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## A SURVEY ON THE KISSING NUMBERS

Peter Boyvalenkov, Stefan Dodunekov, Oleg Musin\*

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ABSTRACT. The maximum possible number of non-overlapping unit spheres that can touch a unit sphere in  $n$  dimensions is called kissing number. The problem for finding kissing numbers is closely connected to the more general problems of finding bounds for spherical codes and sphere packings. We survey old and recent results on the kissing numbers keeping the generality of spherical codes.

**1. Introduction.** How many equal billiard balls can touch (kiss) simultaneously another billiard ball of the same size? This was the subject of a famous dispute between Newton and Gregory in 1694. The more general problem in  $n$  dimensions, how many non-overlapping spheres of radius 1 can simultaneously touch the unit sphere  $\mathbf{S}^{n-1}$ , is called the kissing number problem. The answer  $\tau_n$  is called kissing number, also Newton number, or contact number. In fact, Newton was right, without proof indeed, with his answer  $\tau_3 = 12$ .

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Further generalization of the problem leads to investigation of spherical codes. A spherical code is a non-empty finite subset of  $\mathbf{S}^{n-1}$ . Important parameters of a spherical code  $C \subset \mathbf{S}^{n-1}$  are its cardinality  $|C|$ , the dimension  $n$  (it is convenient to assume that the vectors of  $C$  span  $\mathbf{R}^n$ ) and the maximal inner product

$$s(C) = \max\{\langle x, y \rangle : x, y \in C, x \neq y\}.$$

The function

$$A(n, s) = \max\{|C| : \exists C \subset \mathbf{S}^{n-1} \text{ with } s(C) \leq s\}$$

extends  $\tau_n$  and it is easy to see that  $A(n, 1/2) = \tau_n$ . One also considers the function

$$D(n, M) = \max\{d(C) = \sqrt{2(1 - s(C))} : \exists C \subset \mathbf{S}^{n-1} \text{ with } |C| = M\}$$

which is used in the information theory (cf. [13, 18, 32]).

For  $n \geq 3$  and  $s > 0$ , only a few values of  $A(n, s)$  are known. In particular, only six kissing numbers are known:  $\tau_1 = 2$ ,  $\tau_2 = 6$  (these two are trivial),  $\tau_3 = 12$  (some incomplete proofs appeared in 19th century and Schütte and van der Waerden [41] first gave a detailed proof in 1953, see also [27, 46, 3, 37]),  $\tau_4 = 24$  (finally proved in 2003 by Musin [37]),  $\tau_8 = 240$  and  $\tau_{24} = 196560$  (found independently in 1979 by Levenshtein [30], and by Odlyzko and Sloane [38]).

Note that Kabatiansky and Levenshtein have found an asymptotic upper bound  $2^{0.401n(1+o(1))}$  for  $\tau_n$  [25]. The best currently known lower bound is  $2^{0.2075n(1+o(1))}$  [47].

This survey deals with the above-mentioned values of  $\tau_n$  and mainly with upper and lower bounds in dimensions  $n \leq 24$ . Some interesting advances during the last years are described.

Usually the lower bounds are obtained by constructions. We describe such constructions which often lead to the best known lower bounds. The upper bounds are based on the so-called linear programming techniques [16, 25] and its strengthening [37, 39, 37]. Applications were proposed by Odlyzko and Sloane [38], the first named author [7], and strengthening by the third named author [37] and Pfender [39].

Recently, the linear programming approach was strengthened as the so-called semi-definite programming method was proposed by Bachoc and Vallentin [5] with further applications by Mittelmann and Vallentin [33].

The last few paragraphs of this paper were written after August 5, 2012, the day when the second named author, Stefan Dodunekov, passed away.

## 2. Upper bounds on kissing numbers.

**2.1. The Fejes Tóth bound and Coxeter-Böröczky bound.** Fejes Tóth [22] proved a general upper bound on the minimum distance of a spherical code of given dimension and cardinality. In our notations, the Fejes Tóth bound states that

$$(1) \quad D(n, M) \leq d_{FT} = \left(4 - \frac{1}{\sin^2 \varphi_M}\right)^{1/2}$$

where  $\varphi_M = \frac{\pi M}{6(M-2)}$ . This bound is attained for  $M = 3, 4, 6,$  and  $12$ . This gives four exact values of the function  $D(n, M)$  (but not necessarily implying exact values for  $A(n, s)$ ).

First general upper bounds on the kissing numbers were proposed by Coxeter [14] and were based on a conjecture that was proved later by Böröczky [6]. Thus it is convenient to call this bound the Coxeter-Böröczky bound.

Let the function  $F_n(\alpha)$  be defined as follows:

$$F_0(\alpha) = F_1(\alpha) = 1, \\ F_{n+1}(\alpha) = \frac{2}{\pi} \int_{(1/2) \arccos(1/n)}^{\alpha} F_{n-1}(\beta(t)) dt$$

for  $n \geq 1$ , where  $\beta(t) = \frac{1}{2} \arccos \frac{\cos 2t}{1 - 2 \cos 2t}$ . This function was introduced by Schläfli [40] and is usually referred to as Schläfli function.

In terms of the Schläfli function the Coxeter-Böröczky bound is

$$(2) \quad A(n, s) \leq A_{CB}(n, s) = \frac{2F_{n-1}(\alpha)}{F_n(\alpha)},$$

where  $\alpha = \frac{1}{2} \arccos \frac{s}{1 + (n-2)s}$ .

The bounds  $\tau_n \leq A_{CB}(n, 1/2)$  are weaker than the linear programming bound to be discussed below. On the other hand, we have

$$A(4, \cos \pi/5) = 120 = A_{CB}(4, \cos \pi/5) = \frac{2F_3(\pi/5)}{F_4(\pi/5)}$$

(the lower bound is ensured by the 600-cell). The value  $A(4, \cos \pi/5) = 120$  can be found by linear programming as well [2]. This suggests that the Coxeter-Böröczky bound can be better than the linear programming bounds when  $s$  is close to 1.

**2.2. Pure linear programming bounds.** The linear programming method for obtaining bounds for spherical codes was built in analogy with its counter-part for codes over finite fields which was developed by Delsarte [15]. Delsarte, Goethals and Seidel [16] proved in 1977 the main theorem and it was generalized by Kabatianskii and Levenshtein [25] in 1978.

The Gegenbauer polynomials [1, 44] play important role in the linear programming. For fixed dimension  $n$ , they can be defined by the recurrence  $P_0^{(n)} = 1$ ,  $P_1^{(n)} = t$  and

$$(k + n - 2)P_{k+1}^{(n)}(t) = (2k + n - 2)tP_k^{(n)}(t) - kP_{k-1}^{(n)}(t) \text{ for } k \geq 1.$$

If

$$f(t) = \sum_{i=0}^m a_i t^i$$

is a real polynomial, then  $f(t)$  can be uniquely expanded in terms of the Gegenbauer polynomials as

$$f(t) = \sum_{k=0}^m f_k P_k^{(n)}(t).$$

The coefficients  $f_i$ ,  $i = 0, 1, \dots, k$ , are important in the linear programming theorems.

**Theorem 1** (Delsarte, Goethals and Seidel [16], Kabatianskii and Levenshtein [25]). *Let  $f(t)$  be a real polynomial such that*

$$(A1) \quad f(t) \leq 0 \text{ for } -1 \leq t \leq s,$$

$$(A2) \quad \text{The coefficients in the Gegenbauer expansion } f(t) = \sum_{k=0}^m f_k P_k^{(n)}(t)$$

*satisfy  $f_0 > 0$ ,  $f_k \geq 0$  for  $i = 1, \dots, m$ .*

$$\text{Then } A(n, s) \leq f(1)/f_0.$$

There are two cases, in dimensions eight and twenty four, where only technicalities remain after Theorem 1. The lower bounds  $\tau_8 \geq 240$  and  $\tau_{24} \geq 196560$  are obtained by classical configurations and the upper bounds are obtained by the polynomials

$$f_6^{(8,0.5)}(t) = (t + 1) \left( t + \frac{1}{2} \right)^2 t^2 \left( t - \frac{1}{2} \right)$$

and

$$f_{10}^{(24,0.5)}(t) = (t + 1) \left( t + \frac{1}{2} \right)^2 \left( t + \frac{1}{4} \right)^2 t^2 \left( t - \frac{1}{4} \right)^2 \left( t - \frac{1}{2} \right),$$

respectively (the notations will become clear later). Indeed, one may easily check that these two polynomial satisfy the conditions (A1) and (A2) for the corresponding values of  $n$  and  $s$  and therefore  $\tau_8 \leq \frac{f_6^{(8,0.5)}(1)}{f_0} = 240$  and  $\tau_{24} \leq \frac{f_{10}^{(24,0.5)}(1)}{f_0} = 196560$ .

Together with the Gegenbauer polynomials we consider their adjacent polynomials which are Jacobi polynomials  $P_k^{(\alpha,\beta)}(t)$  with parameters

$$(\alpha, \beta) = \left( a + \frac{n-3}{2}, b + \frac{n-3}{2} \right)$$

where  $a, b \in \{0, 1\}$  (the Gegenbauer polynomials are obtained for  $a = b = 0$ ). Denote by  $t_k^{a,b}$  the greatest zero of the polynomial  $P_k^{(\alpha,\beta)}(t)$ . Then

$$t_{k-1}^{1,1} < t_k^{1,0} < t_k^{1,1}$$

for every  $k \geq 2$ .

Denote

$$\mathcal{I}_m = \begin{cases} [t_{k-1}^{1,1}, t_k^{1,0}], & \text{if } m = 2k - 1, \\ [t_k^{1,0}, t_k^{1,1}], & \text{if } m = 2k, \end{cases}$$

for  $k = 1, 2, \dots$  and  $\mathcal{I}_0 = [-1, t_1^{1,0}]$ .

Then the intervals  $\mathcal{I}_m$  are consecutive and non-overlapping. For every  $s \in \mathcal{I}_m$ , the polynomial

$$f_m^{(n,s)}(t) = \begin{cases} (t-s) \left( T_{k-1}^{1,0}(t,s) \right)^2, & \text{if } m = 2k - 1, \\ (t+1)(t-s) \left( T_{k-1}^{1,1}(t,s) \right)^2, & \text{if } m = 2k, \end{cases}$$

can be used in Theorem 1 for obtaining a linear programming bound. Levenshtein [30] proved that the polynomials  $f_m^{(n,s)}(t)$  satisfy the conditions (A1) and (A2) for all  $s \in \mathcal{I}_m$ . Moreover, all coefficients  $f_i$ ,  $0 \leq i \leq m$ , in the Gegenbauer expansion of  $f_m^{(n,s)}(t)$  are strictly positive for  $s \in \mathcal{I}_m$ . Hence this implies (after some calculations) the following universal bound.

**Theorem 2** (Levenshtein bound for spherical codes [30, 31]). *Let  $n \geq 3$*

and  $s \in [-1, 1)$ . Then

$$A(n, s) \leq \begin{cases} L_{2k-1}(n, s) = \binom{k+n-3}{k-1} \left[ \frac{2k+n-3}{n-1} - \frac{P_{k-1}^{(n)}(s) - P_k^{(n)}(s)}{(1-s)P_k^{(n)}(s)} \right] \\ \text{for } s \in \mathcal{I}_{2k-1}, \\ \\ L_{2k}(n, s) = \binom{k+n-2}{k} \left[ \frac{2k+n-1}{n-1} - \frac{(1+s)(P_k^{(n)}(s) - P_{k+1}^{(n)}(s))}{(1-s)(P_k^{(n)}(s) + P_{k+1}^{(n)}(s))} \right] \\ \text{for } s \in \mathcal{I}_{2k}. \end{cases}$$

In particular, one has  $\tau_8 \leq L_6(8, 1/2) = L_7(8, 1/2) = 240$  and  $\tau_{24} \leq L_{10}(24, 1/2) = L_{11}(24, 1/2) = 196560$ . The Levenshtein bound can be attained in some other cases (cf. the tables in [30, 31, 32]).

The possibilities for existence of codes attaining the bounds  $L_m(n, s)$  were discussed in [10]. In particular, it was proved in [10, Theorem 2.2] that the even bounds  $L_{2k}(n, s)$  can be only attained when  $s = t_k^{1,0}$  or  $s = t_k^{1,1}$ . This follows from a close investigation of the two-point distance distribution

$$A_t = \frac{1}{|C|} \sum_{x \in C} |\{y \in C : \langle x, y \rangle = t\}| = \frac{1}{|C|} |\{(x, y) \in C^2 : \langle x, y \rangle = t\}|$$

of the possible  $(n, L_{2k}(n, s), s)$ -codes.

On the other hand it was proved by Sidelnikov [42] (see also [32, Theorem 5.39]) that the Levenshtein bounds are the best possible pure linear programming bound provided the degree of the improving polynomial is at most  $m$ . This restriction was later extended by Boyvalenkov, Danev and Bumova [9] to  $m + 2$  and the polynomials  $f_m^{(n,s)}(t)$  are still the best.

However, in some cases the Leveshtein bounds are not the best possible pure linear programming bounds. This was firstly demonstrated in 1979 for the kissing numbers by Odlyzko-Sloane [38]. Boyvalenkov, Danev and Bumova [9] proved in 1996 necessary and sufficient conditions for existence of certain improvements.

**Theorem 3** [9]. *The bound  $L_m(n, s)$  can be improved by a polynomial from  $A_{n,s}$  of degree at least  $m + 1$  if and only if  $Q_j(n, s) < 0$  for some  $j \geq m + 1$ . Moreover, if  $Q_j(n, s) < 0$  for some  $j \geq m + 1$ , then  $L_m(n, s)$  can be improved by a polynomial from  $A_{n,s}$  of degree  $j$ .*

For  $s = 1/2$  (the kissing number case) and  $3 \leq n \leq 23$ ,  $n \neq 8$ , the Levenshtein bounds are better than the Coxeter-Böröczky bounds but weaker than these which were obtained by Odlyzko and Sloane [38].

In three dimensions, the Levenshtein bound gives  $\tau_3 \leq L_5(3, 1/2) \approx 13.285$  and it can be improved to  $\tau_3 \leq 13.184$  which is, of course, not enough. Then Anstreicher [3] in 2002 and Musin [37] in 2003 presented new proofs which were based on strengthening the linear programming and using spherical geometry on  $\mathbf{S}^2$ . The Musin approach will be discussed in more details below.

In dimensions four, we have  $\tau_4 \leq L_5(4, 1/2) = 26$  and this can be improved to  $\tau_4 \leq 25.5584$  which implies that  $\tau_4$  is 24 or 25. Then Arestov and Babenko [4] proved in 2000 that the latter bound is the best possible one can find by pure linear programming. Earlier (in 1993), Hsiang [24] claimed a proof that  $\tau_4 = 24$  but that proof was not widely recognized as complete. Musin [37] presented his proof of  $\tau_4 = 24$  in 2003 to finally convince the specialists.

Odlyzko and Sloane [38] use discrete version of the condition (A1) and then apply the usual linear programming for  $s = 1/2$  and  $3 \leq n \leq 24$ . Their table can be seen in [13, Chapter 1, Table 1.5]. Upper bounds for  $25 \leq n \leq 32$  by linear programming were published in [11]. Now the first open case is in dimension five, where it is known that  $40 \leq \tau_5 \leq 44$ . (The story of the upper bounds is:  $\tau_5 \leq L_5(5, 1/2) = 48$ ,  $\tau_5 \leq 46.345$  from [38]  $\tau_5 \leq 45$  from [5] and  $\tau_5 \leq 44.998$  from [33].)

Let  $n$  and  $s$  be fixed, the Levenshtein bound gives  $A(n, s) \leq L_m(n, s)$  and it can be improved as seen by Theorem 3. In [8], the first named author proposed a method for searching improving polynomials  $f(t) = A^2(t)G(t)$ , where  $A(t)$  must have  $m + 1$  zeros in  $[-1, s]$ ,  $G(s) = 0$  and  $G(t)/(t - s)$  is a polynomial of second or third degree polynomial which does not have zeros in  $[-1, s]$ . Moreover, one has  $f_i = 0$ ,  $i \in \{m, m + 1, m + 2, m + 3\}$  for two or three consecutive coefficients in the Gegenbauer expansion of  $f(t)$ . These restrictions leave several unknown parameters which can be found by consideration of the partial derivatives of  $f(1)/f_0$  and numerical optimization methods. This approach was realized (see [26]) by a programme SCOD. In fact, SCOD first checks for possible improvements by Theorem 3 and then applies the above method. It works well for improving  $L_m(n, s)$  for  $3 \leq m \leq 16$  and wide range of  $s$ .

**2.3. Strengthening the linear programming.** The linear programming bounds are based on the following identity

$$(3) \quad |C|f(1) + \sum_{\substack{x, y \in C \\ x \neq y}} f(\langle x, y \rangle) = |C|^2 f_0 + \sum_{i=1}^k \frac{f_i}{r_i} \sum_{j=1}^{r_i} \left( \sum_{x \in C} v_{ij}(x) \right)^2,$$



where  $C \subset \mathbf{S}^{n-1}$  is a spherical code,

$$f(t) = \sum_{i=0}^k f_i P_i^{(n)}(t),$$

$\{v_{ij}(x) : j = 1, 2, \dots, r_i\}$  is an orthonormal basis of the space  $\text{Harm}(i)$  of homogeneous harmonic polynomials of degree  $i$  and  $r_i = \dim \text{Harm}(i)$ . In the classical case (cf. [16, 25]) the sums of the both sides are neglected for polynomials which satisfy (A1) and (A2) and this immediately implies Theorem 1.

Musin [37] strengthened the linear programming approach by proposing the following extension of Theorem 1 which deals with a careful consideration of the left hand side of (3).

**Theorem 4** [37]. *Let  $f(t)$  be a real polynomial such that*

(B1)  $f(t) \leq 0$  for  $t_0 \leq t \leq s$ , where  $-t_0 > s$ ,

(B2)  $f(t)$  is decreasing function in the interval  $[-1, t_0]$ ,

(B3) *The coefficients in the Gegenbauer expansion  $f(t) = \sum_{k=0}^m f_k P_k^{(n)}(t)$*

*satisfy  $f_0 > 0, f_k \geq 0$  for  $i = 1, \dots, m$ .*

*Then*

$$A(n, s) \leq \frac{\max\{h_0, h_1, \dots, h_\mu\}}{f_0},$$

where  $h_m, m = 0, 1, \dots, \mu$ , is the maximum of  $f(1) + \sum_{j=1}^m f(\langle e_1, y_j \rangle)$ ,  $e_1 = (1, 0, \dots, 0)$ , over all configurations of  $m$  unit vectors  $\{y_1, y_2, \dots, y_m\}$  in the spherical cap (opposite of  $y_1$ ) defined by  $-1 \leq \langle y_1, x \rangle \leq t_0$  such that  $\langle y_i, y_j \rangle \leq s$ .

The proof of Theorem 4 follows from (3) in a similar way to the proof of Theorem 1 – neglect the nonnegative sum in the right hand side and replace the sum in the left hand side with its upper bound

$$\sum_{i=1}^{|C|} \sum_{j: \langle y_i, y_j \rangle \leq t_0} f(\langle y_i, y_j \rangle).$$

Now observe that the last expression does not exceed  $\frac{\max\{h_0, h_1, \dots, h_\mu\}}{f_0}$ .

Now the problems are to find  $\mu$ , choose  $t_0$  and a polynomial which minimizes the maximal value of  $h_0, h_1, \dots, h_\mu$ . In [37] good polynomials  $f(t)$  were

found by an algorithm which is similar to the algorithm of Odlyzko and Sloane [38]. One easily sees that  $h_0 = f(1)$  and  $h_1 = f(1) + f(-1)$ . However, the calculation of the remaining  $h_m$ 's usually requires estimations on

$$S(n, M) = \min \max \{s(C) : C \subset \mathbf{S}^{n-1} \text{ is a spherical code, } |C| = M\}$$

(cf. [9, 22, 23, 32, 37, 41]); observe that  $D(n, M) = \sqrt{2(1 - S(n, M))}$ . This approach was successfully applied in dimensions three and four. In [37] the third named author also noted that this generalization does not give better upper bounds on the kissing numbers in dimensions 5, 6, 7 and presumably can lead to improvements in dimensions 9, 10, 16, 17, 18.

For  $n = 3$  and  $s = 1/2$  it is proved that  $\mu = 4$ . Then we can choose  $t_0 = -0.5907$  and find a suitable polynomial of degree 9 (similar to these found in [38, 8, 26]) to show that  $\tau_3 = 12$ . Analogously, for  $n = 4$  and  $s = 1/2$  one has  $\mu = 6$ ,  $t_0 = -0.608$  and certain polynomial of degree 9 gives  $\tau_4 = 24$ . The calculations of  $h_0, h_1$  and  $h_2 = \max_{\varphi \leq \pi/3} \{f(1) + f(\cos \varphi) + f(-\cos(\pi/3 - \varphi))\}$  are easy but the computations of  $h_3, \dots, h_6$  require numerical methods.

**2.4. Semidefinite programming.** Let  $C = \{x_i\} \subset \mathbb{S}^{n-1}$  be a spherical code, let  $I \subset [-1, 1)$  and let

$$s_k(C, I) := \sum_{\substack{\langle x, y \rangle \in I \\ x, y \in C}} \langle x, y \rangle^k = |C| \sum_{t \in I} A_t t^k.$$

Odlyzko and Sloane [38] used in dimension 17 the constraints

$$s_0(C, I_1) \leq |C|, \quad s_0(C, I_2) \leq 2|C|,$$

where  $I_1 = [-1, -\sqrt{3}/2)$  and  $I_2 = [-1, -\sqrt{2/3})$ , to improve on the LP bound. More general, if it is known that the open spherical cap of angular radius  $\varphi$  can contain at most  $m$  points of a code  $C$  with  $S(C) = s$ , where  $\cos \varphi = t = \sqrt{s + (1 - s)/(m + 1)}$ , then  $s_0(C, I) \leq m|C|$ , where  $I = [-1, t)$ .

Pfender [39] found the inequality

$$s_2(C, I) \leq s_0(C, I)s + |C|(1 - s),$$

where  $I = [-1 - \sqrt{s})$ , and used it to improve the upper bounds for the kissing numbers in dimensions 9, 10, 16, 17, 25 and 26. In fact, the discussion in the preceding subsection can be viewed as in the following way: the third named author [34, 35, 36, 37] found a few inequalities for some linear combinations

of  $s_k(C, I)$  for  $0 \leq k \leq 9$ ,  $s = 1/2$  (the kissing numbers' case) and certain  $I = [-1, t_0]$ ,  $t_0 < -1/2$ . In particular, that gave the proof that  $\tau_4 = 24$  [37] and a new solution of the Thirteen spheres problem [35].

This approach can be further generalized by consideration of the three-point distance distribution

$$A_{u,v,t} = \frac{1}{|C|} |\{(x, y, z) \in C^3 : \langle x, y \rangle = u, \langle x, z \rangle = v, \langle y, z \rangle = t\}|$$

(note that  $A_{u,u,1} = A_u$ ). Here one needs to have  $1 + 2uv t \geq u^2 + v^2 + t^2$ . Bachoc and Vallentin [5] developed this to obtain substantial improvements for the kissing numbers in dimensions  $n = 5, 6, 7, 9$  and 10. Some numerical difficulties prevented Bachoc and Vallentin from further calculations but Mittelman and Vallentin [33] were able to overcome this and to report the best known upper bounds so far.

### 3. Lower bounds on kissing numbers.

**3.1. Constructions A and B.** The idea for using error-correcting codes for constructions of good spherical codes is natural for at least two reasons – it usually simplifies the description of codes and makes easier the calculation of the code parameters. Leech and Sloane [29] make systematic description of dense best sphere packings which can be obtained by error-correcting codes and give, in particular, the corresponding kissing numbers.

We describe Constructions A–B following [13]. Let  $C$  be an  $(n, M, d)$ -binary code. Then Construction A uses  $C$  to build a sphere packing in  $\mathbf{R}^n$  by taking centers of spheres  $(x_1, x_2, \dots, x_n)$ ,  $x_i$  are integers, if and only if the  $n$ -tuple

$$(x_1 \pmod{2}, x_2 \pmod{2}, \dots, x_n \pmod{2})$$

belongs to  $C$ .

The largest possible radius of nonoverlapping spheres is  $\frac{1}{2} \min\{2, \sqrt{d}\}$ . The touching points on the sphere with center  $x$  are

$$2^d A_d(x) \text{ if } d < 4, \quad 2n + 16A_4(x) \text{ if } d = 4, \quad 2n \text{ if } d > 4,$$

where  $A_i(x)$  is the number of codewords of  $C$  at distance  $i$  from  $x$ . Suitable choices of codes for Construction A give good spherical codes for the kissing number problem in low dimensions. The record lower bounds for the kissing numbers which can be produced by Construction A are shown in Table 1.

Table 1

Dimension	Best known lower bound	Best known upper bound
3	12	12
4	24	24
5	40	45
6	72	78
7	126	134
8	240	240
9	306	364
10	500	554
11	582	870
12	840	1357
13	1154	2069
14	1606	3183
15	2564	4866
16	4320	7355
17	5346	11072
18	7398	16572
19	10668	24812
20	17400	36764
21	27720	54584
22	49896	82340
23	93150	124416
24	196560	196560

Let in addition all codewords of  $C$  have even weight. Construction B takes centers  $(x_1, x_2, \dots, x_n)$ ,  $x_i$  are integers, if and only if  $(x_1 \pmod{2}, x_2 \pmod{2}, \dots, x_n \pmod{2}) \in C$  and 4 divides  $\sum_{i=1}^n x_i$ . The touching points on the sphere with center  $x$  are now

$$2^{d-1}A_d(x) \text{ if } d < 8, \quad 2n(n-1) + 128A_8(x) \text{ if } d = 8, \quad 2n(n-1) \text{ if } d > 8.$$

This, further development, of Construction A gives good codes for the kissing number problem in dimensions below 24. It is remarkable that Construction B produces the even part of the Leech lattice in dimension 24. The record achievements of Construction B are also indicated in Table 1.

Having the sphere packings (by Constructions A and B, for example) one can take cross-sections to obtain packings in lower dimensions and can build up layers for packings in higher dimensions. This approach is systematically used in [13] (see Chapters 5–8).

**3.2. Other constructions.** Dodunekov, Ericson and Zinoviev [17] proposed a construction which develops the ideas from the above subsection by putting some codes at suitable places (sets of positions in the original codes; this is called concatenation in coding theory). This construction gives almost all record cardinalities for the kissing numbers in dimensions below 24. Ericson and Zinoviev [18, 19, 20] later proposed more precise constructions which give records in dimensions 13 and 14 [20].

**4. A table for dimensions  $n \leq 24$ .** The table of Odlyzko and Sloane [13, 38] covers dimensions  $n \leq 24$ . Lower bounds by constructions via error-correcting codes in many higher dimensions can be found in [18, 13] (see also <http://www.research.att.com/njas/lattices/kiss.html>). The Table 1 reflects our present (July 2012) knowledge in dimensions  $n \leq 24$ .

The lower bounds in the Table 1 follow Table 1.5 from [13] apart from dimensions 13 and 14 taken from [20]. The upper bounds are mainly taken from [33] (dimensions 5-7, 9-23).

Note that recently in [12] new kissing configurations in dimensions 25-31 were found, which improve on the records set in 1982 by the laminated lattices.

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*Peter Boyvalenkov*  
*Institute of Mathematics and Informatics*  
*Bulgarian Academy of Sciences*  
*Acad. G. Bonchev Str., Bl. 8*  
*1113 Sofia, Bulgaria*  
*e-mail: peter@moi.math.bas.bg*

*Oleg Musin*  
*Department of Mathematics*  
*University of Texas at Brownsville*  
*80 Fort Brown, TX 78520, USA*  
*e-mail: oleg.musin@utb.edu*

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