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ON THE POISSON-CHARLIER POLYNOMIALS

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Abstract. In this paper the Poisson-Charlier polynomials are introduced. Some of their recurrence relations are presented. Various families of bilinear and bilateral generating functions for these polynomials are derived. Furthermore, some special cases of the results are presented in this study.

1. Introduction. The Poisson-Charlier polynomials $c_n(\alpha;x)$ are defined explicitly by [4, 8, 9, 12, 14]

$$
\begin{aligned}
c_n(\alpha;x) &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{\alpha}{k}k!x^{-k} \\
&= {}_2F_0\left(-n,-\alpha;-;-\frac{1}{x}\right) \\
(x &> 0,\ \alpha\in\mathbb{N}_0:=\mathbb{N}\cup\{0\})
\end{aligned}
\tag{1.1}
$$

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where ${}_2F_0$ is a case of the generalized hypergeometric function ${}_pF_q$ defined by

$$
\begin{aligned}
{}_pF_q\left[\begin{array}{ll}\alpha_1,\ldots,\alpha_p; & z\\ \beta_1,\ldots,\beta_q; & \end{array}\right] &= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ \cdots\ (\alpha_p)_n}{(\beta_1)_n\ \cdots\ (\beta_q)_n}\frac{z^n}{n!}\\
&= {}_pF_q\left(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z\right).
\end{aligned}
$$

Here, as usual, $(\lambda)_n$ denotes the Pochhammer symbol given by

$$(\lambda)_n := \lambda(\lambda+1)\cdots(\lambda+n-1)\ \ (n\in\mathbb{N})\ \text{ and } (\lambda)_0 := 1.$$

The Poisson-Charlier polynomials have the following generating function (see, for instance, [4]):

$$\sum_{n=0}^{\infty} c_n(\alpha;x)\frac{t^n}{n!} = \left(1-\frac{t}{x}\right)^{\alpha}\exp(t). \tag{1.2}$$

It is well-known that these polynomials are a family of orthogonal polynomials satisfying the following relation:

$$\sum_{\alpha=0}^{\infty}\frac{x^{\alpha}}{\alpha!}c_n(\alpha;x)c_m(\alpha;x) = x^{-n}e^{x}n!\delta_{nm},\quad x>0.$$

It is also known that

$$(-1)^n L_n^{(\alpha-n)}(x) = \frac{x^n}{n!}c_n(\alpha;x), \tag{1.3}$$

which indicates a relationship between the Poisson-Charlier polynomials and the Laguerre polynomials $L_n^{(\alpha)}(x)$ (see [4]). Hence, using the relation (1.3) and taking into account the general properties of the Laguerre polynomials it is possible to obtain some other properties for the Poisson-Charlier polynomials. In addition, setting $\alpha \to (2x)^{1/2}\alpha + x$ and letting $x\to\infty$, the Hermite polynomials are obtained from the Poisson-Charlier polynomials, that is,

$$\lim_{x\to\infty}(2x)^{n/2}c_n\left((2x)^{1/2}\alpha+x;x\right) = (-1)^n H_n(\alpha).$$

We also know another generating function relation for the Poisson-Charlier polynomials as follows (see [8, 10]):

$$\sum_{n=0}^{\infty} c_{n+m}(\alpha;x)\frac{t^n}{n!} = \left(1-\frac{t}{x}\right)^{\alpha}\exp(t)c_m(\alpha;x-t). \tag{1.4}$$

Some other properties may be found in the paper [6].

On the other hand, Srivastava-Daoust (or generalized Lauricella) function (see [11]), which is a generalization of the Kampé de Fériet function in two variables, is defined by

$$F^{A:B^{(1)};...;B^{(n)}}_{C:D^{(1)};...;D^{(n)}}\left(\begin{array}{l}\left[(a):\theta^{(1)},...,\theta^{(n)}\right]:\left[(b^{(1)}):\phi^{(1)}\right];...;\left[(b^{(n)}):\phi^{(n)}\right];\\ \\ \left[(c):\psi^{(1)},...,\psi^{(n)}\right]:\left[(d^{(1)}):\delta^{(1)}\right];...;\left[(d^{(n)}):\delta^{(n)}\right];\end{array} z_1,...,z_n\right)$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\Omega(m_1,\ldots,m_n)\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_n^{m_n}}{m_n!},$$

where, for convenience,

$$\Omega(m_1,\ldots,m_n):=\frac{\prod_{j=1}^{A}(a_j)_{m_1\theta_j^{(1)}+\cdots+m_n\theta_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\psi_j^{(1)}+\cdots+m_n\psi_j^{(n)}}}\frac{\prod_{j=1}^{B^{(1)}}(b_j^{(1)})_{m_1\phi_j^{(1)}}}{\prod_{j=1}^{D^{(1)}}(d_j^{(1)})_{m_1\delta_j^{(1)}}}\cdots\frac{\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\phi_j^{(n)}}}{\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}},$$

the coefficients

$$\theta_j^{(k)}\ (j=1,\ldots,A;\ k=1,\ldots,n),\ \text{and}\ \phi_j^{(k)}\ (j=1,\ldots,B^{(k)};\ k=1,\ldots,n),$$

$$\psi_j^{(k)}\ (j=1,\ldots,C;\ k=1,\ldots,n),\ \text{and}\ \delta_j^{(k)}\ (j=1,\ldots,D^{(k)};\ k=1,\ldots,n)$$

are real constants and $\left(b^{(k)}_{B^{(k)}}\right)$ abbreviates the array of $B^{(k)}$ parameters

$$b_j^{(k)}\ (j=1,\ldots,B^{(k)};\ k=1,\ldots,n)$$

with similar interpretations for other sets of parameters [7].

This paper concerns with the following main objectives:

- obtaining theorems giving multilinear and multilateral generating function relations for the Poisson-Charlier polynomials and discussing their special cases,
- deriving various recurrence relations for the Poisson-Charlier polynomials
- getting a new kind of bilateral generating function between the Poisson-Charlier polynomials and the Srivastava-Daoust function.

2. Multilinear and Multilateral Generating Functions. In this section, firstly we derive several families of bilinear and bilateral generating functions for the Poisson-Charlier polynomials $c_n(\alpha; x)$ which are generated by (1.2) and given explicitly by (1.1) by using the similar method considered in [2, 1, 5, 13].

Theorem 2.1. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, \ldots, y_r)$ of r complex variables $y_1, \ldots, y_r$ $(r \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\psi}(y_1, \ldots, y_r; \zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, \ldots, y_r)\zeta^k$$

where $(a_k \neq 0\,,\ \mu, \psi \in \mathbb{C})$ and

$$\Theta_{n,p}^{\mu,\psi}(\alpha, x; y_1, \ldots, y_r; \xi) := \sum_{k=0}^{[n/p]} a_k c_{n-pk}(\alpha; x)\Omega_{\mu+\psi k}(y_1, \ldots, y_r)\frac{\xi^k}{(n-pk)!}.$$

Then, for $p \in \mathbb{N}$; we have

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}\left(\alpha, x; y_1, \ldots, y_r; \frac{\eta}{t^p}\right) t^n = \left(1 - \frac{t}{x}\right)^{\alpha} \exp(t)\Lambda_{\mu,\psi}(y_1, \ldots, y_r; \eta) \tag{2.1}$$

provided that each member of (2.1) exists.

Proof. For convenience, let S denote the first member of the assertion (2.1) of Theorem 2.1. Then,

$$S = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k c_{n-pk}(\alpha; x)\Omega_{\mu+\psi k}(y_1, \ldots, y_r)\eta^k \frac{t^{n-pk}}{(n-pk)!}.$$

Replacing n by $n + pk$, we may write that

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k c_n(\alpha; x)\Omega_{\mu+\psi k}(y_1, \ldots, y_r)\eta^k \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} c_n(\alpha; x)\frac{t^n}{n!}\sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, \ldots, y_r)\eta^k \\ &= \left(1 - \frac{t}{x}\right)^{\alpha} \exp(t)\Lambda_{\mu,\psi}(y_1, \ldots, y_r; \eta) \end{aligned}$$

which completes the proof. □

By using a similar idea, we also get the next result immediately.

Theorem 2.2. *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, \ldots, y_r)$ of r complex variables $y_1, \ldots, y_r$ ($r \in \mathbb{N}$) and of complex order μ, let*

$$\Lambda_{\mu,p,q}[\alpha, x; y_1, \ldots, y_r; t] := \sum_{n=0}^{\infty} a_n c_{m+qn}(\alpha; x)\Omega_{\mu+pn}(y_1, \ldots, y_r)\frac{t^n}{(nq)!}$$

where ($a_n \neq 0$, $\mu \in \mathbb{C}$) *and*

$$\theta_{n,p,q}(y_1, \ldots, y_r; z) := \sum_{k=0}^{[n/q]} \binom{n}{n-qk} a_k \Omega_{\mu+pk}(y_1, \ldots, y_r) z^k.$$

Then, for $p \in \mathbb{N}$; *we have*

$$\sum_{n=0}^{\infty} c_{m+n}(\alpha; x)\theta_{n,p,q}(y_1, \ldots, y_r; z)\frac{t^n}{n!} = \left(1 - \frac{t}{x}\right)^{\alpha} \exp(t)\Lambda_{\mu,p,q}(\alpha, x-t; y_1, \ldots, y_r; zt^q) \tag{2.2}$$

provided that each member of (2.2) *exists.*

Proof. For convenience, let T denote the first member of the assertion (2.2) of Theorem 2.2. Then,

$$T = \sum_{n=0}^{\infty} c_{m+n}(\alpha; x) \sum_{k=0}^{[n/q]} \binom{n}{n-qk} a_k \Omega_{\mu+pk}(y_1, \ldots, y_r) z^k \frac{t^n}{n!}.$$

Replacing n by $n + qk$, we may write that

$$\begin{aligned}
T &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \binom{n+qk}{n} c_{m+n+qk}(\alpha; x) a_k\, \Omega_{\mu+pk}(y_1, \ldots, y_r) z^k \frac{t^{n+qk}}{(n+qk)!} \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \binom{n+qk}{n} c_{m+n+qk}(\alpha; x)\frac{t^n}{(n+qk)!} \right) a_k \Omega_{\mu+pk}(y_1, \ldots, y_r)(zt^q)^k \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} c_{m+n+qk}(\alpha; x)\frac{t^n}{n!} \right) a_k \Omega_{\mu+pk}(y_1, \ldots, y_r)\frac{(zt^q)^k}{(qk)!} \\
&= \sum_{k=0}^{\infty} \left(1 - \frac{t}{x}\right)^{\alpha} \exp(t) c_{m+qk}(\alpha; x-t)\, a_k \Omega_{\mu+pk}(y_1, \ldots, y_r)\frac{(zt^q)^k}{(qk)!} \\
&= \left(1 - \frac{t}{x}\right)^{\alpha} \exp(t) \sum_{k=0}^{\infty} a_k c_{m+qk}(\alpha; x-t)\, \Omega_{\mu+pk}(y_1, \ldots, y_r)\frac{(zt^q)^k}{(qk)!}
\end{aligned}$$

$$= \left(1-\frac{t}{x}\right)^{\alpha}\exp(t)\Lambda_{\mu,p,q}\left(\alpha,x-t;y_1,\ldots,y_r;zt^q\right)$$

which completes the proof. □

3. Special Cases. As an application of the above theorems, when the multivariable function $\Omega_{\mu+\psi k}(y_1,\ldots,y_r)$, $k\in\mathbb{N}_0$, $r\in\mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set

$$r=1 \text{ and } \Omega_{\mu+\psi k}(y) = P_{\mu+\psi k}(y)$$

in Theorem 2.1, where $P_n(x)$ [12] are the Legendre polynomials generated by

$$\left(1-2xt+t^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{3.1}$$

We are thus led to the following result which provides a class of bilateral generating functions for the Legendre polynomials and the Poisson-Charlier polynomials given explicitly by (1.1).

Corollary 3.3. *If*

$$\Lambda_{\mu,\psi}(y;\zeta) := \sum_{k=0}^{\infty} a_k P_{\mu+\psi k}(y)\zeta^k \ , a_k\neq 0, \quad \mu,\psi\in\mathbb{C} \ ,$$

then, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k c_{n-pk}\left(\alpha;x\right)P_{\mu+\psi k}(y)\frac{\eta^k}{t^{pk}}\frac{t^n}{(n-pk)!} \tag{3.2}$$
$$= \left(1-\frac{t}{x}\right)^{\alpha}\exp(t)\Lambda_{\mu,\psi}(y;\eta)$$

provided that each member of (3.2) *exists.*

Remark 3.1. Using the generating relation (3.1) for the Legendre polynomials and getting $a_k=1$, $\mu=0$, $\psi=1$, we find that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} c_{n-pk}\left(\alpha;x\right)P_k(y)\eta^k\frac{t^{n-pk}}{(n-pk)!}$$
$$= \left(1-\frac{t}{x}\right)^{\alpha}\exp(t)\left(1-2y\eta+\eta^2\right)^{-1/2}.$$

If we set $r = 1$ and $\Omega_{\mu+\psi k}(y) = c_{\mu+\psi k}(\beta; y)$, $(y > 0,\ \beta = 0, 1, 2, \dots)$ in Theorem 2.2, we have the bilinear generating function relation for the Poisson-Charlier polynomials.

Furthermore, for every suitable choice of the coefficients a_i $(i \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\psi k}(y_1, \dots, y_s)$, $(s \in \mathbb{N})$, is expressed as an appropriate product of several simpler functions, the assertions of Theorems 2.1 and 2.2 can be applied in order to derive various families of multilinear and multilateral generating functions for the Poisson-Charlier polynomials.

4. Miscellaneous Properties. In this section we give some properties for the Poisson-Charlier polynomials $c_n(\alpha; x)$ given by (1.1).

Firstly, if we use (1.3) and the relation between Jacobi and Laguerre polynomials [10]

$$L_n^{(\alpha)}(x) = \lim_{\beta\to\infty}\left\{P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right)\right\}$$

we have

$$c_n(\alpha; x) = \lim_{\beta\to\infty}\left\{(-x)^{-n} n! P_n^{(\alpha-n,\beta)}\left(1-\frac{2x}{\beta}\right)\right\}, \tag{4.1}$$

which gives a relationship between the Poisson-Charlier polynomials and the Jacobi polynomials. In addition, if we use (4.1) and the relation between Jacobi and Lagrange polynomials [3]

$$g_n^{(\alpha,\beta)}(x, y) = (y - x)^{-n} P_n^{(-\alpha-n,-\beta-n)}\left(\frac{x+y}{x-y}\right),$$

where the Lagrange polynomials $g_n^{(\alpha,\beta)}(x, y)$ are defined through the generating function

$$(1 - xt)^{-\alpha}(1 - yt)^{-\beta} = \sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x, y)t^n\ ,\ \ (|t| < \min\left\{|x|^{-1}, |y|^{-1}\right\})$$

which occur in certain problems in statistics [4], we get

$$c_n(\alpha; x) = \lim_{\beta\to\infty}\left\{\left(\frac{-x^2}{y\beta}\right)^{-n} n! g_n^{(-\alpha,-\beta-n)}\left(y(1-\frac{\beta}{x}), y\right)\right\}.,$$

which gives a relationship between the Poisson-Charlier polynomials and the Lagrange polynomials.

We now discuss some miscellaneous recurrence relations of the Poisson-Charlier polynomials. By differentiating each member of the generating function relation (1.2) with respect to x and using

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k),$$

we arrive at the following (differential) recurrence relations for the Poisson-Charlier polynomials:

$$\frac{c_n'(\alpha;x)}{n!} = \alpha \sum_{m=0}^{n-1} x^{m-n-1}\frac{c_m(\alpha;x)}{m!}\ , \quad n \geq 1, \tag{4.2}$$

and

$$x^2\frac{c_n'(\alpha;x)}{n!} - x\frac{c_{n-1}'(\alpha;x)}{(n-1)!} = \alpha\frac{c_{n-1}(\alpha;x)}{(n-1)!}. \tag{4.3}$$

If we compare (4.2) and (4.3), we get

$$\sum_{m=0}^{n-1} x^{m-n+1}\frac{c_m(\alpha;x)}{m!} - \sum_{m=0}^{n-2} x^{m-n-1}\frac{c_m(\alpha;x)}{m!} = \frac{c_{n-1}(\alpha;x)}{(n-1)!}.$$

Besides, by differentiating each member of the generating function relation (1.2) with respect to t, we have the recurrence relation

$$xc_{n+1}(\alpha;x) - nc_{n-1}(\alpha;x) = (n-\alpha+x)c_n(\alpha;x),$$

for the Poisson-Charlier polynomials.

5. Another bilateral generating function relation. For a suitably bounded non-vanishing multiple sequence $\{\Omega(m_1;m_2,\ldots,m_s)\}_{m_1,m_2,\ldots,m_s\in\mathbb{N}_0}$ of real or complex parameters, we define a function $\phi_n(u_1;u_2,\ldots,u_s)$ of s (real or complex) variables $u_1;u_2,\ldots,u_s$ by

$$\begin{aligned}\phi_n(u_1;u_2,\ldots,u_s) \ :&= \sum_{m_1=0}^{n}\ \sum_{m_2,\ldots,m_s=0}^{\infty} \frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}} \\ &\quad \times\Omega\left(m_1\theta^{(1)}+\cdots+m_s\theta^{(s)};m_2,\ldots,m_s\right)\frac{u_1^{m_1}}{m_1!}\cdots\frac{u_s^{m_s}}{m_s!}\end{aligned}$$

where,

$$((b))_{m_1\phi} = \prod_{j=1}^{B}(b_j)_{m_1\phi_j} \text{ and } ((d))_{m_1\delta} = \prod_{j=1}^{D}(d_j)_{m_1\delta_j}.$$

Theorem 5.1. *The following bilateral generating function holds true:*

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_n(\alpha; x)\phi_n(u_1; u_2, \ldots, u_s)\frac{t^n}{n!} \\
= \ &\left(1-\frac{t}{x}\right)^{\alpha} \exp(t) \sum_{k,m_1,m_2,\ldots,m_s=0}^{\infty} \frac{((b))_{(m_1+k)\phi}(-\alpha)_k}{((d))_{(m_1+k)\delta}} \\
&\times\Omega((m_1+k)\theta^{(1)} + \cdots + m_s\theta^{(s)}; m_2, \ldots, m_s) \\
&\times\frac{(-u_1t)^{m_1}}{m_1!}\frac{\left(-\dfrac{u_1t}{x-t}\right)^k}{k!}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}.
\end{aligned}$$

Proof. By using the relationship (1.4), it is easily observed that

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_n(\alpha; x)\phi_n(u_1; u_2, \ldots, u_s)\frac{t^n}{n!} \\
= \ &\sum_{n=0}^{\infty} c_n(\alpha; x)\left(\sum_{m_1=0}^{n}\sum_{m_2,\ldots,m_s=0}^{\infty}\frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}}\right. \\
&\left.\times\ \Omega\left(m_1\theta^{(1)} + \cdots + m_s\theta^{(s)}; m_2, \ldots, m_s\right)\frac{u_1^{m_1}}{m_1!}\cdots\frac{u_s^{m_s}}{m_s!}\right)\frac{t^n}{n!} \\
= \ &\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}}.\Omega\left(m_1\theta^{(1)} + \cdots + m_s\theta^{(s)}; m_2, \ldots, m_s\right) \\
&\times\frac{(-u_1t)^{m_1}}{m_1!}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}\left(1-\frac{t}{x}\right)^{\alpha}\exp(t)c_{m_1}(\alpha; x-t) \\
= \ &\left(1-\frac{t}{x}\right)^{\alpha}\exp(t)\sum_{m_1,m_2,\ldots,m_s=0}^{\infty}\frac{((b))_{m_1\phi}}{((d))_{m_1\delta}} \\
&\times\Omega\left(m_1\theta^{(1)} + \cdots + m_s\theta^{(s)}; m_2, \ldots, m_s\right) \\
&\times\frac{(-u_1t)^{m_1}}{m_1!}\frac{u_2^{m_2}}{m_2!}\cdots\frac{u_s^{m_s}}{m_s!}\sum_{k=0}^{m_1}(-1)^k\binom{m_1}{k}\binom{\alpha}{k}k!(x-t)^{-k} \\
= \ &\left(1-\frac{t}{x}\right)^{\alpha}\exp(t)\sum_{k,m_1,m_2,\ldots,m_s=0}^{\infty}\frac{((b))_{(m_1+k)\phi}}{((d))_{(m_1+k)\delta}} \\
&\times\Omega\left((m_1+k)\theta^{(1)} + \cdots + m_s\theta^{(s)}; m_2, \ldots, m_s\right)(-\alpha)_k
\end{aligned}$$

$$\times \frac{(-u_1 t)^{m_1}}{m_1!} \frac{\left(-\frac{u_1 t}{x-t}\right)^k}{k!} \frac{u_2^{m_2}}{m_2!} \cdots \frac{u_s^{m_s}}{m_s!}.$$

Thus, the proof of Theorem 5.1 is completed. □

By appropriately choosing the multiple sequence $\Omega(m_1, m_2, \ldots, m_s)$ in Theorem 5.1, we obtain several interesting results including, for example, the following bilateral generating functions.

I. By letting

$$\Omega\left(m_1\theta^{(1)} + \cdots + m_s\theta^{(s)}; m_2, \ldots, m_s\right)$$
$$= \frac{\prod_{j=1}^{A} (a_j)_{m_1\theta_j^{(1)}+\cdots+m_s\theta_j^{(s)}}}{\prod_{j=1}^{E} (c_j)_{m_1\psi_j^{(1)}+\cdots+m_s\psi_j^{(s)}}} \frac{\prod_{j=1}^{B^{(2)}} (b_j^{(2)})_{m_2\phi_j^{(2)}}}{\prod_{j=1}^{D^{(2)}} (d_j^{(2)})_{m_2\delta_j^{(2)}}} \cdots \frac{\prod_{j=1}^{B^{(s)}} (b_j^{(s)})_{m_s\phi_j^{(s)}}}{\prod_{j=1}^{D^{(s)}} (d_j^{(s)})_{m_s\delta_j^{(s)}}}$$

in Theorem 5.1, we obtain the following result:

Corollary 5.2. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} c_n(\alpha; x) F^{A:B+1;B^{(2)};\ldots;B^{(s)}}_{E:D;D^{(2)};\ldots;D^{(s)}}$$
$$\left(\begin{array}{l} \left[(a) : \theta^{(1)}, \ldots, \theta^{(s)}\right] : \ [-n : 1], [(b) : \phi]; \ [(b^{(2)}) : \phi^{(2)}]; \ \ldots; \ [(b^{(s)}) : \phi^{(s)}]; \\ \left[(c) : \psi^{(1)}, \ldots, \psi^{(s)}\right] : \qquad [(d) : \delta]; \ [(d^{(2)}) : \delta^{(2)}]; \ \ldots; \ [(d^{(s)}) : \delta^{(s)}]; \end{array} \ u_1, u_2, \ldots, u_s\right) \frac{t^n}{n!}$$
$$= \left(1 - \frac{t}{x}\right)^{\alpha} \exp(t) F^{A+B:0;1;B^{(2)};\ldots;B^{(s)}}_{E+D:0;0;D^{(2)};\ldots;D^{(s)}}$$
$$\left(\begin{array}{l} \left[(e) : \varphi^{(1)}, \ldots, \varphi^{(s+1)}\right] : -; \ [-\alpha : 1]; \ [(b^{(2)}) : \phi^{(2)}]; \ \ldots; \ [(b^{(s)}) : \phi^{(s)}]; \\ \left[(f) : \zeta^{(1)}, \ldots, \zeta^{(s+1)}\right] : -; \quad -; \quad [(d^{(2)}) : \delta^{(2)}]; \ \ldots; \ [(d^{(s)}) : \delta^{(s)}]; \end{array} \ (-u_1 t), (-\frac{u_1 t}{x-t}), u_2, \ldots, u_s\right),$$

where the coefficients $e_j, f_j, \varphi_j^{(k)}$ *and* $\Theta_j^{(k)}$ *are given by*

$$e_j = \begin{cases} a_j & (1 \leq j \leq A) \\ b_{j-A} & (A < j \leq A+B), \end{cases}$$

$$f_j = \begin{cases} c_j & (1 \leq j \leq E) \\ d_{j-E} & (E < j \leq E+D), \end{cases}$$

$$\varphi_j^{(k)} = \begin{cases} \theta_j^{(1)} & (1 \leq j \leq A;\ 1 \leq k \leq 2) \\ \theta_j^{(k-1)} & (1 \leq j \leq A;\ 2 < k \leq s+1) \\ \phi_{j-A} & (A < j \leq A+B;\ 1 \leq k \leq 2) \\ 0 & (A < j \leq A+B;\ 2 < k \leq s+1) \end{cases}$$

and

$$\zeta_j^{(k)} = \begin{cases} \psi_j^{(1)} & (1 \leq j \leq E;\ 1 \leq k \leq 2) \\ \psi_j^{(k-1)} & (1 \leq j \leq E;\ 2 < k \leq s+1) \\ \delta_{j-E} & (E < j \leq E+D;\ 1 \leq k \leq 2) \\ 0 & (E < j \leq E+D;\ 2 < k \leq s+1), \end{cases}$$

respectively.

II. Upon setting

$$\Omega\left(m_1\theta^{(1)} + \cdots + m_s\theta^{(s)}; m_2, \ldots, m_s\right) = \frac{(a)_{m_1+\cdots+m_s}(b_2)_{m_2}\ldots(b_s)_{m_s}}{(c_1)_{m_1}\ldots(c_s)_{m_s}}$$

and

$$\phi = \delta = 0 \ \ (\text{that is, } \phi_1 = \cdots = \phi_B = \delta_1 = \cdots = \delta_D = 0)$$

in Theorem 5.1, we obtain the following result:

Corollary 5.3. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} c_n(\alpha; x) F_A^{(s)}\left[a, -n, b_2, \ldots, b_s; c_1, \ldots, c_s; u_1, u_2, \ldots, u_s\right] \frac{t^n}{n!}$$
$$= \ (1 - \frac{t}{x})^{\alpha} \exp(t) F^{1:0;1;1;\ldots;1}_{1:0;0;1;\ldots;1}$$

$$\left(\begin{array}{l} [(a):1,\ldots,1]: \quad -; \quad [-\alpha:1]; \quad [b_2:1]; \quad \ldots; \quad [b_s:1]; \\ \left[(c_1):\psi^{(1)},\ldots,\psi^{(s+1)}\right]: \quad -; \quad -; \quad [c_2:1]; \quad \ldots; \quad [c_s:1]; \\ \qquad (-u_1t), (-\frac{u_1t}{x-t}), u_2, \ldots, u_s \end{array} \right),$$

where $F_A^{(s)}$ is the first kind Lauricella function in s variables and the coefficient $\psi^{(k)}$ is given by

$$\psi^{(k)} = \begin{cases} 1, & (1 \le k \le 2) \\ 0, & (2 < k \le s+1) \end{cases}.$$

III. If we put

$$\Omega\left(m_1\theta^{(1)} + \cdots + m_s\theta^{(s)}; m_2, \ldots, m_s\right) = \frac{(a_1^{(1)})_{m_2} \ldots (a_1^{(s-1)})_{m_s} (a_2^{(1)})_{m_2} \ldots (a_2^{(s-1)})_{m_s}}{(c)_{m_1+\cdots+m_s}}$$

and

$$B = 1,\ b_1 = b,\ \phi_1 = 1 \text{ and } \delta = 0$$

in Theorem 5.1, we obtain the following result:

Corollary 5.4. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} c_n(\alpha;x) F_B^{(s)}\left[-n, a_1^{(1)}, \ldots a_1^{(s-1)}, b, a_2^{(1)}, \ldots, a_2^{(s-1)}; c; u_1, u_2, \ldots, u_s\right] \frac{t^n}{n!}$$

$$= (1-\frac{t}{x})^{\alpha} \exp(t) F_{1:0;0;0;\ldots;0}^{1:0;1;2;\ldots;2}$$

$$\left(\begin{array}{l} \left[(b):\theta^{(1)},\ldots,\theta^{(s+1)}\right]: \quad -; \quad [-\alpha:1]; \quad \left[a^{(1)}:1\right]; \quad \ldots; \quad \left[a^{(s-1)}:1\right]; \\ [(c):1,\ldots,1]: \quad -; \quad -; \quad -; \quad \ldots; \quad -; \\ \qquad (-u_1t), (-\frac{u_1t}{x-t}), u_2, \ldots, u_s \end{array} \right),$$

where $F_B^{(s)}$ is the second kind Lauricella function in s variables and the coefficient

$\theta^{(k)}$ is given by

$$\theta^{(k)} = \begin{cases} 1, & (1 \leq k \leq 2) \\ 0, & (2 < k \leq s+1) \end{cases}.$$

IV. By letting

$$\Omega\left(m_1\theta^{(1)} + \cdots + m_s\theta^{(s)}; m_2, \ldots, m_s\right) = \frac{(a)_{m_1+\cdots+m_s}(b_2)_{m_2}\cdots(b_s)_{m_s}}{(c)_{m_1+\cdots+m_s}}$$

and

$$\phi = \delta = 0,$$

in Theorem 5.1, we obtain the following our last result:

Corollary 5.5. *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} c_n(\alpha; x) F_D^{(s)}\left[a, -n, b_2, \ldots, b_s; c; u_1, u_2, \ldots, u_s\right] \frac{t^n}{n!}$$
$$= (1 - \frac{t}{x})^{\alpha} \exp(t) F_D^{(s+1)}\left[a, -, (-\alpha), b_2, \ldots, b_s; c; (-u_1 t), (-\frac{u_1 t}{x-t}), u_2, \ldots, u_s\right],$$

where $F_D^{(s)}$ is the fourth kind Lauricella function in s variables.

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