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MEAN VALUE THEOREMS FOR ANALYTIC FUNCTIONS

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This work is dedicated to the 118th anniversary of the birth of academician Nikola Obrechkoff.

Abstract. We prove a sharper Evard-Jafari Theorem, various mean value theorems, and an improved version of the Davitt-Powers-Riedel-Sahoo Theorem.

Introduction. The distribution of zeros and critical points of polynomials, or more generally analytic functions, is of paramount interest in analysis. Famous results to that effect are the classical Rolle's Theorem, Lagrange's Mean Value Theorem, and in the case of polynomials – the Budan-Fourier Theorem and Descartes' Rule of Signs. Problems become especially interesting when complex functions are considered. For instance, a direct analogue of Rolle's theorem

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does not hold for functions of a complex variable, as the example $f(z) = e^z - 1$ shows: $f(0) = f(2\pi i) = 0$ but $f'(z) \neq 0$. That branch of complex analysis has long been a favorite among Bulgarian mathematicians, many of whom have made significant contributions. For example, in 1921, Nikola Obrechkoff (at the age of 25 and while still a student) proved the following remarkable results (see [4]):

Suppose $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{R}[x]$ is a polynomial, let V_x be the number of variations of sign of the Fourier sequence, namely

$$V_x \stackrel{\text{def}}{=} V\Big(f(x), f'(x), \cdots, f^{(n)}(x)\Big),$$

and let (a, b) be an interval.

Obrechkoff's Generalization of the Budan-Fourier Theorem. *Corresponding to the interval (a,b) consider the quadrangle $ACBD$ in $\mathbb{C}$, symmetric about the real line, and such that vertex $A \equiv a$, vertex $B \equiv b$, $\angle BAC = \dfrac{\pi}{n - V_a}$, $\angle ABC = \dfrac{\pi}{V_b}$. Then*

$$\#\Big\{\textit{zeros of f(x) in } ABCD\Big\} = V_a - V_b,$$

or is less than that quantity by an even number.

An immediate corollary is

Obrechkoff's Generalization of Descartes' Rule of Signs. *Let $V = V(a_0, a_1, \ldots, a_n)$. Then*

$$\#\left\{\textit{zeros of f(x) in } \left\{z \in \mathbb{C} : \big|\arg(z)\big| < \frac{\pi}{n - V}\right\}\right\} = V,$$

or is less than it by an even number.

An important result, considered a version of Rolle's Theorem for complex polynomials, is the celebrated

Grace-Heawood Theorem. *Let $p(z) \in \mathbb{C}[z]$ have degree n and suppose $p(A) = p(B)$ for $A, B \in \mathbb{C}$. Then there is a critical point in the disk $\overline{\mathcal{D}}\left(\dfrac{A+B}{2}, \dfrac{|A-B|}{2} \cot \dfrac{\pi}{n}\right)$.*

Blagovest Sendov and Hristo Sendov have just announced [8, Theorem 3] (see also [7]) the following very interesting improvement:

Theorem (Bl. Sendov, Hr. Sendov). *Let $p(z)$ be a complex polynomial of degree $n \geq 3$ and suppose (taking without loss of generality $A = -i,\ B = i$) that $p(i) = p(-i) = 0$. Then the double disk*

$$\overline{\mathcal{D}}\left(-\cot\frac{2\pi}{n}, \csc\frac{2\pi}{n}\right) \cup \overline{\mathcal{D}}\left(\cot\frac{2\pi}{n}, \csc\frac{2\pi}{n}\right)$$

contains at least one critical point of $p(z)$.

Finally, we mention what we consider the premier unsolved problem in the analytic theory of polynomials, stated by academician Blagovest Sendov *circa* 1958 (see [6]):

Sendov's Conjecture. *Let $f(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$ be a polynomial of degree $n \geq 2$ and $z_j \in \mathbb{D}_1 = \overline{\mathcal{D}}(0, 1)$ for all $j = 1, \ldots, n$. Then each of the closed disks $\overline{\mathcal{D}}(z_1, 1)$, $\overline{\mathcal{D}}(z_2, 1)$, $\ldots$, $\overline{\mathcal{D}}(z_n, 1)$ contains a critical point of f.*

The interested reader may get acquainted with the depth and beauty of the subject by consulting the monographs [4], [5] and [6].

For the remainder of the paper we shall be considering analytic functions. Throughout, all zeros and critical points are to be counted with their exact multiplicities. The following result (see [4], [5]) will be needed in the sequel:

Proposition 1 (Stronger Rolle's Theorem for real analytic functions). *Between two consecutive zeros of a real analytic function there is an odd number of critical points.*

2. A sharper Evard-Jafari theorem and applications. In 1992, Evard and Jafari published a complex Rolle's Theorem [2, Theorem 2.1], which has not yet been fully appreciated by subsequent authors (for instance, it does not appear in the comprehensive work [6]). The theorem asserts that the zeros of the real and imaginary parts of f' separate the zeros of a holomorphic function f along the line segments connecting pairs of zeros:

The Evard-Jafari Theorem. *Let $f(z) = u(z) + iv(z)$ be holomorphic on the open convex set $D_f \subseteq \mathbb{C}$ and let $A, B \in D_f$ be such that $f(A) = 0 = f(B)$. Then $\exists\, z_1, z_2 \in (A, B)$ such that $\Re\big[f'(z_1)\big] = 0$ and $\Im\big[f'(z_2)\big] = 0$.*

The main result in this paper is to establish a sharper theorem which gives a more detailed understanding of how the zeros of $\Re[f']$ and $\Im[f']$ separate the zeros of f:

Theorem 1 (Sharper Evard-Jafari Theorem). *Let $f(z) = u(z) + iv(z)$ be holomorphic on the open convex set $D_f \subseteq \mathbb{C}$ and let $A, B \in D_f$ be such that $f(A) = 0 = f(B)$. Suppose $A = a_1 + ia_2$, $B = b_1 + ib_2$ and define the real functions*

$$\phi(t) = (b_1 - a_1)u(A + t(B - A)) + (b_2 - a_2)v(A + t(B - A)),$$
$$\psi(t) = (b_1 - a_1)v(A + t(B - A)) - (b_2 - a_2)u(A + t(B - A)), \qquad t \in [0, 1].$$

Suppose $0 = \tau_0 < \tau_1 < \cdots < \tau_m = 1$ ($m \geq 1$) are zeros of $\phi(t)$, with corresponding points $P_j = A + \tau_j(B - A)$, $j = 0, \ldots, m-1$ on the segment $[A, B]$. Consider each interval (τ_j, τ_{j+1}) and each subsegment (P_j, P_{j+1}).

(i) If $\phi(t) \neq 0$, $t \in (\tau_j, \tau_{j+1})$, then there will be an odd number of points $z_{1,j}, \ldots, z_{q,j} \in (P_j, P_{j+1})$ such that $\Re\big[f'(z_{1,j})\big] = 0, \ldots, \Re\big[f'(z_{q,j})\big] = 0$.

(ii) If it is not known that $\phi(t) \neq 0$, $t \in (\tau_j, \tau_{j+1})$, then $\exists\, z_j \in (P_j, P_{j+1})$ such that $\Re\big[f'(z_j)\big] = 0$.

Similarly, let $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n = 1$ ($n \geq 1$) be zeros of $\psi(t)$, with corresponding points $Q_k = A + \sigma_k(B - A)$, $k = 0, \ldots, n-1$ on the segment $[A, B]$.

(iii) If $\psi(t) \neq 0$, $t \in (\sigma_k, \sigma_{k+1})$, then there will be an odd number of points $\zeta_{1,k}, \ldots, \zeta_{r,k} \in (Q_k, Q_{k+1})$ such that $\Im\big[f'(\zeta_{1,k})\big] = 0, \ldots, \Im\big[f'(\zeta_{r,k})\big] = 0$.

(iv) If it is not known that $\psi(t) \neq 0$, $t \in (\sigma_k, \sigma_{k+1})$, then $\exists\, \zeta_k \in (Q_k, Q_{k+1})$ such that $\Im\big[f'(\zeta_k)\big] = 0$.

Proof. The function $\phi(t)$ was introduced in [2, Th. 2.1], and the function $\psi(t)$ is implied by the proof of the theorem. As in [2], upon differentiation one obtains

$$\begin{aligned}
\phi'(t) &= (b_1 - a_1)\left[\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}\right] + (b_2 - a_2)\left[\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}\right] \\
&= (b_1 - a_1)^2\frac{\partial u}{\partial x} + (b_1 - a_1)(b_2 - a_2)\frac{\partial u}{\partial y} + (b_2 - a_2)(b_1 - a_1)\frac{\partial v}{\partial x} + (b_2 - a_2)^2\frac{\partial v}{\partial y} \\
&= \Big[(b_1 - a_1)^2 + (b_2 - a_2)^2\Big]\frac{\partial u}{\partial x},
\end{aligned}$$

the last step being a consequence of the Cauchy-Riemann equations.

Similarly, $\psi'(t) = \Big[(b_1 - a_1)^2 + (b_2 - a_2)^2\Big]\dfrac{\partial v}{\partial x}$. Thus,

$$\phi'(t) = 0 \implies \frac{\partial u}{\partial x} = 0, \quad \psi'(t) = 0 \implies \frac{\partial v}{\partial x} = 0.$$

Now suppose that τ_j and τ_{j+1} are two consecutive zeros of $\phi(t)$. Since u and v are harmonic functions, $\phi(t)$ is real analytic, and by Proposition 1 $\exists$ an odd number of critical points $t_{1,j}, \dots, t_{q,j}$ such that $\phi'(t_{s,j}) = 0$, $s = 1, \dots, q$. Putting $z_{s,j} = A + t_{s,j}(B - A)$ establishes (i). If it is not known that τ_j and τ_{j+1} are consecutive zeros, then the usual Rolle's theorem yields (ii). The proofs of cases (iii) and (iv) are similar. The original Evard-Jafari Theorem is a special case of Theorem 1, with $m = n = 1$ and no additional information on the zeros of $\phi(t)$ and $\psi(t)$, $t \in (0,1)$. $\square$

We illustrate the theorem with two examples.

Example 1. Consider the function $f(z) = (z^2-1)(z-i) = z^3 - iz^2 - z + i$. Then $f(x,y) = x^3 - 3xy^2 + 2xy - x + i(3x^2y - y^3 - x^2 + y^2 - y + 1)$,

$$\Re(f') = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 2y - 1,$$
$$\Im(f') = \frac{\partial v}{\partial x} = 6xy - 2x.$$

One easily calculates:

on $[-1, i]$: $\quad \phi(t) = t(1-t), \ \psi(t) = 4t(t-1)^2;$

on $[1, i]$: $\quad \phi(t) = 4t(1-t), \ \psi(t) = 4t(t-1)^2;$

on $[-1, 1]$: $\quad \phi(t) = 8t(t-1)(2t-1), \ \psi(t) = 8t(t-1)^2.$

Thus, on the segment $[-1, 1]$ the function $\phi(t)$ has an additional zero $t = 1/2$, corresponding to $z = 0$, and so the curve $\Re(f') = 0$ must separate the zeros -1 and 1 in such a way that there will be an odd number of intersection points along the subsegments $[-1, 0]$ and $[0, 1]$. Indeed, the hyperbola $3x^2 - 3y^2 + 2y - 1 = 0$ intersects the segment $[-1, 1] = [-1, 0] \cup [0, 1]$ twice. (The function $\psi(t)$ has no extra zeros on any segment, and $\Im(f') = 0$ reduces to the lines $x = 0$ and $y = 1/3$ which of course also separate the zeros of f.)

Example 2 (cf. [2, Example (i)]). Let $f(z) = e^z - 1$. The zeros are $2k\pi i$, $k \in \mathbb{Z}$, and $u(x,y) = e^x \cos y - 1$, $v(x,y) = e^x \sin y$. Take for instance $A = 2\pi i$, $B = 4\pi i$. Then $\phi(t) = 2\pi \sin(2\pi t)$, $\psi(t) = 2\pi\Big(\cos(2\pi t) - 1\Big)$. The only zeros of $\psi(t)$ are 0 and 1, while $\phi(t)$ has an additional zero for $t = 1/2$, which corresponds to $z = 3\pi i$. Solving $\Re(f') = 0$ gives the lines $y = (2k+1)\frac{\pi}{2}$, and two of them separate $2\pi i$ and $4\pi i$; solving $\Im(f') = 0$ gives the lines $y = k\pi$ and one of them separates $2\pi i$ and $4\pi i$.

Applying Theorem 1 to the functions

$$\lambda(z) = f(z) - f(A) - \frac{f(B) - f(A)}{B - A}(z - A)$$

and

$$\mu(z) = \big[f(B) - f(A)\big]z - (B - A)f(z) - Af(B) + Bf(A),$$

respectively, we obtain two mean value theorems in the spirit of Lagrange:

Corollary 1 (Two Complex Mean Value Theorems)**.** *Let $f(z)$ be holomorphic on the open convex set $D_f \subseteq \mathbb{C}$ and let $A, B \in D_f$. Then $\exists\, z_{1,j} \in (A, B),\ j = 1, \cdots, m_1$ and $\exists\, z_{2,k} \in (A, B),\ k = 1, \cdots, m_2$ (where m_1, m_2 are determined as in Theorem 1) such that*

$$\Re\big[f'(z_{1,j})\big] = \Re\left[\frac{f(B) - f(A)}{B - A}\right],$$
$$\Im\big[f'(z_{2,k})\big] = \Im\left[\frac{f(B) - f(A)}{B - A}\right],$$

and
$\exists\, w_{1,j} \in (A, B),\ j = 1, \cdots, n_1$ and $\exists\, w_{2,k} \in (A, B),\ k = 1, \cdots, n_2$ (where n_1, n_2 are determined as in Theorem 1) such that

$$\Re\big[(B - A)f'(w_{1,j})\big] = \Re\big[f(B) - f(A)\big],$$
$$\Im\big[(B - A)f'(w_{2,k})\big] = \Im\big[f(B) - f(A)\big].$$

Corollary 1 generalizes [2, Theorem 2.2]. In general, $\{z_{1,j}\} \neq \{w_{1,j}\}$ and $\{z_{2,k}\} \neq \{w_{2,k}\}$, but the two will coincide if $\Im(B - A) = 0$.

Similarly, by considering

$$\Lambda(z) = f(z) - f(A) - \frac{f(B) - f(A)}{g(B) - g(A)}\big[g(z) - g(A)\big]$$

and

$$\mathrm{M}(z) = \big[f(B) - f(A)\big]g(z) - \big[g(B) - g(A)\big]f(z) - g(A)f(B) + g(B)f(A),$$

respectively, we obtain two more general results:

Corollary 2 (Two Complex Generalized (Cauchy-type) Mean Value Theorems)**.** *Let $f(z)$, $g(z)$ be holomorphic on the open convex set $D_{f,g} \subseteq \mathbb{C}$ and let $A, B \in D_{f,g}$ (in the first case $g(A) \neq g(B)$ must be required). Then*
$\exists\, z_{1,j} \in (A, B),\ j = 1, \ldots, m_1$ and $\exists\, z_{2,k} \in (A, B),\ k = 1, \ldots, m_2$ (where m_1, m_2 are determined as in Theorem 1) such that

$$\Re\big[f'(z_{1,j})\big] = \Re\left[\frac{f(B) - f(A)}{g(B) - g(A)}g'(z_{1,j})\right],$$

$$\Im\big[f'(z_{2,k})\big] = \Im\left[\frac{f(B)-f(A)}{g(B)-g(A)}g'(z_{2,k})\right],$$

and

$\exists\, w_{1,j} \in (A,B)$, $j = 1,\ldots,n_1$ *and* $\exists\, w_{2,k} \in (A,B)$, $k = 1,\ldots,n_2$ *(where* n_1, n_2 *are determined as in Theorem* 1*) such that*

$$\Re\Big[\Big(g(B)-g(A)\Big)f'(w_{1,j})\Big] = \Re\Big[\Big(f(B)-f(A)\Big)g'(w_{1,j})\Big],$$
$$\Im\Big[\Big(g(B)-g(A)\Big)f'(w_{2,k})\Big] = \Im\Big[\Big(f(B)-f(A)\Big)g'(w_{2,k})\Big].$$

3. Flett's Mean Value Theorem and its extensions in $\mathbb{R}$ and $\mathbb{C}$. We now turn our attention to a theorem discovered by Flett in 1958 [3], which unfortunately has not received much attention in textbooks, even though its proof is a good exercise and its geometric interpretation is illustrative.

Flett's Mean Value Theorem. *Suppose that I is an open interval, $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable, $[a,b] \subset I$, and $f'(a) = f'(b)$. Then $\exists\, \xi \in (a,b)$ such that*

$$f'(\xi) = \frac{f(\xi)-f(a)}{\xi - a}.$$

The example $f(z) = e^z - z$ shows that Flett's theorem fails in $\mathbb{C}$. Motivated by Evard and Jafari's paper, Davitt, Powers, Riedel, and Sahoo have proved [1] extensions of this theorem for both real and complex functions. First they generalize Flett's Theorem [1, Th. 1] by removing the condition $f'(a) = f'(b)$:

Proposition 2 (Extension of Flett's Mean Value Theorem)**.** *Suppose that I is an open interval, $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable and $[a,b] \subset I$. Then $\exists\, \xi \in (a,b)$ such that*

$$f'(\xi) = \frac{f(\xi)-f(a)}{\xi - a} + \frac{1}{2}\frac{f'(b)-f'(a)}{b-a}\big(\xi - a\big).$$

For $\alpha, \beta \in \mathbb{C}$, define $\big\langle \alpha, \beta \big\rangle = \Re\big(\alpha\bar{\beta}\big)$. Theorem 2 in [1] generalizes Proposition 2 to holomorphic functions as follows:

The Davitt-Powers-Riedel-Sahoo Mean Value Theorem. *Let $D_f \subset \mathbb{C}$ be open and convex, $f : D_f \to \mathbb{C}$ – holomorphic, $A, B \in D_f$. Then $\exists\, z_1, z_2 \in (A, B)$ such that*

$$\Re\big[f'(z_1)\big] = \frac{\Big\langle B-A, f(z_1)-f(A)\Big\rangle}{\big\langle B-A, z_1-A\big\rangle} + \frac{1}{2}\frac{\Re\big[f'(B)-f'(A)\big]}{B-A}(z_1-A),$$
$$\Im\big[f'(z_2)\big] = \frac{\Big\langle B-A, -i\big[f(z_2)-f(A)\big]\Big\rangle}{\big\langle B-A, z_2-A\big\rangle} + \frac{1}{2}\frac{\Im\big[f'(B)-f'(A)\big]}{B-A}(z_2-A).$$

The expressions on the right-hand sides are unnecessarily complicated. We state a better version of the theorem and provide a detailed proof (cf. [1, Proof of Th. 2]).

Theorem 2. *Let $f(z) = u(z) + iv(z)$ be holomorphic on the open convex set $D_f \subseteq \mathbb{C}$ and let $A = a_1 + ia_2 \in D_f$, $B = b_1 + ib_2 \in D_f$. Then $\exists\, z_1, z_2 \in (A, B)$ such that*

$$\Re\big[f'(z_1)\big] = \Re\left[\frac{f(z_1)-f(A)}{z_1-A} + \frac{1}{2}\frac{f'(B)-f'(A)}{B-A}(z_1-A)\right],$$
$$\Im\big[f'(z_2)\big] = \Im\left[\frac{f(z_2)-f(A)}{z_2-A} + \frac{1}{2}\frac{f'(B)-f'(A)}{B-A}(z_2-A)\right].$$

P r o o f. Define $\phi(t)$ and $\psi(t)$, $t \in [0,1]$ as in Theorem 1; recall that

$$\phi'(t) = \big|B-A\big|^2\Re\big[f'(z)\big],\ \psi'(t) = \big|B-A\big|^2\Im\big[f'(z)\big].$$

Proposition 2, applied to $\phi(t)$ on $[0,1]$, gives $\exists\, t_1 \in (0,1)$ such that

$$\phi'(t_1) = \frac{\phi(t_1)-\phi(0)}{t_1} + \frac{1}{2}\big[\phi'(1)-\phi'(0)\big]t_1.$$

Upon setting $z_1 = A + t_1(B-A)$ this reduces to

$$\Re\big[f'(z_1)\big] = \frac{\phi(t_1)-\phi(0)}{t_1\big|B-A\big|^2} + \frac{1}{2}\frac{\phi'(1)-\phi'(0)}{\big|B-A\big|^2}t_1.$$

But

$$\frac{\phi(t_1)-\phi(0)}{t_1\big|B-A\big|^2} = \frac{(b_1-a_1)\big[u(z_1)-u(A)\big]+(b_2-a_2)\big[v(z_1)-v(A)\big]}{t_1\big|B-A\big|^2}$$

and

$$\frac{f(z_1)-f(A)}{z_1-A} = \frac{u(z_1)-u(A)+i\big[v(z_1)-v(A)\big]}{t_1\big[b_1-a_1+i(b_2-a_2)\big]}$$
$$= \frac{\Big[u(z_1)-u(A)+i\big[v(z_1)-v(A)\big]\Big]\big[b_1-a_1-i(b_2-a_2)\big]}{t_1\big|B-A\big|^2},$$

hence

$$\frac{\phi(t_1)-\phi(0)}{t_1\big|B-A\big|^2} = \Re\left[\frac{f(z_1)-f(A)}{z_1-A}\right].$$

Also

$$\frac{\phi'(1)-\phi'(0)}{\big|B-A\big|^2}t_1 = \Big(\Re\big[f'(B)\big]-\Re\big[f'(A)\big]\Big)t_1 = \Re\Big[\Big(f'(B)-f'(A)\Big)t_1\Big]$$
$$= \Re\left[\Big(f'(B)-f'(A)\Big)\frac{z_1-A}{B-A}\right],$$

and the first part of Theorem 2 follows. The proof of the second part is similar and we omit it. □

With the additional assumption $f'(A)=f'(B)$, one obtains as a corollary

Theorem 3 (A Complex Flett's Mean Value Theorem)**.** *Let $f(z)$ be holomorphic on the open convex set $D_f \subseteq \mathbb{C}$. Suppose $f'(A)=f'(B)$ for two points $A,B \in D_f$. Then $\exists\, z_1, z_2 \in (A,B)$ such that*

$$\Re\big[f'(z_1)\big] = \Re\left[\frac{f(z_1)-f(A)}{z_1-A}\right],$$
$$\Im\big[f'(z_2)\big] = \Im\left[\frac{f(z_2)-f(A)}{z_2-A}\right].$$

In closing, we mention that Theorem 3 may be applied for instance to the function

$$\nu(z) = (B-A)f(z) - \frac{1}{2}\big[f'(B)-f'(A)\big](z-A)^2$$

to obtain one more mean value theorem:

Corollary 3. *Let $f(z)$ be holomorphic on the open convex set $D_f \subseteq \mathbb{C}$ and let $A,B \in D_f$. Then $\exists\, w_1, w_2 \in (A,B)$ such that*

$$\Re\big[(B-A)f'(w_1)\big] = \Re\Big[(B-A)\frac{f(w_1)-f(A)}{w_1-A} + \frac{1}{2}\big[f'(B)-f'(A)\big](w_1-A)\Big],$$

$$\Im\big[(B-A)f'(w_2)\big] = \Im\Big[(B-A)\frac{f(w_2)-f(A)}{w_2-A} + \frac{1}{2}\big[f'(B)-f'(A)\big](w_2-A)\Big].$$

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