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BOOK REVIEWS

NIKOLA OBRECHKOFF, **Selected Papers**. Prof. Marin Drinov Academic Publishing House, 2006; vol I, XII + 361 pp., € 20, ISBN 10 954-322-120-0(I), ISBN 13 978-954-322-120-2(I); vol II, XIV + 474 pp., € 20, ISBN 10 954-322-121-9(II), ISBN 13 978-954-322-121-9(II).

The distinguished Bulgarian mathematician with world recognition Nikola Obrechkoff was born on March 6, 1896, in Varna, Bulgaria, in a family of military officer. He received his primary education in his native town and in 1915 he finished the Second Boys' High School in Sofia.

In 1920 Obrechkoff graduated from the University of Sofia and immediately was appointed as an assistant professor at the Department of Differential and Integral Calculus of the Faculty of Physics and Mathematics of the University of Sofia. In 1922 he was elected as associate professor at the Department of Algebra at the same Faculty. Three years later he became an extraordinary professor and in 1928, at the age of 32, he was elected to the position of full professor and Head of the Department of Algebra. He held this position until his death in August 1963.

In short time Obrechkoff gained a very high scientific popularity. On October 13, 1932, he received the degree of Doctor of Mathematics from the Unversity of Palermo, Italia, and on July 20, 1933, a second doctoral degree, this time from the Sorbonne in Paris, after defending his Thesis "*Sur la sommation des séries divergentes*". The President of the Jury was Emil Borel and its members were Poll Montel and George Valiron.

Before reaching forty, Obrechkoff was already an internationally known figure, a welcome guest in major scientific centres of the mathematical thought and an active participant in international scientific events. His prestige grew up

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steadily and his achievements became widely recognized. His name leaded the list of the best known experts in his areas of investigations and his works were frequently quoted and referred to.

Nikola Obrechkoff left a rich scientific heritage of over 250 publications in Bulgarian and international journals and editions, including four monographs and several textbooks. The impressive scope and depth of his work will remain as a great example in the mathematical science: *classical algebra, number theory, mathematical analysis, probability theory, integral geometry, topology, equations of mathematical physics, mechanics*. Some of his achievements in these fields remained unsurpassed even till now.

Obrechkoff's favorite research area was the distribution of roots of algebraic equations. His first remarkable success was the generalization of the classical theorems of Budan-Fourier and Descartes for the number of the roots of an algebraic equation with real coefficients lying in a given interval. Obrechkoff gave an upper bound for the complex roots of such an equation with amplitudes in an interval depending on the number of sign-variations of its coefficients. This was the beginning of his very productive interest in this field which lasted to his death. Obrechkoff was recognized as a world expert on the distribution of zeros of polynomials. His numerous important results in this area became a starting point for the research of many mathematicians all over the world, e.g. for I.J. Schoenberg, S. Lipka and M. Marden. Obrechkoff had a number of followers in Bulgaria as well. If one can speak about Bulgarian School in the zero distribution of algebraic polynomials, then he is one of its founders.

In 1963, a few months before Obrechkoff's death, the crown of his studies in the field, the monograph "*Zeros of Polynomials*" was published in Bulgarian. Nearly at the same time "Akademie Verlag", Berlin, published his monograph "*Verteilung and Berechnung der Nullstellen reeller Polynome*". Forty years after Obrechkoff's death, in 2003, the first of these two monographs appeared in English as an edition of "Prof. Marin Drinov Academic Publishing House".

The Bulgarian Academy of Sciences marked 110 years of the birth of Nikola Obreshkoff by publishing a number of his papers in the present two volumes which contain the main achievements in the different fields of his scientific activity. Below we present some of the results included in the volumes.

In 1943 Obreshkoff published a version of the classical Taylor formula. It involves the Cesàro arithmetical means and is used later in other of his papers.

Obrechkoff has results about classical orthogonal polynomials. His paper on the asymptotics of Jacobi polynomials is referred in the well-known monograph of G. Szegö. In his paper on Bessel polynomials he defined their associated functions and on the basis of a formula of Christofel-Darboux type he proved

that every complex function holomorphic in a disk, centered at the origin, can be extended in a series of Bessel polynomials.

In a paper published in 1940, Obrechkoff proved a theorem of Tauberian type for the classical Laplace transform. It seems that this was the first result of such kind. He studied also an integral transform with the Whittaker confluent hypergeometric function as a kernel, and found an inversion formula for it. As a corollary a generalization of a classical theorem for the absolutely monotonic functions due to S.N. Bernstein is obtained.

In a paper from 1958, Obrechkoff introduced and investigated an integral transform which is an essential generalization of the Laplace transform. Due to the fact that the paper was published in Bulgarian, it remained unknown for the mathematical community. Later, several authors gave such generalizations and all of them turned out to be particular cases of Obrechkoff's one. Now, after the publication of I. Dimovski and V. Kiryakova in 1975, Obrechkoff's priority is restored.

The problem of asymptotics of the derivatives as well as of the integral representations of differentiable functions of one real variable has been in the attention of such outstanding mathematicians from the previous century as G. Hardy, J. Littlewood and E. Landau. It is remarkable that Obrechkoff succeeded to give a number of interesting results in this field having as special cases and corollaries theorems due to the above listed authors. One of his main tools was a new formula for the Newton quotients discovered by him.

In a paper from 1940, Obrechkoff gave a precise quadrature formula of a quite general form generalizing well-known classical formulas of this type and especially that of Newton. He used his new formula for the Cesàro means for the partial sums of the Taylor series of a differentiable function. A contribution to the numerical analysis is his modification of a method, due to Laguerre, which preserves the rate of convergence even in the case of multiple roots.

In the 1950's, when Obrechkoff reached creative maturity, he directed his interest and talent to the rather challenging area of arithmetics known as *Diophantine analysis*. For a short period of time he proved new and interesting theorems about the approximation of linear forms. In 1957 he found the exact value of Borel's constant, a problem that had remained unsolved for over than 50 years, which marked one of the peaks of Obrechkoff's work at all.

A fact confirming Obrechkoff's wide scientific abilities is that he paid attention to a problem from probability theory. In connection to Poisson's distribution he introduced and studied a system of polynomials named by him polynomials of Charlier. He found an interesting representation of the two-dimensional Poisson distribution, studied the asymptotic behaviour of the solu-

tions of systems of linear recurrence equations, considered integral as well as recurrence equations arising in the *Renewal theory* and proved theorems of Liouville type for their bounded solutions, investigated the convolution of density functions and its Fourier-Laplace transform and studied the asymptotic behaviour of the tail probabilities for the corresponding cumulative distribution function around the mean value.

Nearly one third of Obrechkoff's works are in one of the most actively elaborated region of investigations in the classical analysis in the 20-th century namely the summation of divergent series. Obrechkoff was extremely skillful at the summation of such series and was recognized as one of the greatest experts in this field. Being familiar to perfection with the classical methods of Riemann, Cesàro, Riesz, Euler-Knopp, Borel, Mittag-Leffler, and Hausdorff, he established deep results on the summation of the Taylor, Dirichlet, Fourier, and Laplace series. He introduced also the absolute summation by the typical means of Riesz and proved a number of precise Tauberian theorems. These essential contributions, included in the second volume, became a starting point for the investigations of many mathematicians in England, Germany and India.

Obrechkoff was the first who studied, by his own method, the summability of the Taylor series of a holomorphic function at points of regularity on the boundary of the region of summation under suitable assumptions about the behaviour of the function around its singular points. As corollaries he obtained results for the summations of Borel, Mittag-Leffler, and Euler-Knopp.

One of the most significant Obrechkoff's results are related to the summation of the Fourier series by the Cesàro methods. As early as in his first publications he announced results, particular cases of which are theorems of Lebesgue, Hardy-Littlewood, Pollard and others. Undoubtedly, one of the highest peaks of his scientific works is the final solution of the problem for the summation by Cesàro's method of the derived Fourier series of a L-integrable function.

The **Selected Papers** of N. Obrechkoff are recommended to everybody who is interested in the achievements in classical analysis, Diophantine approximations, and probability theory due to one of the heading specialists in these fields during the last century. The reader will be greatly impressed by Obrechkoff's technical ability and the richness of his results. Doubtless, some of these results are real pearls of the treasure-house of mathematical sciences.

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