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BI-CHARACTERISTIC CURVES OF BODY AND SURFACE WAVES AND APPLICATION IN GEOPHYSICS

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Abstract. In this paper is given a new approach to 3D medelling of elastic piecewise homogeneous media, in particular Earth crust and upper Mantle. The method is based on the principle of tomography with Earthquake as a source of the signal and at least three receiver stations on the surface.

The wave propagation in such media is described by a system of three strongly coupled hyperbolic equations with piece-wise constant coefitients. The characteristic set and bi-characteristic curves are computed in a homogeneous half-space with free boundary as well as the formulae of reflection and diffraction of the bi-characteristics on the internal boundaries of the media. Applications of the characteristic set and bi-characteristic curves for the inverse problem in geophysics and Earth modelling are given.

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1. Introduction. Strong ground motion, in particular earthquakes, has been always subject of human fear, interest and in the last two centuries – analytic study. The main obstacle in exploring seismic mechanism is, generally speaking, the scale of the object in study. Our planet is too large to be examined by probes – as a matter of fact, our best technology provides samples of Earth structure in just few kilometers deep, apart from the cost of drilling. There are very few places as San Andreas fault in California, which is close to the surface and in fact geologist can examine the fault directly, providing best data for the features of the seismic source and the ground structure. For the rest of the planet we use methods as tomography – several well – posed receivers record the data from a specific source. Then the data are processed with different methods like inverse tomography, for instance, and approximation of the ground structure is obtained. For shallow and relatively small areas explosives are used as a source. For larger areas the data records of actual earthquakes are employed. Accurate records of the exact time and shape of the wave, provided by different seismic stations, allow determination of the depth and other features of the source with methods as time-frequency analysis and others. Then tomography could be used to determine the ground structure.

Unfortunately the tomography is not exact method even when the source of the signal is known and under control. In seismology and geophysics it is even more umprecise for neither the source of the wave, nor the media is known, and we have only the seismogram as a data. In fact it is one set of data to determine two unknown quantities. That is why a kind of iterative process is employed – some assumption over the source features is used to model the media, then the media model is used to approximate the source features.

For the purposes of geophysics it is sufficient if Earth is considered as an elastic body that is a continuum, in other words the matter is continuously distributed in space. Furthermore, since seismicity has relatively local effect, the planet can be approximated with no loss of generality by a half-space Ω with free surface $z = 0$ (in Cartesian coordinates) and axis z positive downword. If the elastic parameters depend only on vertical coordinate z then the wave propagating in solid media satisfy

$$
\begin{aligned}
\rho\frac{\partial^2 u_x}{\partial t^2} &= X + (\lambda+2\mu)\frac{\partial^2 u_x}{\partial x^2} + \mu\frac{\partial^2 u_x}{\partial y^2} + \mu\frac{\partial^2 u_x}{\partial z^2} + \\
&\quad + (\lambda+\mu)\frac{\partial^2 u_y}{\partial x\partial y} + (\lambda+\mu)\frac{\partial^2 u_z}{\partial x\partial z} + \frac{\partial\mu}{\partial z}\frac{\partial u_x}{\partial z} + \frac{\partial\mu}{\partial z}\frac{\partial u_z}{\partial x} \\
\rho\frac{\partial^2 u_y}{\partial t^2} &= Y + \mu\frac{\partial^2 u_y}{\partial x^2} + (\lambda+2\mu)\frac{\partial^2 u_y}{\partial y^2} + \mu\frac{\partial^2 u_y}{\partial z^2} + \\
&\quad + (\lambda+\mu)\frac{\partial^2 u_x}{\partial x\partial y} + (\lambda+\mu)\frac{\partial^2 u_z}{\partial y\partial z} + \frac{\partial\mu}{\partial z}\frac{\partial u_y}{\partial z} + \frac{\partial\mu}{\partial z}\frac{\partial u_z}{\partial y} \\
\rho\frac{\partial^2 u_z}{\partial t^2} &= Z + \mu\frac{\partial^2 u_z}{\partial x^2} + \mu\frac{\partial^2 u_z}{\partial y^2} + (\lambda+2\mu)\frac{\partial^2 u_z}{\partial z^2} + \\
&\quad + (\lambda+\mu)\frac{\partial^2 u_x}{\partial x\partial z} + (\lambda+\mu)\frac{\partial^2 u_y}{\partial y\partial z} + \frac{\partial\lambda}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + \\
&\quad + 2\frac{\partial\mu}{\partial z}\frac{\partial u_z}{\partial z}
\end{aligned} \tag{1}
$$

where (u_x, u_y, u_z) is the displacement function. λ, μ and ρ are piecewise continuous functions of z. The boundary conditions at the free surface $z = 0$ are as follows:

$$
\begin{aligned}
\sigma_{zz} &= (\lambda+2\mu)\frac{\partial u_z}{\partial z} + \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0 \\
\sigma_{zx} &= \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \\
\sigma_{zy} &= \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = 0
\end{aligned} \tag{2}
$$

and in addition the continuity of the functions $u_x, u_y, u_z, \sigma_{zz}, \sigma_{zx}$ and σ_{zy} in Ω is required.

Coefficients ρ, λ and μ depend on the geological properties of the rock. Though little information we have about exact ground structure, geological surveys near the surface show that the Earth crust is heterogeneous and consists of areas with homogeneous rock. That is why in any realistic model of the Earth crust and upper mantle the coefficients ρ, λ and μ are piecewise continuous functions and the results for the wave front set [2, Theorem 8.3.1, p. 271] are not applicable. On the other hand a reasonable approximation of the real Earth is 3-dimensional structure of homogeneous blocks in welded contact $\{B_{i,j,k}\}$, where i and j are integers, and k is a natural number. Blocks $B_{i,j,k}$ and $B_{i+1,j,k}$ are neighbours along x axis, $B_{i,j,k}$ and $B_{i,j+1,k}$ along y, and $B_{i,j,k}$ and $B_{i,j,k+1}$ – along z axis. Without loss of generality we suppose the source S of the seismic signal is in block $B_{0,0,0}$.

Numerical methods as finite differences, finite elements and hybrid one are applicable if the domain of system (1), (2) is relatively small. Otherwise the grid is too large and computational time is too costly or the approximation error – too high. There are also different analytical approaches to system (1), (2). One is based on the fundamental solution of (1) in integral form and the presumption that so called Rayleigh and Love modes give good approximation of the solution when the distance from the source is large enough compared to the wavelength (see [1] or [6]). Some other methods are briefly mentioned in section "Brief review of some 2D methods". All of them and others are well described in [5] and are based on one standart approach in geophysics, namely in case the body forces are neglected, the solutions of (1) are considered as a plane harmonic waves propagating along the positive x axis

$$u(x,t) = F(z).e^{i(\omega t - kx)}, \tag{3}$$

where ω is the angular frequency and k is the wavenumber corresponding to the phase velocity c, i.e. $k = \omega/c$.

The main disadvantage of this approach is that the plane wave (3) is two-dimensional one, living in the plane $y = 0$ only. This way all information on y coordinate is lost and it is impossible to build reasonable 3D model on plane wave (3).

On the other hand, if we build realistic 3-D model of the ground structure, plane wave in the form (3) is a significant restriction. The information lost in splitting the system (1) is a serious obstacle to the true to life modeling. This is the reason a new approach is suggested in this paper, approach build on the information from system (1) itself.

Three major components of the ground motion modeling can be distinguished – the source of the motion, the media, or the ground structure, and the equation of motion. The last component is given by system (1). Let us focus on the first two.

There are different models corresponding to very different sources of seismic waves. In this paper we consider point sources that produce an impulse in a certain direction, namely alongside vector $\xi_0 = (\xi_1^0, \xi_2^0, \xi_3^0)$. If the source is more complicated it can be represented as a vector field on given curve, that is the fault. In this case the procedure that is described below for a point source can be easily adapted.

The ground structure of the Earth is really complicated. That is why the standard approximation is used – the real ground is approximated by parallelepiped-like grid, mostly rectangular, i.e. a set of parallelepipeds with common sides what we call "blocks". In each block the elasticity coefficients, i.e. the coefficients of system (1), are supposed constant. The method proposed in this paper works as well some of the sides of the blocks are not plane ones, but any smooth and convex surface which is not paralell to the bicharacteristics of system (1), i.e. if there are no gliding rays.

Considering the ground model above, the system (1) is just as a strongly coupled system of three linear PDEs with piece-wise constant coefficients. Since the system (1) has constant coefficients within every block, the wavefront of the solution is a subset of the characteristic set of system (1). Furthermore, as the principal part is real with constant coefficients, the wavefront set is invariant under the bicharacteristic flow. Having in mind the source model described above, a point source with seismic impulse in some direction, actually the singularities of the solution carry all the information about the wave. Briefly speaking, the solution propagates over bicharacteristic curves within every homogeneous block. Then we can use so called train solutions – the solution in the first block, the one containing the source, determine the boundary conditions for its neighbouring blocks, etc. According to geometrical optics and microlocal analysis, if bicharacteristic curve reflects off the sides of every block the angle of incidence to the surface is equal to the angle of reflection. As for refraction at a surface, it is computed in the usual way, more details and exact coputations are given in Section 3 below. Therefore, if we know the position of the source, the direction of the seismic impulse and media structure we can compute the point s_0 where bicharacteristic curve has contact with the surface $z = 0$. The point s_0 is in fact the centre of the surface waves in the plane $z = 0$ generated by the section of the wave front and the plane $z = 0$. When actual measurement of the seismic waves is done, the coordinates of the point s_0 can be triangulated using the data of several stations. This way verification of the media model could be done. Exact coordinates of the epicenter of an eqrthquake and the centre of the surface waves r_0 can be computed using different and quite reliable techniques, like time-frequency analysis, based on the data from seismic stations. Given a certain 3-D media model, we can compute the point s_0. If the points s_0 and r_0 coincide within the error of the computations, then the media model is plausible.

For practical purposes 3-D models to be tested with the procedure mentioned above can be generated using Monte Carlo or other well known methods. Of course, like any other inverse problem, this algorithm has multiple solutions in the sense that many models can cover the requirement $s_0 \sim p_0$. Unfortunately, this is the best result we can hope to, given the complexity of the object to study and the information we have from the seismograms.

Another application of the bicharacteristics is in all 3-D models. Strictly speaking, approximating the Earth structure by a grid of rectangular parallelepipeds introduce artificial singularities at the vertices of the parallelepipeds as the boundary data there are not harmonised. These singularities "propagate" over the bicharacteristic curves and therefore the model should be chosen such that the bicharacteristics do not contain a vertex of the parallelepiped. In coupling coefficient models the "coupling" of the coefficients should be done at one side of the bicharacteristic cone only since the solution is not smooth at the cone.

2. Characteristic set and bicharacteristic strip in homogeneous block $B_{i,j,k}$. As it is well known, the characteristic set of a linear scalar operator $L(x,D) = \sum_{|\alpha|\le 2} a_\alpha(x) D^\alpha$ is given by the zeroes of its principal symbol $p_L(x,\xi) = \sum_{|\alpha|=2} a_\alpha(x)\xi^\alpha$. In the case of linear strongly coupled system the characteristic set contains the zeroes of the determinant of the characteristic matrix of the system.

Each element of the characteristic matrix is the principal symbol of the corresponding equation with respect to the corresponding argument. For instance, if the system is $L_i(u_1, u_2, \ldots, u_n) = 0$, $i = 1, \ldots, n$, then the element (k,m) of the characteristic matrix is the principal part of L_k with respect to u_m. Then the characteristic set of system (1) in every the block $B_{i,j,k}$ is given by the equation

$$0 = p(x,\xi) =$$

$$= \begin{vmatrix} \rho\tau^2 - (\lambda+2\mu){\xi_1}^2 - \mu{\xi_2}^2 - \mu{\xi_3}^2 & -(\lambda+\mu)\xi_1\xi_2 & -(\lambda+\mu)\xi_1\xi_3 \\ -(\lambda+\mu)\xi_1\xi_2 & \rho\tau^2 - \mu{\xi_1}^2 - (\lambda+2\mu){\xi_2}^2 - \mu{\xi_3}^2 & -(\lambda+\mu)\xi_2\xi_3 \\ -(\lambda+\mu)\xi_1\xi_3 & -(\lambda+\mu)\xi_2\xi_3 & \tau^2 - \mu{\xi_1}^2 - \mu{\xi_2}^2 - (\lambda+2\mu){\xi_3}^2 \end{vmatrix}.$$

If we denote by $\alpha = -(\lambda+\mu)^{-1}\left[\rho\tau^2 - \mu\left({\xi_1}^2+{\xi_2}^2+{\xi_3}^2\right)\right]$ we have

$$0 = \begin{vmatrix} \alpha+{\xi_1}^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \alpha+{\xi_2}^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \alpha+{\xi_3}^2 \end{vmatrix}$$

$$= \alpha\begin{vmatrix} \alpha+{\xi_2}^2 & \xi_2\xi_3 \\ \xi_2\xi_3 & \alpha+{\xi_3}^2 \end{vmatrix} + \begin{vmatrix} {\xi_1}^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \alpha+{\xi_2}^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \alpha+{\xi_3}^2 \end{vmatrix} =$$

$$= \alpha\begin{vmatrix} \alpha+{\xi_2}^2 & \xi_2\xi_3 \\ \xi_2\xi_3 & \alpha+{\xi_3}^2 \end{vmatrix} + \alpha\begin{vmatrix} {\xi_2}^2 & \xi_1\xi_3 \\ \xi_1\xi_3 & \alpha+{\xi_3}^2 \end{vmatrix} + \begin{vmatrix} {\xi_1}^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & {\xi_2}^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \alpha+{\xi_3}^2 \end{vmatrix} =$$

$$= \alpha\begin{vmatrix} \alpha+{\xi_2}^2 & \xi_2\xi_3 \\ \xi_2\xi_3 & \alpha+{\xi_3}^2 \end{vmatrix} + \alpha\begin{vmatrix} {\xi_1}^2 & \xi_1\xi_3 \\ \xi_1\xi_3 & \alpha+{\xi_3}^2 \end{vmatrix} + \begin{vmatrix} {\xi_1}^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & {\xi_2}^2 \end{vmatrix} + \begin{vmatrix} {\xi_1}^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & {\xi_2}^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & {\xi_3}^2 \end{vmatrix} =$$

$$= \alpha^2\left(\alpha+{\xi_1}^2+{\xi_2}^2+{\xi_2}^2\right).$$

These simple calculations show that the characteristic set of system (1) consists of two subsets

$$\begin{aligned} p_1(x,\xi) &= \rho\tau^2 - \mu\left({\xi_1}^2+{\xi_2}^2+{\xi_3}^2\right) = 0 \\ p_2(x,\xi) &= \rho\tau^2 - (\lambda+2\mu)\left({\xi_1}^2+{\xi_2}^2+{\xi_3}^2\right) = 0 \end{aligned} \tag{4}$$

since $(\lambda+\mu) > 0$.

Therefore the wave propagating in homogeneous block $B_{i,j,k}$ is actually a composition of two waves. This result corresponds to the theory of P (primary) and S (secondary) body waves. P wave corresponds to the set defined by $p_2(x,\xi) = 0$, ans S wave – to the one defined by $p_1(x,\xi) = 0$.

Another important object in our study is the bicharacteristic strip of the linear strongly coupled system (1), since the characteristic set of a operatop with real principal part $p(x,\xi)$ and constant coefficients is invariant under the bicharacteristic flow [2, Chapter 8]. By definition if $p(x^0,\xi^0) = 0$ then the bicharacteristic strip at point (x^0,ξ^0) is defined by the Hamilton equations

$$\frac{dx}{ds} = \frac{\partial p(x,\xi)}{\partial \xi}, \frac{d\xi}{ds} = \frac{\partial p(x,\xi)}{\partial x}$$

with initial data $(x, \xi) = (x_0, \xi_0)$ for $t = 0$. The bicharacteristic strip l_1 generated by $p_1(x, \xi)$ through point (x^0, ξ^0) is

$$(5)\qquad \left|\begin{array}{ll} x_i = 2\xi_i^0.s + x_i^0, & i = 1, 2, 3 \\ t = 2c\sqrt{(\xi_1^0)^2 + (\xi_2^0)^2 + (\xi_3^0)^2}.s + t^0, & \\ \xi_i = \xi_i^0, & i = 1, 2, 3 \\ \tau = \tau^0 = c\sqrt{(\xi_1^0)^2 + (\xi_2^0)^2 + (\xi_3^0)^2} & \end{array}\right.$$

as $p_1(x^0, \xi^0) = 0$. Constant $c = \sqrt{\mu/\rho}$ for bicharacteristics generated by $p_1(x, \xi)$ and $c = \sqrt{(\lambda + 2\mu)/\rho}$ for ones generated by $p_2(x, \xi)$.

The values of ξ_1^0, ξ_2^0 and ξ_3^0 are determined by the features of the seismic source. Without loss of generalization we can assume source of the seismic wave to be a point one with direction of the impulse $\xi_1^0, \xi_2^0, \xi_3^0$.

The restriction of the bicharacteristic strip into R^4 is named bicharacteristic curve. It is applicable to 3D modeling of the Earth, for, generally speaking, the singularities propagate over the bicharacteristic curves. In other words, singularity that is generated by an Earthquake in block $B_{0,0,0}$ propagate over the bicharacteristic curve in $B_{0,0,0}$ until it intersects at point (x_1, y_1, z_1) the boundary to the neighbouring block, $B_{1,0,0}$ for instance. Continuous boundary conditions mean that at point (x_1, y_1, z_1) system (1) in the block $B_{1,0,0}$ has singularity, that propagates over the bicharacteristic curves in $B_{1,0,0}$, etc. For computational purpose it is convenient to write (5) in the form

$$(6)\qquad \left|\begin{array}{l} x_1 = c^{-1}\xi_1^0.(t - t^0) + x_1^0 \\ x_2 = c^{-1}\xi_2^0.(t - t^0) + x_2^0 \\ x_3 = c^{-1}\xi_3^0.(t - t^0) + x_3^0 \end{array}\right.$$

since $t - t^0 = 2c\sqrt{(\xi_1^0)^2 + (\xi_2^0)^2 + (\xi_3^0)^2}.s = 2c|\xi^0|$ and without loss of generality we may assume $|\xi^0| = 1$.

3. Reflection and refraction. All calculations above are made for system (1) with constant coefficients, i.e. for homogeneous block or half space. Geologically the Earth crust and upper mantle consist of sub-domains $B_{i,j,k}$ containing the same material. In terms of our model $\Omega = \cup B_{i,j,k}$ and the coefficients of (1) are constants in each $B_{i,j,k}$, or in other words realistic system (1) is the one with piecewise constant coefficients. Without loss of generality we assume the boundary of $B_{i,j,k}$ to be piecewise smooth:

$$\partial B_{i,j,k} = \cup\{F_{i,j,k,l}(x,y,z) = 0, \quad l = 1,\ldots,N_m\},$$

where $F_{i,j,k,l}(x,y,z) = 0$ is smooth surface in R^3 and N_m is a finite number.

Equation (5) describes the bicharacteristic curves of (1) in each $B_{i,j,k}$ and their behavior on the boundary $\partial B_{i,j,k}$ is studied by geometrical optics and microlocal analysis. Meeting a surface $F_{i,j,k,l}(x,y,z) = 0$ at point p_b where the coefficients of (1) are not smooth, the bicharacteristic curve b can be reflected or refracted. In both cases there is singularity at boundary point p_b. It propagates over the bicharacteristics as well and this way the well-known formula for reflection and refraction from geometrical optics are obtained. When bicharacteristic curve is reflected the angle θ_{in} of incidence to the surface $F_{i,j,k,l}(x,y,z) = 0$ is equal to the angle of reflection θ_{rl}, since in the same block the equation (6) has the same coefficients. As for refraction at a surface, the match of the boundary conditions of the neighbouring blocks at the two sides of the boundary lead to the well-known formula from geometric opticsths $v_1 \sin\theta_r r = v_2 \sin\theta_{in}$, where θ_{rr} is the angle of refraction, v_1 is the speed of the wave in the "incidence" block and v_2 is the one in "refraction" block.

Let

$$\vec{n} = (n_1,n_2,n_3) = \left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right]^{-1/2} \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)(p_b)$$

be the normal unit vector to surface $F_{i,j,k,l} = 0$ at the point of incidence p_b, $\xi^{in} = \left(\xi_1^{in}, \xi_2^{in}, \xi_3^{in}\right)$ be the unit vector along the incidental bicharacteristic curve, $\xi^{rr} = (\xi_1^{rr}, \xi_2^{rr}, \xi_3^{rr})$ be the unit vector along refracted one, and $\xi^{rl} = \left(\xi_1^{rl}, \xi_2^{rl}, \xi_3^{rl}\right)$ be the unit vector along reflected one.

The speed of the wave is a physical feature of every material and it is preliminary known. For instance, the velocity of the P-wave in homogeneous isotropic medium is $v_P = \sqrt{(\lambda+2\mu)/\rho}$, for S-wave it is $v_S = \sqrt{\mu/\rho}$.

Quantities $\sin\theta_{in} = \sin\theta_{rl}$ and $\sin\theta_{rr}$ are easy to compute using scalar, or dot product $\cos\theta = \xi\cdot\vec{n}$ of unit vectors ξ and the normal unit vector $\vec{n}$, for instance

$$\sin^2\theta_{in} = 1 - \left(\xi_1^{in} n_1 + \xi_2^{in} n_2 + \xi_3^{in} n_3\right)^2$$

Then equations of refraction and reflection from geometrical optics yield

$$\begin{aligned}
&\xi_1^{rr} n_1 + \xi_2^{rr} n_2 + \xi_3^{rr} n_3 = \left[1 - \left(\frac{v_2}{v_1} \right)^2 \left(1 - [\xi_1^{in} n_1 + \xi_2^{in} n_2 + \xi_3^{in} n_3]^2 \right) \right]^{1/2} \\
(7)\qquad & \\
&\xi_1^{rl} n_1 + \xi_2^{rl} n_2 + \xi_3^{rl} n_3 = \left[1 - \left(1 - [\xi_1^{in} n_1 + \xi_2^{in} n_2 + \xi_3^{in} n_3]^2 \right) \right]^{1/2}.
\end{aligned}$$

In addition, the incidental bicharacteristic curve, the refracted one and the normal to the surface vector lie on the same plane and give us the relation

$$(8)\qquad \begin{vmatrix} n_1 & n_2 & n_3 \\ \xi_1^{in} & \xi_2^{in} & \xi_3^{in} \\ \xi_1^{rr} & \xi_2^{rr} & \xi_3^{rr} \end{vmatrix} = 0.$$

The same relation is valid for vector ξ^{rl}.

Finally, since we consider vectors ξ^{in}, ξ^{rr} and ξ^{rl} be unit ones, we obtain

$$\begin{aligned}
&(\xi_1^{rr})^2 + (\xi_2^{rr})^2 + (\xi_3^{rr})^2 = 1 \\
(9)\qquad & \\
&(\xi_1^{rl})^2 + (\xi_2^{rl})^2 + (\xi_3^{rl})^2 = 1
\end{aligned}$$

Equations (7), (8) and (9) define uniquely vectors of refraction ξ^{rr} and reflection ξ^{rl} up to the sigh.

4. 3-D modeling of Earth crust and upper mantle. A key point in every approximation is the error, or in terms of mathematical modeling how accurate is the model compared to the real object. Using bicharacterstic curves, described in the previous section, it is possible to define the following criterion for 3D model of the Earth crust and upper mantle.

Definition. *Let $\{B_{i,j,k}\}$ be a set of blocks and the source of seismic wave be a point one at the point S with direction alongside vector ξ^0. Let P is the point of the Earth surface belonging to the bicharacteristic curves generates by system* (1)*, set of blocks $\{B_{i,j,k}\}$ and source S. Given set of blocks $B_{i,j,k}$ is plausible if the point P coincides with the epicenter E of the surface waves generated by the earthquake.*

Since seismic stations record both surface and body waves, point E is a subject of triangulation if there are enough sensors in the region.

Computing the bi-characteristic curves in all set $\{B_{i,j,k}\}$ arises an important question. At the boudaries between two blocks – surfaces $F_{i,j,k,l}(x,y,z) = 0$ – is the bicharacteristic curve reflected, refracted, or both? The answer comes from so-called reflection and refraction index. It is a physical feature of the material that build the block. How to compute refraction and reflection index is well descrinbed in [1], [3] or in [4].

Furthermore, the body waves records are useful to determine the block structure of the closest to the seismic stations blocks. Wave front in a homogeneous block is a subset of the characteristic set of system (1), therefore it has constant speed by (4).

Using bi-characteristic curves and the characteristic set we can compute arrival time for P- and S-waves. In combination with the criteria from the Definition, we can generate and test plausible 3-D models of the Earth crust and upper mantle.

5. Brief review of some 2D methods. All the following methods are well described in [5].

In case the body forces are neglected, one standart approach in geophysics is to considere solutions of (1) in the form of a plane harmonic waves propagating along the positive x axis

$$u(x,t) = F(z).e^{i(\omega t - kx)},$$

where ω is the angular frequency and k is the wavenumber corresponding to the phase velocity c, i.e. $k = \omega/c$.

Then system (1), (2) reduces to a system

$$\begin{aligned}
&\frac{\partial}{\partial z}\left[\mu\frac{\partial F_x}{\partial z} - ik\mu F_z\right] - ik\lambda\frac{\partial F_z}{\partial z} + F_x[\omega^2\rho - k^2(\lambda + 2\mu)] = 0 \\
&\frac{\partial}{\partial z}\left[(\lambda + 2\mu)\frac{\partial F_z}{\partial z} - ik\lambda F_x\right] - ik\mu\frac{\partial F_x}{\partial z} + F_z(\omega^2\rho - k^2\mu) = 0
\end{aligned} \tag{10}$$

with boundary conditions at surface $z = 0$

$$\begin{aligned}
\sigma_{zz} &= (\lambda + 2\mu)\frac{\partial F_z}{\partial z} - ik\lambda F_x = 0, \\
\sigma_{zx} &= \mu\left(\frac{\partial F_x}{\partial z} - ikF_z\right) = 0,
\end{aligned}$$

which solutions are so called P-SV waves, and a single equation

$$\frac{\partial}{\partial z}\left(\mu\frac{\partial F_y}{\partial z}\right)+F_y(\omega^2\rho-k^2\mu)=0 \tag{11}$$

with boundary conditions at $z=0$

$$\mu\frac{\partial F_y}{\partial z}=0,$$

which solutions are called SH waves.

If the ground structure is approximated by paralel to the surface $z=0$ homogeneous layers, in other words, if parameter functions λ and μ are step-wise ones – constants in each layer, then one widely used and efficient method of solving system (10) and equation (11) is so called Multimodal method, or modal summation teqnique. The idea is to compute the solution of (10) or (11) in each layer, that is to solve the equation (or system) with constant coefficients, and then use the welded contact on the boundary, i.e. the boundary conditions on the common boundary are the same. For instance, if we consider Love modes in model with N layers, and $(u_y)_m$ is the solution of (11) in the m-th layer, then

$$\begin{bmatrix} k_m(u_y)_m \\ (\sigma_{zy})_m \end{bmatrix}=a_m\begin{bmatrix} k_{m-1}(u_y)_{m-1} \\ (\sigma_{zy})_{m-1} \end{bmatrix}$$

where constant 2x2 matrix a_m depends only on ρ_m,α_m,μ_m and density of the layer d_m. Denoting $A=a_{N-1}.a_{N-2}\ldots a_1$ we have

$$\begin{bmatrix} k_m(u_y)_N \\ (\sigma_{zy})_N \end{bmatrix}=A\begin{bmatrix} k_{m-1}(u_y)_0 \\ (\sigma_{zy})_0 \end{bmatrix}.$$

Then the eigenfunctions, which are basis of the space of solutions of equation (11), can be easily computed. Similar procedure can be applied to system (10).

For ground model with lateral heterogeneous media another approaches are employed as so called Ray theory and various variants of mode coupling, like WKBJ method, etc. In Ray theory for instance the basic assumption is that the solution of the elastic equations of motion has the form $u(x,\omega)=A(x,x_0,\omega).e^{i\omega\theta(x,x_0)}/\sqrt{J(x,x_0)}$, where $\theta(x,x_0)$ is the phase function, representing the time that wave takes to travel from point x_0 to point x, and J is the geometric decay of the wavefront (see [7]). Further assuming that A has the form $A(x,x_0,\omega)=S(\omega).\sum A_i(x,x+0)\omega^{-i}$, where $S(\omega)$ is the wavefront of the source time function. Then function $A(x,x_0,\omega)$ is approximated by $S(\omega).A_0(x,x_0)$, and respectively, $u(x,\omega) \approx S(\omega).A_0(x,x_0).e^{i\omega\theta(x,x_0)}/\sqrt{J(x,x_0)}$, which the wavefront can be easily computed. As the wavefront propagates on the bicharacteristics so-called ray tracking and techniques from the geometrical optics allow the computation of rays and the following wavefronts. Unfortunately this method is quite sensitive to small local perturbations of the velocity filed and it works well if the dimensions of the heterogeneities is much larger than the dominated wavelenght of the considered waves.

Mode coupling uses idea similar to the one Multimodal method, described above. It is focused on the study of the surface waves, giving the fact that they, mostly fundamental, first and few higher modes (eigenfunctions), represent the longest and strongest part of a seismic signal generated by an earthquake. Different methods as WKBJ, Invariant Imbedding Technique and Coupling Coefficients Method implement this idea in practice.

The main disadvantage of all mentioned above methods is the limitation of the plane wave described in (3). It is in fact two-dimensional one, living in the plane $y = 0$ only. That is why the system (1) decomposes to system (10) and equation (11). Roughly speaking, the plane wave (3) is like restriction operator on plane $z = 0$. The residue is equation (11), which is independent on x and z coordinates but some information is lost. This assumption is not a problem at all if we suppose the source and the receiver of the signal to be situated in this plane, and assume that everything meaningful happens within this plane. In other words, if we consider in fact 2-dimensional model then a plane wave in the form (3) is acceptable.

On the other hand, if we build realistic 3-D model of the ground structure, plane wave in the form (3) is a significant restriction. The information lost in splitting the system (1) is a serious obstacle to the true to life modeling. This is the reason a new approach to be suggested in this paper, approach build on the information from system (1) itself.

REFERENCES

[1] K. Aki, P. G. Richards. Quantitative Seismology, 2nd edition. University Science Books, 2002, ISBN 978-1-891389-63-4.

[2] L. Hörmander. The analysis of linear partial differential operators. I: Distribution theory and Fourier analysis. Berlin Heidelberg-New York-Tokyo, Springer-Verlag, 1983.

[3] W. M. Ewing, W. S. Jardetzky, F. Press. Elastic waves in layered media. Lamont Geological Observatory Contribution No. **189**. McGraw-Hill Series in the Geological Sciences. New York-Toronto-London, McGraw-Hill Book Company, Inc., 1957.

[4] W. L. Pilant. Elastic waves in the earth. Developments in Solid Earth Geophysics, vol. **11**. Elsevier Scientific Pub. Co., 1979.

[5] G. F. Panza, F. Romanelli, F. Vaccari. Seismic wave propagation in laterally heterogeneous anelastic media: Theory and applications to seismic zonation. *Advances in Geophysics* **43** (2001), 1–95.

[6] A. Levshin. Surface and Channel Seismic Waves. Nauka, Moscow, 1973, (in Russian).

[7] V. M. Babich, V. S. Buldyrev, I. A. Molotkov. Space-time Ray Method: Linear and Nonlinear Waves. Leningrad, Leningrad University Publishing House, 1985 (in Russian).

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