* r' *with degree* $> d+1$.

We call $\hat{r}$ *the essential remainder.*

For $f \in \mathbb{R}[x]$, we use the notation $f_z(x) := f(x+z) - f(z)$. Note that $f_z(0) = 0$. For $g_i \in \mathbb{R}[x], i = 1, \dots, l$, let $\langle g_1, \dots, g_l \rangle^{\sim} = \{\sum_i \tau_i g_i \mid \tau_i \in \mathbb{R}[[x]]\}$.

Theorem 5.3. *For a global minimizer* z *of* (POP), *let* $L = f - \sum_{i=1}^{l} \lambda_i g_i - \sum_{j=1}^{m} \mu_j h_j$ *with* $\lambda_i \geq 0, \mu_j \in \mathbb{R}$ *satisfying* $\nabla L(z) = 0$ *and* $\lambda_i g_i(z) = 0$. *Suppose that for a local order, an essential remainder* $\tilde{r}$ *of modified Mora's division of*

$$L_z \text{ by } \{\lambda_i g_{i,z}, h_{j,z} \bigm| \lambda_i \nabla g_i(z) \neq 0\}.$$

satisfies the following:

(1) $\Gamma = \Gamma(\tilde{r})$ *meets all coordinate axes of appearing variables in* $\tilde{r}$.

(2) $\forall \gamma \in \Gamma$, $\tilde{r}_\gamma \in \operatorname{rint} \sum \mathbb{R}[x]^2_{\frac{\gamma}{2}}$

(3) $\tilde{r}$ *has a regular Newton polyhedron.*

Then we have $f \in \widetilde{M}(g_{1,z}, \dots, g_{l,z}) + \langle h_{1,z}, \dots, h_{m,z} \rangle^{\sim}$.

Proof. For a global minimizer z, let $I = \{i \mid \lambda_i \nabla g_i(z) \neq 0\}$. By the modified Mora's division, there exist $u, p_i, q_j, \tilde{r}, w \in \mathbb{R}[x]$ such that $u(0) = 0$ and

$$(1+u)L_z = \sum_{i \in I} p_i \lambda_i g_{i,z} + \sum_{j=1}^{m} q_j h_{j,z} + \tilde{r} + w,$$

where $\mathrm{LT}(L_z) \geq \mathrm{LT}(p_i \lambda_i g_{i,z}), \mathrm{LT}(q_i h_{j,z})$ in the local order, each monomial of $\tilde{r}$ can not be divided by $\mathrm{LT}(g_{i,z}), \mathrm{LT}(h_{j,z})$ and $w \in \langle g_{i,z}, h_{j,z} \rangle_{i,j}$ and the least degree of $w \geq d+2$, where d is the number given in the definition of the modified Mora's division. Since $L(z) = 0, \nabla L(z) = 0$, we have $\deg(\mathrm{LT}(L_z)) \geq 2$. Then the least degree of the monomials of $p_i \lambda g_{i,z} \geq 2$ for all $i \in I$. Thus the least degree of monomials in $p_i \geq 1$ and hence $p_i(0) = 0$ for all $i \in I$.

Further by the Division theorem in $\mathbb{R}[[x]]$ [6, Theorem 6.4.1], there exist $p', q', r' \in \mathbb{R}[[x]]$ such that

$$w = \sum_{i \in I} p'_i \lambda_i g_{i,z} + \sum_j q'_j h_{j,z} + r',$$

where each monomial of r' can not be divided by $\mathrm{LT}(\lambda_i g_i), \mathrm{LT}(h_j)$ and the least degree of $r' \geq d+2$. Similarly, we have $p'_i(0) = 0$. Then

$$\begin{aligned} f_z &= \sum_{i=1}^{l} \lambda_i g_{i,z} + \sum_{j=1}^{m} \mu_j h_{j,z} + L_z \\ &= \sum_{i \in I} \lambda_i \left(1 + \frac{p_i + p'_i}{1+u}\right) g_{i,z} + \sum_{j=1}^{m} \left(\mu_j + \frac{q_j + q'_j}{1+u}\right) h_{j,z} \\ &\qquad + \sum_{i \notin I} \lambda_i g_{i,z} + \frac{\tilde{r} + r'}{1+u}. \end{aligned}$$

Since $\tilde{r}+r'$ is contained in $\sum \mathbb{R}[x]^2$ by Theorem 4.21, we have $f_z \in \widetilde{M}(g_{1,z}, \ldots, g_{l,z}) + \langle h_{1,z}, \ldots, h_{m,z} \rangle^{\sim}$. $\square$

Example 5.4.

$$\begin{aligned} &\min\ f = x^3 + y^3 + z^2 + w^4 + 2 \\ &\text{s.t.}\quad g = 2 - x^4 - y^4 - z^4 - w^4 \geq 0 \end{aligned}$$

The optimal is $a = (-1, -1, 0, 0)$. We have

$$\nabla f(a) = \frac{3}{4} \nabla g(a), \quad \nabla^2 \left(f - \frac{3}{4} g\right)(a) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $\nabla^2 \left(f - \frac{3}{4} g\right)(a)$ is not positive definite on the subspace

$$\nabla g(a)^{\perp} = \left\langle \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \end{bmatrix} \right\rangle^{\perp} = \left\langle \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle,$$

and hence the second order condition is not satisfied. Let $>$ be the anti-graded rex order. We have

$$\begin{aligned} f_a &= 3x + 3y - 3x^2 - 3y^2 + z^2 + x^3 + y^3 + w^4 \\ g_a &= 4x + 4y - 6x^2 - 6y^2 + 4x^3 + 4y^3 - x^4 - y^4 - z^4 - w^4, \end{aligned}$$

and the remainder of f_a by g_a is

$$\begin{aligned} r = 3y^2 + z^2 + \frac{1}{4}x^3 - \frac{9}{4}x^2y + \frac{9}{4}xy^2 - \frac{17}{4}y^3 \\ - \frac{3}{4}x^4 + \frac{3}{2}x^3y - \frac{3}{2}xy^3 + \frac{9}{4}y^4 + \frac{3}{4}z^4 + \frac{7}{4}w^4 + \frac{3}{8}x^5 - \frac{3}{8}x^4y \\ + \frac{3}{8}xy^4 + \frac{3}{8}xz^4 + \frac{3}{8}xw^4 - \frac{3}{8}y^5 - \frac{3}{8}yz^4 - \frac{3}{8}yw^4. \end{aligned}$$

By eliminating terms of r contained in $\mathrm{LT}\langle g_a \rangle = \langle x \rangle$, we obtain

$$r_0 = 3y^2 + z^2 - \frac{17}{4}y^3 + \frac{9}{4}y^4 + \frac{3}{4}z^4 + \frac{7}{4}w^4 - \frac{3}{8}y^5 - \frac{3}{8}yz^4 - \frac{3}{8}yw^4.$$

For $\Gamma := \Gamma(r_0)$,

$$r_{0,\Gamma} = 3y^2 + z^2 + \frac{7}{4}w^4$$

and $\deg r_{0,\Gamma} = 4$, Then the essential remainder $\widehat{r} = r_0$ and $\widehat{r}_\Gamma = r_{0,\Gamma} \in \operatorname{rint} \sum \mathbb{R}[x,y,z,w]^2_{\frac{1}{2}\Gamma}$. Since the Newton diagram of $\tilde{r}$ satisfies the conditions of Theorem 4.19, $\tilde{r}$ has a regular Newton polyhedron. By Theorem 5.3, we have $f \in \widetilde{M}(g)$.

REFERENCES

[1] S. Burgdorf, C. Scheiderer, M. Schweighofer. Pure states, nonnegative polynomials and sums of squares. *Comment. Math. Helv.* **87**, *1* (2012), 113–140.

[2] M. D. Choi, T. Y. Lam, B. Reznick. Sums of squares of real polynomials. K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992), 103–126, Proc. Sympos. Pure Math., **58**, Part 2, Amer. Math. Soc., Providence, RI, 1995.

[3] J. Cimprič, M. Marshall, T. Netzer. Closure of quadratic modules. *Israel J. Math.* **183** (2011), 445–474.

[4] D. Cox, J. Little, D. O'Shea. Using algebraic geometry, Second edition. Graduate Texts in Mathematics, vol. **185**, Springer, New York, 2005.

[5] M. Ghasemi, M. Marshall. Lower bounds for a polynomials in terms of its coefficients. *Arch. Math. (Basel)* **95**, *4* (2010), 343–353.

[6] G.-M. Greuel, G. Pfister. A Singular Introduction to Commutative Algebra, 2nd extended edition. Springer, Berlin, 2008.

[7] M. Marshall. Positive Polynomials and Sums of Squares. Mathematical Surveys and Monographs, vol. **146**. American Mathematical Society, Providence, RI, 2008.

[8] J. Nie. Optimality conditions and finite convergence of Lasserre's hierarchy, *Math. Program.* **146**, *1–2* (2014), 97–121.

[9] H. H. Scheafer. Topological Vector Spaces, 2nd edition. Graduate Texts in Mathematics, Springer, New York, 1999.

[10] V. A. Vasil'ev. Asymptotic exponential integrals, Newton's diagram, and the classification of minimal points. *Funktsional. Anal. Prilozh.* **11**, *3* (1977), 1–11 (in Russian); English translation in *Funct. Anal. Appl.* **11** (1977), 163–172.

Graduate School of Marine Science and Technology
Tokyo University of Marine Science and Technology
Etchujima 2-1-8, Koto, Tokyo 135-8533, Japan
e-mail: `yoshi-s@kaiyodai.ac.jp`

Received August 28, 2014
Revised March 23, 2015