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FINITE TIME BLOW UP OF THE SOLUTIONS TO NONLINEAR KLEIN-GORDON EQUATION WITH ARBITRARY HIGH POSITIVE INITIAL ENERGY*

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Abstract. The global behaviour of the weak solutions of the Cauchy problem to nonlinear Klein-Gordon equation in $\mathbb{R}^n \times \mathbb{R}^+$ is investigated. Finite time blow up of the solutions with arbitrary high positive initial energy is proved under general structural conditions on the initial data.

1. Introduction. In this paper we study the Cauchy problem for nonlinear Klein-Gordon equation

$$u_{tt} - \Delta u + mu = a|u|^{p-1}u \qquad \text{for} \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \qquad \text{for} \qquad x \in \mathbb{R}^n, \tag{2}$$

where a and m are positive constants and

$$1 < p < \infty \quad \text{for} \quad n = 1, 2; \qquad 1 < p < \frac{n+2}{n-2} \quad \text{for} \quad n \geq 3, \tag{3}$$

$$u_0(x) \in \mathrm{H}^1(\mathbb{R}^n), \quad u_1(x) \in \mathrm{L}^2(\mathbb{R}^n). \tag{4}$$

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In the last decades a great effort of research work has been invested for studying problem (1), (2), see [1, 2, 5, 7, 9, 10, 12] and the references therein. It is well known that the Cauchy problem for (1), (2) can be solved locally in time for initial data satisfying (3), (4), see for example [2, 9, 10, 12]. Global existence for small initial data is due to [13] (see also [2, 5, 7]).

The finite time blow up of solutions with negative initial energy is proved in [1, 8]. For positive subcritical initial energy ($0 < E(0) < d$) sharp conditions for global existence or finite time blow are obtained in [11, 16, 17] by the potential well method.

In the case of supercritical initial energy ($E(0) > d$) there are only few results. For nonlinear Klein-Gordon equation with general nonlinear term $f(u)$ satisfying $f(u) \geq (2+\varepsilon)\displaystyle\int_0^u f(s)\,ds$ and arbitrary high positive initial energy, the first finite time blow up result is given in [14]. Let us mention that for (1) in bounded domains with $m = 0$ the first blow up result for supercritical initial energy is proved in [3]. For damped Klein-Gordon equation similar blow up results are obtained in [4, 15, 16]. In the papers mentioned above ([14, 16]) the blow up results for (1) are proved under the following assumptions on the initial data:

$$\int_{\mathbb{R}^n} u_0(x)u_1(x)\,dx \geq 0, \qquad \int_{\mathbb{R}^n} u_0^2(x)\,dx \neq 0 \tag{5}$$

$$\frac{p-1}{2(p+1)}\int_{\mathbb{R}^n} u_0^2(x)\,dx > E(0) > 0, \qquad I(u_0) < 0. \tag{6}$$

Here initial energy $E(0)$ and Nehari functional $I(u_0)$ are defined below in (7) and (8) respectively. In all these papers key arguments in proving the blow up results are the concave method of Levine [8] and the sign preserving property of the Nehari functional $I(u(\cdot,t))$.

The aim of this paper is to prove finite time blow up of the solutions to (1), (2) under conditions more general then conditions (5), (6). These new conditions depend not only on the initial profile $u_0(x)$ but also on the initial velocity $u_1(x)$. Moreover they guarantee sign preserving properties of the Nehari functional $I(u(\cdot,t))$ under the flow of (1), (2) without condition $I(u_0) < 0$.

The paper is organized as follows. In Section 2 some definitions and preliminaries are given. The main results of the paper are stated in Section 3. Section 4 deals with the proof of the main results, while in Section 5 an explicitly choice of the initial data satisfying all conditions of the theorems is presented. Let us emphasize that the constructed initial data have arbitrary high positive energy.

2. Preliminaries. For convenience, without loss of generality, we set $m = 1$ in (1). Further on we use the following short notations:

$$\|u\| = \|u(\cdot,t)\|_{\mathrm{L}^2(\mathbb{R}^n)}, \quad \|u\|_1 = \|u(\cdot,t)\|_{\mathrm{H}^1(\mathbb{R}^n)},$$

$$(u,v) = (u(\cdot,t), v(\cdot,t)) = \int_{\mathbb{R}^n} u(x,t)v(x,t)\,dx.$$

Let us recall some important functionals and definitions related to problem (1), (2).

- Conservation law:

$$E(0) = E(t) \quad \text{for every} \quad t \in [0, T_m), \quad \text{where} \tag{7}$$
$$E(t) := E(u(\cdot,t)) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|_1^2 - \frac{a}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}(x,t)\,dx,$$

  T_m is the maximal existence time of the solution $u(x,t)$ to (1), (2), $u(x,t) \in \mathrm{C}([0,T_m);\mathrm{H}^1(\mathbb{R}^n)) \bigcap \mathrm{C}^1([0,T_m);\mathrm{L}^2(\mathbb{R}^n)) \bigcap \mathrm{C}^2([0,T_m);\mathrm{H}^{-1}(\mathbb{R}^n))$;

- Nehari functional:

$$I(u) = \|u\|_1^2 - a\int_{\mathbb{R}^n} |u|^{p+1}\,dx; \tag{8}$$

- Nehari manifold:

$$\mathcal{N} = \{u \in \mathrm{H}^1(\mathbb{R}^n) : \ \|u\|_1 \neq 0, \ I(u) = 0\};$$

- Critical energy constant d and functional J:

$$d = \inf_{u\in\mathcal{N}} J(u), \qquad J(u) = \frac{1}{2}\|u\|_1^2 - \frac{a}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}\,dx.$$

When the function u depends on x and t we use the short notation $I(u(t)) = I(u(\cdot,t))$.

Note that the critical energy constants d and the sign of the Nehari functional (8) are crucial for global solvability or finite time blow up in the framework of the potential well method. For subcritical initial energy, i.e. $0 < E(0) < d$, and $I(u_0) > 0$ problem (1), (2) has a unique global solution defined for every $t \in [0,\infty)$ while for $I(u_0) < 0$ the weak solution of (1), (2) blows up in a finite time, see [11, 16, 17].

In case of supercritical initial energy, i.e. $E(0) > d$, the sign preserving property of the Nehari functional is a key argument for proving finite time blow up results. In Theorem 1 and Theorem 2 we give sufficient conditions $I(u(t))$ to be negative under the flow of (1), (2).

We need the following result from [6, 8].

Lemma 1. *Suppose $\Phi(t)$ is a twice differentiable and positive function for $t \geq b \geq 0$, which satisfies the inequality*

$$\Phi''(t)\Phi(t) - \gamma(\Phi'(t))^2 \geq 0$$

for every $t \geq b$ and some $\gamma > 1$. If $\Phi(b) > 0$ and $\Phi'(b) > 0$, then $\Phi(t) \to +\infty$ for $t \to t^\star$, $t^\star \leq b + \dfrac{\Phi(b)}{(\gamma - 1)\Phi'(b)}$.

3. Main results.

Theorem 1 (Sign preserving property of $I(u(t))$)**.** *Suppose* (3) *and* (4) *hold and*

$$\|u_0\|_1 \neq 0, \quad (u_0, u_1) \geq 0, \tag{9}$$

$$\sqrt{\frac{p-1}{p+1}}(u_0, u_1) \geq E(0) > 0. \tag{10}$$

Then for the solution $u(x,t)$ of (1), (2) *the inequality $I(u(t)) < 0$ holds for every $t \in [0, T_m)$. Moreover if $t_b = \sqrt{\dfrac{p+1}{p-1}} < T_m$ then*

$$I(u(t)) \leq -\frac{p+1}{2}(u_t, u_t) \quad \textit{for all} \quad t \in [t_b, T_m). \tag{11}$$

Under some additional assumptions on (u_0, u_1) we have the following theorem.

Theorem 2 (Sign preserving property of $I(u(t)))$)**.** *Suppose* (3), (4), (9) *hold and*

$$\sqrt{\frac{p-1}{p+1}}\|u_0\|^2 \geq (u_0, u_1) \geq 0, \tag{12}$$

$$\frac{(p-1)}{2(p+1)}\|u_0\|^2 + \frac{1}{2}\frac{(u_0, u_1)^2}{\|u_0\|^2} \geq E(0) > 0. \tag{13}$$

Then for the solution $u(x,t)$ *of* (1), (2) *the inequality* $I(u(t)) < 0$ *holds for every* $t \in [0, T_m)$. *Moreover if* $t_b = \frac{(p+1)}{2(p-1)} \frac{(u_0, u_1)}{\|u_0\|^2} < T_m$ *then*

$$I(u(t)) \leq -\frac{p+1}{2}(u_t, u_t) \quad \text{for all} \quad t \in [t_b, T_m). \tag{14}$$

As a consequence of Theorem 1 and Theorem 2 we have the following finite time blow up result for problem (1), (2).

Theorem 3 (Finite time blow up). *Suppose* (3) *and* (4) *hold,* $\|u_0\|_1 \neq 0$ *and either*

$$(u_0, u_1) \geq 0, \qquad \sqrt{\frac{p-1}{p+1}}(u_0, u_1) \geq E(0) > 0 \tag{15}$$

or

$$\sqrt{\frac{p-1}{p+1}}\|u_0\|^2 \geq (u_0, u_1) \geq 0, \ \frac{(p-1)}{2(p+1)}\|u_0\|^2 + \frac{1}{2}\frac{(u_0, u_1)^2}{\|u_0\|^2} \geq E(0) > 0. \tag{16}$$

Then every weak solution of problem (1), (2) *blows up for a finite time* $t_* < \infty$, *where*

$$\text{either} \qquad t_* \leq t_b := \begin{cases} \sqrt{\dfrac{p+1}{p-1}} & \text{if (15) holds,} \\ \dfrac{(p+1)}{2(p-1)}\dfrac{(u_0, u_1)}{\|u_0\|^2} & \text{if (16) holds,} \end{cases}$$

$$\text{or} \qquad t_* \leq T_b := t_b + \frac{2\langle u(t_b), u(t_b)\rangle}{(p-1)\langle u(t_b), u_t(t_b)\rangle}.$$

Remark 1. Let us compare the result in Theorem 3 with the corresponding results in [14, 16] for equation (1). For $\sqrt{\frac{p-1}{p+1}}\|u_0\|^2 \geq (u_0, u_1)$ it is clear that (16) is an weaker one than (6) in [14, 16], i.e. when (6) is satisfied then (16) also holds. When

$$\sqrt{\frac{p-1}{p+1}}\|u_0\|^2 < (u_0, u_1) \tag{17}$$

then condition (15) is more general in comparison with (6). Indeed, if (6) is fulfilled then from (17) it follows that

$$\sqrt{\frac{p-1}{p+1}}(u_0, u_1) > \sqrt{\frac{p-1}{p+1}}\sqrt{\frac{p-1}{p+1}}\|u_0\|^2 = \frac{p-1}{p+1}\|u_0\|^2 > \frac{p-1}{2(p+1)}\|u_0\|^2 > E(0)$$

and (15) holds too. Moreover, assumption $I(u_0) < 0$ in (6) is superfluous for the blow up result.

4. Proof of the main results.

We need the following lemma.

Lemma 2. *Suppose* (3), (4) *hold and* $(u_0, u_1) \geq 0$. *If* $u(x,t)$ *is a weak solution of problem* (1), (2) *then the functions* $\phi(t) = \|u\|^2$ *and* $\phi'(t) = 2(u, u_t)$ *are strictly increasing ones in* $(0,T)$, $T < T_m$, *provided* $I(u(t)) < 0$ *for every* $t \in [0,T]$. *Moreover, the function* $\phi(t)$ *is strictly convex in* $(0,T)$ *and the inequality*

$$\|u(\cdot,t)\|^2 \geq \|u(\cdot,s)\|^2 + 2(t-s)(u(\cdot,s), u_t(\cdot,s)) \tag{18}$$

holds for every $0 \leq s \leq t \leq T$.

Proof. Since $\phi'(t) = 2(u, u_t)$, $\phi''(t) = 2(u_t, u_t) - 2I(u(t)) > 0$ and $(u_0, u_1) \geq 0$ it follows that $\phi(t)$ is a strictly convex function, $\phi'(t) > 0$ and $\phi(t)$ is a strictly increasing function of t. Inequality (18) is a consequence of the convexity of $\phi(t)$. $\square$

Proof of Theorem 1. From the conservation law (7) we have

$$\frac{1}{p+1} I(u(t)) = E(0) - \frac{1}{2}\|u_t\|^2 - \frac{p-1}{2(p+1)}\|u\|_1^2. \tag{19}$$

For $t = 0$ by means of (10) and (19) we get the following chain of inequalities:

$$\begin{aligned}\frac{1}{p+1} I(u(0)) &\leq \sqrt{\frac{p-1}{p+1}}(u_0, u_1) - \frac{1}{2}\|u_1\|^2 - \frac{p-1}{2(p+1)}\|u_0\|_1^2 \\ &= -\frac{1}{2}\left\|\sqrt{\frac{p-1}{p+1}}u_0 - u_1\right\|^2 - \frac{p-1}{2(p+1)}\|\nabla u_0\|^2 \leq 0.\end{aligned}$$

An equality in the above chain of inequalities is possible iff $\|u_0\| = \|\nabla u_0\| = \|u_1\| = 0$ which contradicts (9) and consequently $I(u(0)) < 0$.

Suppose that $I(u(t)) < 0$ for every $t \in [0, t_0)$ and $I(u(t_0)) = 0$ for some $0 < t_0 < T_m$. From Lemma 2, (10) and (19) we have

$$\begin{aligned}0 =& \frac{1}{p+1} I(u(t_0)) \leq \sqrt{\frac{p-1}{p+1}}(u_0, u_1) - \frac{1}{2}\|u_t(t_0)\|^2 - \frac{p-1}{2(p+1)}\|u(t_0)\|_1^2 \\ =& -\frac{1}{2}\left\|\sqrt{\frac{p-1}{p+1}}u(t_0) - u_t(t_0)\right\|^2 - \sqrt{\frac{p-1}{p+1}}\left((u(t_0), u_t(t_0)) - (u_0, u_1)\right) \\ & - \frac{p-1}{2(p+1)}\|\nabla u(t_0)\|^2 \leq 0\end{aligned}$$

and an equality is possible iff $t_0 = 0$ and $\|u(0)\| = \|\nabla u(0)\| = \|u_t(0)\| = 0$, i.e. $\|u_0\|_1 = \|u_1\| = 0$ which contradicts (9). Thus $I(u(t)) < 0$ for every $t \in [0, T_m)$.

If $t_b < T_m$ then repeating the above calculations we obtain from Lemma 2 the following inequalities for every $t \in [t_b, T_m)$

$$\begin{aligned}\frac{1}{p+1} I(u(t)) \leq & \sqrt{\frac{p-1}{p+1}}(u_0, u_1) - \frac{1}{2}\|u_t(t)\|^2 - \frac{p-1}{2(p+1)}\left(\|u_0\|^2 + 2t(u_0, u_1)\right) \\ \leq & -\sqrt{\frac{p-1}{p+1}}(u_0, u_1)\left(t\sqrt{\frac{p-1}{p+1}} - 1\right) - \frac{1}{2}\|u_t(t)\|^2 \leq -\frac{1}{2}\|u_t(t)\|^2.\end{aligned}$$

Thus Theorem 1 is proved.

P r o o f o f T h e o r e m 2. As in the proof of Theorem 1, for $t = 0$ from (12), (13) and (19) we have

$$\begin{aligned}&\frac{1}{p+1} I(u(0)) = E(0) - \frac{1}{2}\|u_1\|^2 - \frac{p-1}{2(p+1)}\|u_0\|_1^2 \\ &\leq \frac{(p-1)}{2(p+1)}\|u_0\|^2 + \frac{1}{2}\frac{(u_0, u_1)^2}{\|u_0\|^2} - \frac{1}{2}\|u_1\|^2 - \frac{p-1}{2(p+1)}\|u_0\|_1^2 \\ &= -\frac{1}{2}\left\|u_1 - \frac{(u_0, u_1)}{\|u_0\|^2}u_0\right\|^2 - \frac{(u_0, u_1)^2}{\|u_0\|^2} + \frac{1}{2}\frac{(u_0, u_1)^2}{\|u_0\|^2} - \frac{p-1}{2(p+1)}\|\nabla u_0\|^2 \leq 0\end{aligned}$$

and an equality is possible iff $\|u_0\|_1 = \|u_1\| = 0$ which contradicts (9). Hence $I(u(0)) < 0$.

Let us suppose that $I(u(t)) < 0$ for every $t \in [0, t_0)$ and $I(u(t_0) = 0)$ for some $t_0 \in (0, T_m)$. From Lemma 2, (12), (13), (19) we get the following impossible chain of inequalities

$$\begin{aligned}0 =& \frac{1}{p+1} I(u(t_0)) = E(t_0) - \frac{1}{2}\|u_t(t_0)\|^2 - \frac{p-1}{2(p+1)}\|u(t_0)\|_1^2 \\ =& E(0) - \frac{1}{2}\left\|u_t(t_0) - \frac{(u_0, u_1)}{\|u_0\|^2}u(t_0)\right\|^2 - \frac{(u_0, u_1)(u(t_0), u_t(t_0))}{\|u_0\|^2} \\ & - \frac{1}{2}\left(\frac{p-1}{p+1} - \frac{(u_0, u_1)^2}{\|u_0\|^4}\right)\|u(t_0)\|^2 - \frac{p-1}{2(p+1)}\|\nabla u(t_0)\|^2 \\ \leq & \frac{p-1}{2(p+1)}\|u_0\|^2 + \frac{1}{2}\frac{(u_0, u_1)^2}{\|u_0\|^2} - \frac{(u_0, u_1)^2}{\|u_0\|^2} - \frac{1}{2}\left(\frac{p-1}{p+1} - \frac{(u_0, u_1)^2}{\|u_0\|^4}\right)\|u_0\|^2 = 0\end{aligned}$$

and an equality is possible iff $t_0 = 0$ and $\|u(0)\| = \|\nabla u(0)\| = \|u_t(0)\| = 0$ i.e. iff $\|u_0\|_1 = \|u_1\| = 0$ which contradicts (9). Thus $I(u(t)) < 0$ for every $t \in [0, T_m)$.

If $t_b < T_m$ then repeating the above calculations we obtain from Lemma 2 the following inequalities for every $t \in [t_b, T_m)$

$$\begin{aligned}
\frac{1}{p+1} I(u(t)) \leq & \frac{p-1}{2(p+1)}\|u_0\|^2 + \frac{1}{2}\frac{(u_0,u_1)^2}{\|u_0\|^2} - \frac{1}{2}\|u_t(t)\|^2 - \frac{p-1}{2(p+1)}\|u(t)\|^2 \\
& - \frac{p-1}{2(p+1)}\|\nabla u(t)\|^2 \\
\leq & - \frac{p-1}{2(p+1)} 2(u_0,u_1)t + \frac{1}{2}\frac{(u_0,u_1)^2}{\|u_0\|^2} - \frac{1}{2}\|u_t(t)\|^2 \\
= & - (u_0,u_1)\left(t\frac{p-1}{p+1} - \frac{1}{2}\frac{(u_0,u_1)}{\|u_0\|^2}\right) - \frac{1}{2}\|u_t(t)\|^2 \leq -\frac{1}{2}\|u_t(t)\|^2.
\end{aligned}$$

Theorem 2 is proved.

Proof of Theorem 3. Let us suppose by contradiction that solution $u(x,t)$ of (1), (2) is defined for every $t \in [0,\infty)$. From Lemma 2 and (11) or (14) it follows that the function $\phi(t) = \|u\|^2$ satisfies the following differential inequalities

$$\begin{aligned}
&\phi''(t) = 2\|u_t\|^2 - 2I(u(t)) \geq (p+3)\|u_t\|^2 \\
&\phi(t)\phi''(t) - \frac{p+3}{4}(\phi'(t))^2 \geq (p+3)\left(\|u_t\|^2\|u\|^2 - (u,u_t)^2\right) \geq 0
\end{aligned}$$

for every $t \in [t_b, T_m)$. Since $\phi(t_b) = \|u(t_b)\|^2 > 0$, $\phi'(t_b) = 2(u(t_b), u_t(t_b)) > 0$ and $\frac{p+3}{4} > 1$ from Lemma 1 it follows that

$$\|u(t)\| \to \infty \quad \text{for} \quad t \to t_*, \quad t_* \leq t_b + \frac{4\phi(t_b)}{(p-1)\phi'(t_b)} = t_b + \frac{2\|u(t_b)\|^2}{(p-1)(u(t_b), u_t(t_b))}.$$

This contradicts our assumption and Theorem 3 is proved.

5. Construction of initial data with arbitrary high positive energy. In this section we give an explicit choice of initial data with arbitrary high positive energy, satisfying all conditions of Theorem 3, i.e. (15) and/or (16). Moreover, we show for which of these data assumption (6) in [14, 16] does not hold but (15) and/or (16) are fulfilled.

Let w, v be fixed functions such that $w(x) \in \mathrm{H}^1(\mathbb{R}^n)$, $v(x) \in \mathrm{L}^2(\mathbb{R}^n)$, $\|w\|_1 \neq 0$, $\|v\| \neq 0$, $(w,v) = 0$. We choose initial data in the following way

$$u_0(x) = rw(\sigma x), \quad u_1(x) = r(qw(\sigma x) + \mu v(\sigma x)). \tag{20}$$

The constants $r > 0$, $\sigma > 0$, $q \geq 0$, $\mu > 0$ are defined bellow so that conditions (15) and (16) hold, $E(0) \geq K$ for some arbitrary positive fixed constant $K > 0$ but (6) fails.

Straightforward computations give us

$$\|u_0\|^2 = \frac{r^2}{\sigma^n}\|w\|^2, \quad \|\nabla u_0\|^2 = \frac{r^2}{\sigma^{n-2}}\|\nabla w\|^2, \quad \|u_1\|^2 = \frac{r^2 q^2}{\sigma^n}\|w\|^2 + \frac{r^2\mu^2}{\sigma^n}\|v\|^2,$$
$$(u_0, u_1) = \frac{r^2 q}{\sigma^n}\|w\|^2, \quad \int_{\mathbb{R}^n} |u_0|^{p+1}(x)\,dx = \frac{r^{p+1}}{\sigma^n}\int_{\mathbb{R}^n} |w|^{p+1}(x)\,dx,$$

$$E(0) = \frac{r^2}{2\sigma^n}\left((q^2+1)\|w\|^2 + \sigma^2\|\nabla w\|^2 + \mu^2\|v\|^2 - \frac{2ar^{p-1}}{p+1}\int_{\mathbb{R}^n}|w|^{p+1}(x)\,dx\right).$$

Now conditions $\|u_0\|_1 \neq 0$, $(u_0, u_1) \geq 0$ in Theorem 3 and (5) in [14, 16] are satisfied. One has to check conditions (6), (15), (16) and the additional assumption

$$E(0) \geq K. \tag{21}$$

From the choice of the data it follows that conditions (6) and (21), (15) and (21), and (16) and (21) are equivalent to

$$\frac{2(p+1)\sigma^n K}{r^2} \leq R(\sigma, q) + (p+1)\mu^2\|v\|^2 < (p-1)\|w\|^2, \tag{22}$$

$$\frac{2(p+1)\sigma^n K}{r^2} \leq R(\sigma, q) + (p+1)\mu^2\|v\|^2 \leq 2\sqrt{p^2-1}\,q\|w\|^2, \tag{23}$$

$$\begin{gathered} \frac{2(p+1)\sigma^n K}{r^2} \leq R(\sigma, q) + (p+1)\mu^2\|v\|^2 \leq \left(p-1+(p+1)q^2\right)\|w\|^2, \\ 0 \leq q \leq \sqrt{\frac{p-1}{p+1}}, \end{gathered} \tag{24}$$

respectively, where

$$R(\sigma, q) = (p+1)(q^2+1)\|w\|^2 + (p+1)\sigma^2\|\nabla w\|^2 - 2ar^{p-1}\int_{\mathbb{R}^n}|w|^{p+1}(x)\,dx.$$

First step: Let σ and K be arbitrary positive fixed constants.

Second step: In case (23) the constant q is fixed as

$$q \geq \frac{1}{2}\sqrt{\frac{p-1}{p+1}} \tag{25}$$

and in case (24) the constant q is fixed as

$$0 \leq q \leq \sqrt{\frac{p-1}{p+1}}.$$

Inequality (25) guarantees that

$$(p-1)\|w\|^2 \le 2\sqrt{p^2-1}q\|w\|^2 \le \left(p-1+(p+1)q^2\right)\|w\|^2.$$

Third step: The constant r is fixed so that

$$r > \max\left\{\left(\frac{2(p+1)\sigma^n K}{(p-1)\|w\|^2}\right)^{\frac{1}{2}},\ \left(2a\int_{\mathbb{R}^n}|w|^{p+1}(x)\,dx\right)^{-\frac{1}{p-1}} \times \right.$$
$$\left. \times\ \left((p+1)(q^2+1)\|w\|^2+(p+1)\sigma^2\|\nabla w\|^2\right)^{\frac{1}{p-1}}\right\}. \tag{26}$$

Inequality (26) gives us that

$$R(\sigma,q) < 0 < \frac{2(p+1)\sigma^n K}{r^2} < (p-1)\|w\|^2,$$

hence one can choose the constant $\mu \ge 0$ such that the inequalities in the lhs of (22), (23) and (24) to be satisfied.

Fourth step: We define the constants μ_0, μ_1, μ_2 and μ_3 in the following way:

$$\mu_0 = \left\{\frac{2(p+1)\sigma^n K}{r^2} - R(\sigma,q)\right\}^{\frac{1}{2}}(p+1)^{-\frac{1}{2}}\|v\|^{-1},$$

$$\mu_1 = \left\{(p-1)\|w\|^2 - R(\sigma,q)\right\}^{\frac{1}{2}}(p+1)^{-\frac{1}{2}}\|v\|^{-1},$$

$$\mu_2 = \left\{2\sqrt{p^2-1}q\|w\|^2 - R(\sigma,q)\right\}^{\frac{1}{2}}(p+1)^{-\frac{1}{2}}\|v\|^{-1},$$

$$\mu_3 = \left\{\left(p-1+(p+1)q^2\right)\|w\|^2 - R(\sigma,q)\right\}^{\frac{1}{2}}(p+1)^{-\frac{1}{2}}\|v\|^{-1}.$$

The constant μ_0 is defined such that the inequality in the lhs of (22), (23) and (24) becomes an equality. Similarly, constants μ_1, μ_2, μ_3 are determined so that the inequalities in the right hand side of (22), (23) and (24) become equalities respectively.

Fifth step:

- If the constant μ is fixed so that $\mu_0 \le \mu < \mu_1$, then (22), (23) and (24) are satisfied with additional condition $q \ge \dfrac{1}{2}\sqrt{\dfrac{p-1}{p+1}}$ in case (23) and $0 \le q \le \sqrt{\dfrac{p-1}{p+1}}$ in case (24).

- If the constant μ is fixed as $\mu_1 \le \mu < \mu_2$, then (23) is satisfied for every $q \ge \frac{1}{2}\sqrt{\frac{p-1}{p+1}}$; (24) holds for $0 \le q \le \sqrt{\frac{p-1}{p+1}}$ but (22) fails. Note that for $\frac{1}{2}\sqrt{\frac{p-1}{p+1}} \le q \le \sqrt{\frac{p-1}{p+1}}$ inequalities (23) and (24) are fulfilled.

- For μ fixed as $\mu_2 < \mu \le \mu_3$ condition (24) holds for $0 \le q \le \sqrt{\frac{p-1}{p+1}}$, but (22) and (23) are not satisfied for $\frac{1}{2}\sqrt{\frac{p-1}{p+1}} \le q \le \sqrt{\frac{p-1}{p+1}}$.

When $\mu = \mu_0$ it follows that $E(0) = K$, while for $\mu > \mu_0$ we get $E(0) > K$.

In this way we find a wide class of initial data (20) with arbitrary high positive energy which satisfy all conditions of Theorem 3. When $\mu \in [\mu_1, \mu_3]$ then for initial data, defined in (20), Theorem 3 holds but the blow up results in [14, 16] fail. Thus we prove that the blow up results in Theorem 3 are better than the corresponding ones in [14, 16].

REFERENCES

[1] J. M. Ball. Finite time blow up in nonlinear problem. In: Nonlinear Evolution equations (Ed. M. G. Grandall) New York, Academic Press, 1978, 189–205.

[2] T. Cazenave, A. Haraux. An Introduction to Semilinear Evolution Equations. Oxford Lect. Ser. Math. Its Appl., vol. **13**, Oxford, Oxford Univ. Press, 1998.

[3] F. Gazzola, M. Squassina. Global solutions and finite time blow up for damped semilinear wave equations. *Ann. Inst. Henri Poincaré, Analyse non Linéaire* **23** (2006), 185–207.

[4] V. Georgiev, G. Todorova. Existence of a solution of the wave equation with nonlinear damping and source terms. *J. Diff. Equations* **109** (1994), 295–308.

[5] J. Ginibre, G. Velo. The global Cauchy problem for the nonlinear Klein-Gordon equation. *Ann. Inst. Henri Poincaré, Analyse non Linéaire* **6** (1989), 15–35.

[6] V. K. Kalantarov, O. A. Ladyzhenskaya. The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types. *J. Soviet Math.* **10**, *1* (1978), 53–70.

[7] S. Klainerman. Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space-time dimensions. *Commun. Pure Appl. Math.* **38** (1985), 631–641.

[8] H. A. Levine. Instability and nonexistence of global solutions to nonlinear wave equations of the form Putt Au + F(u). *Trans. Amer. Math. Soc.* **192** (1974), 1–21.

[9] J.-L. Lions. Quelques méthodes de résolution des problémes aux limites non linéaires. Paris, Dunod, 1969.

[10] H. Pecher. L^p-Abschätzungen und klassiche Lösungen für nichtlineare Wellengeichungen. I. *Math. Z.* **150**, *2* (1976), 159–183.

[11] L. E. Payne, D. H. Sattinger. Saddle points and instability of nonlinear hyperbolic equations. *Israel J. Math.* **22** (1975), 273–303.

[12] W. Strauss. Nonlinear Wave Equations. CBMS Regional Conference Series in Mathematics, vol. **73**. Providence, RI, Amer. Math. Soc., 1989.

[13] W. Strauss. Nonlinear scattering theory at low energy. *J. Funct. Anal.* **41**, *3* (1981), 110–133.

[14] Y. Wang. A sufficient condition for finite time blow up of the nonlinear Klein-Gordon equations with arbitrary positive initial energy. *Proc. Amer. Math. Soc.* **136** (2008), 3477–3482.

[15] R. Xu. Global existence, blow up and asymptotic behaviour of solutions for nonlinear Klein-Gordon equation with dissipative term. *Math. Methods Appl. Sci.* **33**, *7* (2010), 831–844.

[16] R. Xu, Y.Ding. Global solutions and finite time blow up for damped Klein-Gordon equation. *Acta Math. Sci. Ser. B Engl. Ed.* **33**, *3* (2013), 643–652.

[17] J. Zhang. Sharp conditions of global existence for nonlinear Schrödinger and Klein-Gordon equations. *Nonlinear Analysis* **48** (2002), 191–207.

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