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BERNSTEIN INEQUALITY FOR $1 - D$ HAMILTONIANS WITHOUT RESONANCES

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Abstract. We consider 1-D Laplace operator with short range potential $W(x)$ and prove the Bernstein inequality for this perturbed Laplacian. It is shown that non resonance assumption at zero and sufficiently fast decay of the potential at infinity guarantee that the Hamiltonian obeys the Bernstein inequality.

1. Introduction.

The study of (local and global in time) well - posedness of initial values problem for the Schrödinger equation with power nonlinearity

$$\left| \begin{array}{l} iu_t + \mathcal{H}u = \pm u|u|^{p-1}, \quad (t,x) \in \mathbb{R} \times \mathbb{R} \\ (u)|_{t=0} = f. \end{array} \right.$$

requires the systematic use of Sobolev (Besov) spaces associated with the perturbed Hamiltonian $\mathcal{H} = -\partial_x^2 + W(x)$. Here and below $W : \mathbb{R} \to \mathbb{R}$ is assumed to be a real-valued potential, $W \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and W is decaying sufficiently rapidly at infinity, namely following [10] we require

$$\|\langle x \rangle^\gamma W\|_{L^1(\mathbb{R})} < \infty, \ \gamma > 3, \tag{1}$$

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or equivalently we assume $W \in L^1_\gamma(\mathbb{R})$, where

$$L^p_\gamma(\mathbb{R}) = \{f \in L^p_{loc}(\mathbb{R}); \langle x\rangle^\gamma f(x) \in L^p(\mathbb{R})\},\ \langle x\rangle^2 = 1 + x^2.$$

The study of the decay (in time) of the orbit $e^{itH}f$ of the Schrödinger group is important initial step to treat corresponding nonlinear Schrödinger equation (NLS). In the case of free Hamiltonian $\mathcal{H}_0 = -\partial_x^2$ one can exploit the Strichartz estimate

$$\|e^{it\mathcal{H}_0} f\|_{L^p((0,\infty);\dot{B}^s_q(\mathbb{R}))} \leq C\|f\|_{\dot{H}^s(\mathbb{R})}. \tag{2}$$

where (p,q) is admissible couple (i.e. $2/p + 1/q = 1/2$, $4 \leq p \leq \infty$) and $\dot{B}^s_q(\mathbb{R})$, $\dot{H}^s(\mathbb{R})$ are homogeneous Besov and Sobolev spaces on $\mathbb{R}$.

Application of similar estimates for the NLS with perturbed Hamiltonian is more complicated in the presence of a point spectrum of $\mathcal{H}$.

We shall impose for simplicity in this work the assumption that the point spectrum of $\mathcal{H} = -\partial_x^2 + W(x)$ is empty, i.e.

$$\mathcal{H}f - zf = 0, f \in L^2(\mathbb{R}), z \in \mathbb{C} \Longrightarrow f = 0. \tag{3}$$

The functional calculus for the perturbed operator $\mathcal{H} = -\partial_x^2 + W$ is defined for any function $g \in L^\infty_{loc}(\mathbb{R})$ by the relation

$$g(-\partial_x^2 + W) = \frac{1}{2\pi \mathrm{i}} \int_0^\infty g(\lambda) E_{a.c.}(d\lambda), \tag{4}$$

where

$$\begin{aligned} E_{a.c.}(d\lambda) &= \\ = \lim_{\varepsilon \searrow 0} &\left[(\lambda + \mathrm{i}\varepsilon + \partial_x^2 - W)^{-1} - (\lambda - \mathrm{i}\varepsilon + \partial_x^2 - W)^{-1}\right] d\lambda. \end{aligned} \tag{5}$$

The functional calculus enables one to introduce a Paley-Littlewood partition of unity

$$1 = \sum_{j \in \mathbf{Z}} \varphi\left(\frac{t}{2^j}\right), t > 0$$

for an appropriate non - negative cutoff $\varphi \in C_0^\infty(\mathbf{R}_+)$, such that supp$\varphi \subseteq [1/2, 2]$.

The homogeneous Besov spaces $\dot{B}^s_p(\mathbb{R})$ for $p, 1 \leq p \leq \infty$ and $s \geq 0$ can be defined as the closure of $S(\mathbb{R})$ functions f with respect to the norm

$$\|f\|_{\dot{B}^s_p(\mathbb{R})} = \left(\sum_{j=-\infty}^\infty 2^{2js} \left\|\varphi\left(\frac{\sqrt{-\partial_x^2}}{2^j}\right) f\right\|^2_{L^p(\mathbb{R})}\right)^{1/2}. \tag{6}$$

The perturbed Hamiltonian can be associated with similar homogeneous Besov and Sobolev spaces. We shall denote by $\dot{B}^s_{p,\mathcal{H}}(\mathbb{R})$ the homogeneous Besov spaces associated with the perturbed Hamiltonian $\mathcal{H} = -\partial_x^2 + W$ as the closure of $S(\mathbb{R})$ functions f with respect to the norm

$$\|f\|_{\dot{B}^s_{p,\mathcal{H}}(\mathbb{R})} = \left(\sum_{j=-\infty}^{\infty} 2^{2js} \left\| \varphi\left(\frac{\sqrt{-\partial_x^2 + W}}{2^j} \right) f \right\|^2_{L^p(\mathbb{R})} \right)^{1/2}. \tag{7}$$

The application of the homogeneous Sobolev and Besov spaces in the free case typically involves Bernstein inequality. Then for any $1 \leq p \leq q \leq \infty$ and any $\varphi \in C_0^\infty(\mathbf{R}_+)$, such that $\mathrm{supp}\varphi \subseteq [1/2, 2]$ there exists a constant $C > 0$ so that we have

$$\left\| \varphi\left(\frac{\sqrt{-\partial_x^2}}{M} \right) f \right\|_{L^q(\mathbb{R})} \leq C M^{1/p-1/q} \|f\|_{L^p(\mathbb{R})}. \tag{8}$$

Our main goal in this work shall be the proof of this inequality for the perturbed Hamiltonian.

Theorem 1. *Suppose the condition (1) is fulfilled, the operator $\mathcal{H}$ has no point spectrum and 0 is not a resonance for $\mathcal{H}$. Then for any $1 \leq p \leq q \leq \infty$ and any $\varphi \in C_0^\infty(\mathbf{R}_+)$, such that $\mathrm{supp}\varphi \subseteq [1/2, 2]$ there exists a constant $C > 0$ so that we have*

$$\left\| \varphi\left(\frac{\sqrt{-\partial_x^2 + W}}{M} \right) f \right\|_{L^q(\mathbb{R})} \leq C M^{1/p-1/q} \|f\|_{L^p(\mathbb{R})} \tag{9}$$

for $M > 0$ and $f \in S(\mathbb{R})$.

2. Estimates for the Jost functions, transmission and reflection coefficients

In this section we recall some classical results concerning the spectral decomposition of the perturbed Hamiltonian. Recall that the Jost functions are solutions $f_\pm(x,\tau) = e^{\pm i\tau x} m_\pm(x,\tau)$ of $-\triangle_W u = \tau^2 u$ with

$$\lim_{x \to +\infty} m_+(x,\tau) = 1 = \lim_{x \to -\infty} m_-(x,\tau).$$

We set $x^+ := \max\{0, x\}$, $x^- := \max\{0, -x\}$ and $\langle x \rangle = \sqrt{1+x^2}$. We will denote by $L^p_s(\mathbb{R})$ the space with norm

$$\|u\|_{L^p_s} = \|\langle x\rangle^s f\|_{L^p_x}. \tag{10}$$

The assumption (1) on the decay of the potential can be rewritten as

$$W \in L^1_\gamma(\mathbb{R}),\ \gamma > 3.$$

The following lemma is well known.

Lemma 1. *Assume $W \in L^1_3(\mathbb{R})$. Then we have:*

a) For any $\tau \in \overline{\mathbb{C}_\pm}$ with $\mathbb{C}_\pm = \{\tau \in \mathbb{C}; \mathrm{Im}\tau \gtrless 0\}$ we have $m_\pm(\cdot,\tau) \in C^2(\mathbb{R},\mathbb{C})$;
b) For any $x \in \mathbb{R}$ we have $m_\pm(x,\cdot) \in C^1(\mathbb{R},\mathbb{C})$;
c) There exist constants C_1 and $C_2 > 0$ such that for any $x,\tau \in \mathbb{R}$:

$$|m_\pm(x,\tau) - 1| \le C_1\langle x^\mp\rangle\langle\tau\rangle^{-1}\ ; \tag{11}$$

$$|\partial_\tau m_\pm(x,\tau)| \le C_2\langle x\rangle^2. \tag{12}$$

See Lemma 1 p. 130 [5].

The estimate for partial derivatives in τ of $m_\pm(x,\tau)$ can be improved, since we admit $W \in L^1_\gamma(\mathbb{R})$ with $\gamma > 3$.

Lemma 2. *Suppose $W \in L^1_\gamma(\mathbb{R})$ with $\gamma > 1$. Then we have*

a) For any $x \in \mathbb{R}$ and $k = 0,1,2$ we have

$$\gamma > k+1 \Longrightarrow m_\pm(x,\cdot) \in C^k(\mathbb{R},\mathbb{C});$$

b) There exists a constant $C > 0$ such that for any $x \in \mathbb{R}$ and $\tau \in \mathbb{R}\setminus\{0\}$:

$$\gamma > 1 \Longrightarrow |m_\pm(x,\tau) - 1| \le \min\left(\frac{C\langle x_\mp\rangle}{\langle x_\pm\rangle^{\gamma-1}}, \frac{C\langle x_\mp\rangle}{\langle x_\pm\rangle^{\gamma}|\tau|}\right); \tag{13}$$

$$\gamma > 2 \Longrightarrow |\partial_\tau m_\pm(x,\tau)| \le \min\left(\frac{C\langle x_\mp\rangle^2}{\langle x_\pm\rangle^{\gamma-2}}, \frac{C\langle x_\mp\rangle^2}{\langle x_\pm\rangle^{\gamma-1}|\tau|}\right); \tag{14}$$

$$\gamma > 3 \Longrightarrow |\partial^2_\tau m_\pm(x,\tau)| \le \min\left(\frac{C\langle x_\mp\rangle^3}{\langle x_\pm\rangle^{\gamma-3}}, \frac{C\langle x_\mp\rangle^3}{\langle x_\pm\rangle^{\gamma-2}|\tau|}\right). \tag{15}$$

P r o o f. We shall choose for determinacy the sign + for the function $m_\pm$ in the left sides of (13) – (15) and (37) -(39), since the argument is similar for the term m_-. The integral equation satisfied by $m_+(x,\tau)$ is

$$m_+(x,\tau) = 1 + \int_x^\infty D(t-x,\tau)W(t)m_+(t,\tau)dt \tag{16}$$

due to Lemma 1 p. 130 [5]. Set

$$v(x) = \left|\frac{m_+(x,\tau)-1}{\langle x_-\rangle}\right|.$$

Our goal is to apply for $v(x)$ a Gronwall type inequality on the real line. In fact, we shall use the estimates

$$|\partial_\tau^k D(t,\tau)| \le C\langle t\rangle^{1+k},\ k=0,1,2,\cdots \tag{17}$$

fulfilled for any $\tau \in \overline{\mathbb{C}_+}$. If $\tau \in \overline{\mathbb{C}_+}\setminus\{0\}$ then we can add the following inequalities

$$|\partial_\tau^k D(t,\tau)| \le \frac{C\langle t\rangle^k}{|\tau|},\ k=0,1,2,\cdots \tag{18}$$

In particular, we can write

$$|D(t-x,\tau)| \le \langle t-x\rangle \le \langle t\rangle + \langle x_-\rangle,$$

since $x<t$. Then (16) and the definition of v implies

$$v(x) \le C\int_x^\infty \frac{(\langle t\rangle + \langle x_-\rangle)\langle t_-\rangle}{\langle x_-\rangle}|W(t)|v(t)dt + C\int_x^\infty \frac{(\langle t\rangle + \langle x_-\rangle)}{\langle x_-\rangle}|W(t)|dt. \tag{19}$$

Now we can use the estimates

$$x<t \implies \frac{(\langle t\rangle + \langle x_-\rangle)}{\langle x_-\rangle} \le C\langle t\rangle,\quad \frac{(\langle t\rangle + \langle x_-\rangle)\langle t_-\rangle}{\langle x_-\rangle} \le C\langle t\rangle. \tag{20}$$

Indeed, if $0<x<t$, then $\langle x_-\rangle = \langle t_-\rangle = 1$ and the inequalities in (20) are obvious. If $x<0<t$, then $\langle t_-\rangle = 1$ and $\langle x_-\rangle = \langle x\rangle$. The desired inequalities are reduced to

$$\frac{(\langle t\rangle + \langle x\rangle)}{\langle x\rangle} \le C\langle t\rangle$$

and this is again an obvious inequality. Finally, if $x<t<0$, then

$$\langle x_-\rangle = \langle x\rangle \ge \langle t_-\rangle = \langle t\rangle$$

and we have

$$\frac{(\langle t\rangle + \langle x_-\rangle)}{\langle x_-\rangle} \leq \frac{(\langle t\rangle + \langle x_-\rangle)\langle t_-\rangle}{\langle x_-\rangle} = \frac{(\langle t\rangle + \langle x\rangle)\langle t\rangle}{\langle x\rangle} \leq \frac{2\langle x\rangle)\langle t\rangle}{\langle x\rangle} = 2\langle t\rangle.$$

This observation shows that (20) is verified.

Turning to (19), we can use the assumption $\gamma > 1$ and we can write

$$\left|\int_x^\infty D(t-x,\tau)W(t)dt\right| \leq \int_x^\infty \langle t\rangle |W(t)|dt \quad \leq \quad \frac{C\|W\|_{L^1_\gamma(\mathbb{R})}}{\langle x_+\rangle^{\gamma-1}} \tag{21}$$

and we arrive at

$$v(x) \leq a(x) + \int_x^\infty b(t)v(t)dt, \tag{22}$$

where

$$a(x) = \frac{C\|W\|_{L^1_\gamma(\mathbb{R})}}{\langle x_+\rangle^{\gamma-1}}, \quad b(x) = C\,\langle x\rangle\,|W(x)|.$$

Applying the Gronwall lemma, we find

$$v(x) \leq \frac{C}{\langle x_+\rangle^{\gamma-1}},$$

so we have established the estimate

$$\gamma > 1 \Longrightarrow |m_\pm(x,\tau) - 1| \leq \frac{C\langle x_\mp\rangle}{\langle x_\pm\rangle^{\gamma-1}} \tag{23}$$

for $\tau \in \overline{C_+}$. We can complete the proof of (13) by using (18) in the place of (17).

To prove that $m_+(x,\cdot)$ is continuous in $\overline{C_+}$ we take $\tau_1, \tau_2 \in C_+$ we proceed in a similar way, we shall omit the details.

The existence and continuity of $\partial_\tau m_+(x,\cdot)$ can be deduced (as in Lemma 1 p. 130 [5]) using the fact that $m_+(x,\cdot)$ is analytic in the upper plane $\mathbb{C}_+$ and satisfies appropriate (uniform in τ) bounds.

Finally, the second derivative $\partial^2_\tau m_+(x,\tau)$ can be estimated in a similar way using the integral equation

$$\begin{aligned} \partial^2_\tau m_+(x,\tau) \quad &= \quad \int_x^\infty D(t-x,\tau)W(t)\partial^2_\tau m_+(t,\tau)dt + \\ + \int_x^\infty \partial^2_\tau D(t-x,\tau)W(t)m_+(t,\tau)dt \quad &+ \quad 2\int_x^\infty \partial_\tau D(t-x,\tau)W(t)\partial_\tau m_+(t,\tau)dt \end{aligned} \tag{24}$$

in the place of (16).

Setting

$$v_2(x) = \left| \frac{\partial_\tau^2 m_+(x,\tau)}{\langle x_- \rangle^3} \right|,$$

we quote (17) to obtain

$$|\partial_\tau^2 D(t-x,\tau)| \leq \langle t-x \rangle^3 \leq C\left(\langle t \rangle^3 + \langle x_- \rangle^3\right),$$

for $x < t$. Then (24) and the definition of v_2 implies

$$\begin{aligned} v_2(x) \quad \leq & \int_x^\infty \frac{C(\langle t \rangle + \langle x_- \rangle)\langle t_- \rangle^3}{\langle x_- \rangle^3} |W(t)| v_2(t) dt \\ & + \int_x^\infty \frac{C(\langle t \rangle^3 + \langle x_- \rangle^3)}{\langle x_- \rangle^3} |W(t)||m_+(t,\tau)| dt \\ & + \int_x^\infty \frac{C(\langle t \rangle^2 + \langle x_- \rangle^2)}{\langle x_- \rangle^3} |W(t)||\partial_\tau m_+(t,\tau)| dt. \end{aligned} \tag{25}$$

As before we use the estimates

$$x < t \implies \frac{(\langle t \rangle + \langle x_- \rangle)\langle t_- \rangle^3}{\langle x_- \rangle^3} \leq C\langle t \rangle, \tag{26}$$

$$x < t \implies \frac{(\langle t \rangle^3 + \langle x_- \rangle^3)|m_+(t,\tau)|}{\langle x_- \rangle^3} \leq C\langle t \rangle^3. \tag{27}$$

and

$$x < t \implies \frac{(\langle t \rangle^2 + \langle x_- \rangle^2)|\partial_\tau m_+(t,\tau)|}{\langle x_- \rangle^3} \leq C\langle t \rangle^3. \tag{28}$$

We skip the proof, since it is similar to the proof of (20).

Turning to (25), we can write now

$$\int_x^\infty \frac{(\langle t \rangle^3 + \langle x_- \rangle^3)|m_+(t,\tau)|}{\langle x_- \rangle^3} |W(t)| dt \leq C \int_x^\infty \langle t \rangle^3 |W(t)| dt \leq C \frac{\|W\|_{L^1_3(\mathbb{R})}}{\langle x_+ \rangle^{\gamma-3}},$$

$$\int_x^\infty \frac{(\langle t \rangle^2 + \langle x_- \rangle^2)|\partial_\tau m_+(t,\tau)|}{\langle x_- \rangle^3} |W(t)| dt \leq C \int_x^\infty \langle t \rangle^3 |W(t)| dt \leq C \frac{\|W\|_{L^1_3(\mathbb{R})}}{\langle x_+ \rangle^{\gamma-3}},$$

and we arrive at

$$v_2(x) \leq a_2(x) + \int_x^\infty b(t) v_2(t) dt, \tag{29}$$

where

$$a_2(x) = C\frac{\|W\|_{L^1_3(\mathbb{R})}}{\langle x_+\rangle^{\gamma-3}}, \quad b(x) = C\,\langle x\rangle\,|W(x)|.$$

Applying Gronwall lemma, we find

$$v_2(x) \leq \frac{C}{\langle x_+\rangle^{\gamma-3}},$$

so (15) is established.

This completes the proof of the Lemma. $\square$

In the next step we will find asymptotic expansion for $m_\pm(x,\tau)$. To this end we shall introduce appropriate classes of symbols $a(x,\tau)$ defined in a domain

$$U_\delta = \mathbb{R} \times ((-\infty, -\delta) \cup (\delta, +\infty))$$

for some $\delta > 0$.

Definition 1. *Given any real number $\delta > 0$ and any integer $L \geq 1$, we shall denote by $S^2_L(U_\delta)$ the linear space of all functions $a(x,\tau) \in C^2(U_\delta)$ such that*

$$\sup_{(x,\tau)\in U_\delta} \sum_{k=0}^{2} \left|\tau^L \partial_\tau^k a(x,\tau)\right| \leq C < \infty. \tag{30}$$

When there is no risk of confusion we shall skip the order 2 of the $\tau-$ derivatives in (30) and shall write simply $S_L(U_\delta)$ instead of $S^2_L(U_\delta)$.

Usually, the remainders of our asymptotic expansions shall be in a class of type $S_L(U_\delta)$). It is clear that

$$\cup_{L\geq 1} S_L(U_\delta)$$

is an algebra, since

$$a_1(x,\tau) \in S_{L_1}(U_\delta),\ a_2(x,\tau) \in S_{L_2}(U_\delta) \Longrightarrow a_1(x,\tau)a_2(x,\tau) \in S_{L_1+L_2}(U_\delta).$$

Appropriate subclass, taking into account the "hidden" oscillations inside the symbols, are introduced below.

Definition 2. *Given any integer $L \geq 1$, we shall denote by $S^{W,\pm}_L(\mathbb{R} \times (\mathbb{R} \setminus \{0\}))$ (or simply by $S^{W,\pm}_L$, when there is no risk of misunderstanding) the linear space generated by all functions $a(x,\tau)$ of the form*

$$a(x,\tau) = \frac{1}{\tau^L}\int_{D^\pm(x)} \chi(x,t) e^{\mathrm{i}\tau(xv_0+\langle v,t\rangle)} \left(\prod_{j=1}^{L} W(t_j)\right) dt, \tag{31}$$

where

$$D^{+}(x)=\{t\in\mathbb{R}^{L};x<\min(t_1,\cdots,t_L)\},\tag{32}$$

$$D^{-}(x)=\{t\in\mathbb{R}^{L};x>\max(t_1,\cdots,t_L)\},$$

$v_0\in\mathbb{R}, v\in\mathbb{R}^L$ *and* $\chi(x,\cdot)\in L^{\infty}(\mathbb{R}^L)$ *is such that*

$$\sup_{x\in\mathbb{R}}\|\chi(x,\cdot)\|_{L^{\infty}(\mathbb{R}^L)}\leq C<\infty.$$

Again one can check the multiplication rule

$$a_1(x,\tau)\in S^{W,\pm}_{L_1},\ a_2(x,\tau)\in S^{W,\pm}_{L_2}\Longrightarrow a_1(x,\tau)a_2(x,\tau)\in S^{W,\pm}_{L_1+L_2}.\tag{33}$$

Lemma 3. *Suppose $W\in L^1_{\gamma}(\mathbb{R})$ with $\gamma>1$. There exists a constant $C>0$ such that for $k=1,2$ any $x\in\mathbb{R}$ and $\tau\in\mathbb{R}$ with $|\tau|\geq 1$ we have the following asymptotic expansion:*

$$m_{\pm}(x,\tau)=\left(\sum_{j=0}^{k}m_j^{\pm}(x,\tau)\right)+\mathfrak{R}_k^{m,\pm}(x,\tau),\tag{34}$$

where:

a) $m_0^{\pm}(x,\tau)=1$, *and the recurrence relation*

$$m_j^{\pm}(x,\tau)=\int_{t\gtrless x}D(\pm(t-x),\tau)W(t)m^{\pm}_{j-1}(t,\tau)dt,\ j=1,2,\tag{35}$$

holds with

$$D(t,\tau)=\frac{e^{2it\tau}-1}{2i\tau}=\int_0^t e^{2iy\tau}dy;\tag{36}$$

b) $m_j(x,\tau)\in S^{W,\pm}_j$, $j=1,2$;

c) The remainders $\mathfrak{R}_k^{m,\pm}(x,\tau)$ *obey the estimates*

$$\gamma>1,k=1,2\Longrightarrow|\mathfrak{R}_k^{m,\pm}(x,\tau)|\leq\frac{C}{\langle x_{\pm}\rangle^{(k+1)\gamma}|\tau|^{k+1}};\tag{37}$$

$$\gamma>2,k=1,2\Longrightarrow|\partial_{\tau}\mathfrak{R}_k^{m,\pm}(x,\tau))|\leq\frac{C\langle x_{\mp}\rangle}{\langle x_{\pm}\rangle^{\gamma(k+1)-1}|\tau|^{k+1}};\tag{38}$$

$$\gamma>3,k=1,2\Longrightarrow|\partial^2_{\tau}\mathfrak{R}_k^{m,\pm}(x,\tau))|\leq\frac{C\langle x_{\mp}\rangle^2}{\langle x_{\pm}\rangle^{\gamma(k+1)-2}|\tau|^{k+1}}.\tag{39}$$

d) If $\gamma > 3$, then $\mathfrak{R}_k^{m,+}(x,\tau) \in S_{k+1}$ for $x > 0$, $k = 1,2$ and $\mathfrak{R}_k^{m,-}(x,\tau) \in S_{k+1}$ for $x < 0$, $k = 1,2$.

Proof.
We make the substitution

$$m_+(x,\tau) = 1 + \mathfrak{R}_0^{m,+}(x,\tau)$$

in the integral equation (16) and follow the same line already presented in the proof of the previous Lemma, so we shall skip the details.
□
The transmission coefficient $T(\tau)$ and the reflection coefficients $R_\pm(\tau)$ are defined by the formula

$$T(\tau)m_\mp(x,\tau) = R_\pm(\tau)e^{\pm 2\mathrm{i}\tau x}m_\pm(x,\tau) + m_\pm(x,-\tau). \tag{40}$$

From [5] and from [10] we have the following lemma.

Lemma 4. *We have the following properties of the transmissions and reflection coefficients.*

a) $T, R_\pm \in C(\mathbb{R})$.
b) There exists $C_1, C_2 > 0$ such that:

$$|T(\tau) - 1| + |R_\pm(\tau)| \le C_1\langle\tau\rangle^{-1} \tag{41}$$

$$|T(\tau)|^2 + |R_\pm(\tau)|^2 = 1. \tag{42}$$

c) If $T(0) = 0$, (i.e. zero is not a resonance point), then for some $\alpha \in \mathbb{C} \setminus \{0\}$ and for some $\alpha_+, \alpha_- \in \mathbb{C}$

$$T(\tau) = \alpha\tau + o(\tau), \quad 1 + R_\pm(\tau) = \alpha_\pm\tau + o(\tau), \tag{43}$$

for $\tau \in \mathbb{R}$, $\tau \to 0$.

We can use the stronger decay assumption $W \in L^1_\gamma(\mathbb{R})$, $\gamma > 3$, to get some more precise bounds.

Lemma 5. *Suppose $W \in L^1_\gamma(\mathbb{R})$ with $\gamma > 3$ and $T(0) = 0$. Then we have:*

a) $T, R_\pm \in C^2(\mathbb{R})$;

b) There exists $C>0$ such that for any $\tau \in \mathbb{R}$ we have:

$$\sum_{k=0}^{2} \left| \frac{d^k}{d\tau^k} T(\tau) \right| + \left| \frac{d^k}{d\tau^k} R_{\pm}(\tau) \right| \leq C, \tag{44}$$

$$|\langle\tau\rangle\,[T(\tau)-1]| + |\langle\tau\rangle R_{\pm}(\tau)| + \sum_{k=1}^{2} \left| \langle\tau\rangle \frac{d^k}{d\tau^k} T(\tau) \right| + \left| \langle\tau\rangle \frac{d^k}{d\tau^k} R_{\pm}(\tau) \right| \leq C; \tag{45}$$

Proof.

The proof is based on the relations

$$\frac{\tau}{T(\tau)} = \tau - \frac{1}{2i} \int_{\mathbb{R}} W(t) m_{+}(t,\tau) dt \tag{46}$$

and

$$R_{\pm}(\tau) = \frac{T(\tau)}{2i\tau} \int_{\mathbb{R}} e^{\mp 2it\tau} W(t) m_{\mp}(t,\tau) dt \tag{47}$$

so we can skip the details.

□

We can improve the asymptotic expansions of $T(\tau), R_{\pm}(\tau)$ for $|\tau| \geq 1/10$.

Before stating the precise result we shall introduce class of symbols

$$a(\tau) \in C^2(\{\tau \in \mathbb{R}, |\tau| > 1/10\}),$$

such that

$$\sum_{k=0}^{2} |\partial_\tau^k a(\tau)| \leq \frac{C}{|\tau|^L}, |\tau| > \frac{1}{10}. \tag{48}$$

We shall denote the class of symbols satisfying this property by $S^2_L(\mathbb{R})$ or $S_L(\mathbb{R})$.

Appropriate subclass (denoted by $S^{2,W}_L(\mathbb{R})$ or by $S^W_L(\mathbb{R})$) is defined by linear combinations of functions $a(\tau) \in C^2(\{\tau \in \mathbb{R}, |\tau| > 1/10\})$, having the representation formula

$$a(\tau) = \frac{1}{\tau^L} \int_{\mathbb{R}^L} \chi(t) e^{\mathrm{i}\tau\langle v,t\rangle} \left(\prod_{j=1}^{L} W(t_j) \right) dt, \tag{49}$$

for some vector $v \in \mathbb{R}^L$ and some $\chi(t) \in L^\infty(\mathbb{R}^L)$.

Lemma 6. *Suppose $W \in L^1_\gamma(\mathbb{R}) \cap L^\infty_\beta$ with $\gamma, \beta > 3$ and $T(0) = 0$. Then there exists a constant $C > 0$ such that for any $x \in \mathbb{R}$ and $\tau \in \mathbb{R}$ with $|\tau| \geq 1$ we have the following asymptotic expansion:*

$$T(\tau) = \left(\sum_{j=0}^{2} T_j(\tau)\right) + \mathfrak{R}_2^T(x,\tau), \quad R^\pm(\tau) = \left(\sum_{j=0}^{2} R_j^\pm(\tau)\right) + \mathfrak{R}_2^{R,\pm}(\tau), \tag{50}$$

where $T_0(\tau) = 1, R_0^\pm(\tau) = 0$,

$$T_1(\tau), R_1^\pm \in S_1^W(\mathbb{R}), \ T_2(\tau), R_2^\pm \in S_2^W(\mathbb{R}) \tag{51}$$

and the remainders $\mathfrak{R}_2^T(x,\tau)$, $\mathfrak{R}_2^{R,\pm}(\tau)$ belong to $S_3(\mathbb{R})$, i.e. they obey the estimates

$$\sum_{k=0}^{2} \left|\partial_\tau^k \mathfrak{R}_2^T(x,\tau)\right| + \left|\partial_\tau^k \mathfrak{R}_2^{R,\pm}(x,\tau)\right| \leq \frac{C}{|\tau|^3} < \infty. \tag{52}$$

Proof.
The proof is based on the relations

$$T(\tau) = \frac{1}{1-\Psi(\tau)}, \ \Psi(\tau) = \tau^{-1}\Phi(\tau) = \frac{1}{2\mathrm{i}\tau}\int_{\mathbb{R}} W(t)m_+(t,\tau)dt. \tag{53}$$

and we omit the details. $\square$

3. Functional calculus and Bernstein inequality for perturbed Hamiltonian

The functional calculus in (4), (5) can be connected with the Jost functions. In fact the kernel $(\lambda \pm \mathrm{i}0 + \partial_x^2 - W)^{-1}(x,y)$ of the operator $(\lambda \pm \mathrm{i}0 + \partial_x^2 - W)^{-1}$ satisfies the relation

$$(\lambda \pm \mathrm{i}0 + \partial_x^2 - W)^{-1}(x,y) = \begin{cases} \frac{f_-(x,\pm\sqrt{\lambda})f_+(y,\pm\sqrt{\lambda})}{w(\pm\sqrt{\lambda})}, & \text{if } x < y; \\ \frac{f_-(y,\pm\sqrt{\lambda})f_+(x,\pm\sqrt{\lambda})}{w(\pm\sqrt{\lambda})}, & \text{otherwise.} \end{cases} \tag{54}$$

where the Wronskian $w(\tau)$ is defined by the relation

$$w(\tau) := (\partial_x f_+)(x,\tau)f_-(x,\tau) - f_+(x,\tau)\partial_x f_-(x,\tau). \tag{55}$$

Using the well - known identity (see p.144, [5])

$$\frac{1}{T(\tau)} = \frac{w(\tau)}{2\mathrm{i}\tau} \tag{56}$$

we get

(57)
$$g(-\partial_x^2+W)(x,y)=\int_0^\infty \tau g(\tau^2)\left[\frac{f_-(x,\tau)f_+(y,\tau)}{w(\tau)}-\frac{f_-(x,-\tau)f_+(y,-\tau)}{w(-\tau)}\right]\frac{d\tau}{\pi\mathbf{i}}$$
$$=-\frac{1}{2\pi}\int_{\mathbf{R}} T(\tau)g(\tau^2)f_-(x,\tau)f_+(y,\tau)d\tau$$

for $x<y$. These relations imply that for any $h(\tau)\in L^1(0,\infty)$ one can extend h as an even function $\widetilde{h}(\tau)\in L^1(\mathbb{R})$ so that the relation

$$h\left(\sqrt{-\partial_x^2+W}\right)(x,y)=-\frac{1}{2\pi}\int_{\mathbb{R}}\widetilde{h}\,(\tau)\,T(\tau)m_+(y,\tau)m_-(x,\tau)e^{-\mathrm{i}\tau(x-y)}d\tau. \tag{58}$$

holds for $x<y$.

The proof of the Bernstein inequality is based on few technical estimates that we state and prove now.

Theorem 2. *Suppose the condition (1) is fulfilled, the operator H has no point spectrum and 0 is not a resonance point for H. If φ is an even non - negative function, such that $\varphi\in C_0^\infty(\mathbf{R}\setminus\{0\})$, then for any $M>0$ we have*

a) if $0<M<1$, then

$$\left|\varphi\left(\frac{\sqrt{-\partial_x^2+W}}{M}\right)(x,y)\right|\leq\frac{CM^2}{\langle M(x-y)\rangle^2}+\frac{CM^2}{\langle M(x+y)\rangle^2}; \tag{59}$$

b) if $M\geq 1$, then we have

$$\left|\varphi\left(\frac{\sqrt{-\partial_x^2+W}}{M}\right)(x,y)-\varphi\left(\frac{\sqrt{-\partial_x^2}}{M}\right)(x,y)\right| \leq \tag{60}$$
$$\leq\frac{C}{\langle M^{1/3}(x-y)\rangle^2}+\frac{C}{\langle M^{1/3}(x+y)\rangle^2}\quad .$$

P r o o f. We shall assume $x<y$ for determinacy. Then (58) implies

(61)
$$\varphi\left(\frac{\sqrt{-\partial_x^2+W}}{M}\right)(x,y)=c\int_{\mathbb{R}}\varphi\left(\frac{\tau}{M}\right)T(\tau)m_+(y,\tau)m_-(x,\tau)e^{-\mathrm{i}\tau(x-y)}d\tau=$$
$$=cM\int_{\mathbb{R}}\varphi\,(\tau)\,T(M\tau)m_+(y,M\tau)m_-(x,M\tau)e^{-\mathrm{i}M\tau(x-y)}d\tau.$$

First, we shall prove (59) so we can assume $0 < M < 1$. We shall use the second identity in (61) that is

(62)
$$\varphi\left(\frac{\sqrt{-\partial_x^2+W}}{M}\right)(x,y) = cM\int_{\mathbb{R}} \varphi(\tau)\, T(M\tau) m_+(y, M\tau) m_-(x, M\tau) e^{-\mathrm{i}M\tau(x-y)} d\tau.$$

Since $x < y$, we have three different cases.

(Case A) $$x < 0 < y,$$

(Case B) $$0 \le x < y,$$

(Case C) $$x < y \le 0.$$

In the (Case A), we can use two integration by parts in the oscillatory integral representing $\varphi\left(\sqrt{-\partial_x^2+W}/M\right)(x,y)$ and we use the estimates of Lemmas 1 and 4 . In fact, we have (for $x < 0 < y$)

$$\sum_{k=0}^{2} |\partial_\tau^k m_+(y, M\tau)| \le C, \ \sum_{k=0}^{2} |\partial_\tau^k m_-(x, M\tau)| \le C \tag{63}$$

and also

$$\left|\frac{T(M\tau)}{M\tau}\right| + \sum_{k=1}^{2} |\partial_\tau^k T(M\tau)| \le C \tag{64}$$

due to Lemma 1 and Lemma 4. One can apply integration by parts two times in (62), so we get (59).

In the (Case B) we can use (40) and we can write the identity

$$T(\tau) m_-(x,\tau) = R_+(\tau) e^{2\mathrm{i}\tau x} m_+(x,\tau) + m_+(x,-\tau), \tag{65}$$

so (62) becomes now

$$\varphi\left(\frac{\sqrt{-\partial_x^2+W}}{M}\right)(x,y) = cMI_1(x,y;M) + cMI_2(x,y;M), \tag{66}$$

where

$$I_1(x,y;M) = \int_{\mathbb{R}} \varphi(\tau)\, m_+(y, M\tau)(R_+(M\tau)+1) m_+(x, M\tau) e^{\mathrm{i}M\tau(x+y)} d\tau,$$

$$I_2(x,y;M) =$$
$$= \int_{\mathbb{R}} \varphi(\tau)\, m_+(y,M\tau)(m_+(x,-M\tau) - e^{2\mathrm{i}Mx\tau} m_+(x,M\tau)) e^{-\mathrm{i}M\tau(x-y)} d\tau.$$

For both integrals we can perform two integrations by parts and apply the estimates of Lemma 1 and Lemma 4 for $0 < x < y$ as follows. By Lemma 1 we can write

$$\sum_{k=0}^{2} |\partial_\tau^k m_+(y,M\tau)| \le C, \ \sum_{k=0}^{2} |\partial_\tau^k m_+(x,\pm M\tau)| \le C. \tag{67}$$

We can add the obvious (but crucial) estimate

$$\left| \frac{m_+(y,-M\tau) - e^{2\mathrm{i}Mx\tau} m_+(x,M\tau)}{M\tau} \right| \le C. \tag{68}$$

The estimates of Lemma 4 imply

$$\left|\frac{T(M\tau)}{M\tau}\right| + \sum_{k=1}^{2} |\partial_\tau^k T(M\tau)| \le C, \ \left|\frac{R(M\tau)+1}{M\tau}\right| + \sum_{k=1}^{2} |\partial_\tau^k R_+(x,M\tau)| \le C. \tag{69}$$

In this way we get

$$|I_1(x,y;M)| \le \frac{CM}{\langle M(x+y)\rangle^2}, \ |I_2(x,y;M)| \le \frac{CM}{\langle M(x-y)\rangle^2}$$

and we obtain (59) in the (Case B).

In the (Case C) we follow the argument used in the (Case B), but this time we replace (65) by the following relation

$$T(\tau) m_+(y,\tau) = R_-(\tau) e^{-2\mathrm{i}\tau y} m_-(y,\tau) + m_-(y,-\tau), \tag{70}$$

and derive (59) using two integrations by parts.

This completes the proof of (59).

Next, we turn to (60) so we can assume $M \ge 1$.

In the (Case A) we have $x < 0 < y$. We shall use the first identity in (61) so we start with

$$\varphi\left(\frac{\sqrt{-\partial_x^2 + W}}{M}\right)(x,y) \quad - \quad \varphi\left(\frac{\sqrt{-\partial_x^2}}{M}\right)(x,y) =$$
$$= \quad c \quad M \int_{\mathbb{R}} \varphi(\tau) \left[T(M\tau) m_+(y,M\tau) m_-(x,M\tau) - 1\right] e^{-\mathrm{i}M\tau(x-y)} d\tau. \tag{71}$$

Note that the symbol

$$a(x,y,\tau) = T(\tau)m_+(y,\tau)m_-(x,\tau) - 1$$

can be expand using Lemmas 3 and 6 as follows

$$a(x,y,\tau) = a_1(x,y,\tau) + a_2(x,y,\tau) + \mathfrak{R}_3(x,y,\tau),$$

where

$$a_1(x,y,\tau) = T_1(\tau) + m_1^+(y,\tau) + m_1^-(x,\tau)$$

$$a_2(x,y,\tau) = T_2(\tau) + m_2^+(y,\tau) + m_2^-(x,\tau) +$$

$$+T_1(\tau)m_1^+(y,\tau) + T_1(\tau)m_1^-(x,\tau) + m_1^+(y,\tau)m_1^-(x,\tau)$$

and the remainder $\mathfrak{R}_3(x,y,\tau)$ is in $S_3(\mathbb{R}\times\mathbb{R}\times\{|\tau|\geq\delta>0\})$, i.e. satisfies the estimate

$$\sum_{k=0}^{2} |\partial_\tau^k \mathfrak{R}_3(x,y,\tau)| \leq \frac{C}{|\tau|^3} \tag{72}$$

for $|\tau|\geq\delta$.

We have to estimate each of the oscillatory integrals

$$\int_{\mathbb{R}} \varphi\,(\tau)\, a_j(x,y,M\tau) e^{-\mathrm{i}M\tau(x-y)} d\tau, \;\; j=1,2,$$

and

$$\int_{\mathbb{R}} \varphi\,(\tau)\, \mathfrak{R}_3(x,y,M\tau) e^{-\mathrm{i}M\tau(x-y)} d\tau.$$

Since the remainder satisfies (72) we can use integration by parts and deduce

$$\left| \int_{\mathbb{R}} \varphi\,(\tau)\, \mathfrak{R}_3(x,y,M\tau) e^{-\mathrm{i}M\tau(x-y)} d\tau \right| \leq \frac{C}{\langle M^{1/3}(x-y)\rangle^2}.$$

The symbols $a_j(x,y,M\tau), j=1,2$ can be represented as oscillatory integrals of type (49) so we are in position to use integration by parts and deduce

$$\left| \int_{\mathbb{R}} \varphi\,(\tau)\, a_j(x,y,M\tau) e^{-\mathrm{i}M\tau(x-y)} d\tau \right| \leq \frac{C}{\langle M^{1/3}(x-y)\rangle^2}$$

with $j=1,2$ thus we obtain (60) in the (Case A).

In the (Case B) we can use (40) and we can write the identity (65) so (71) becomes now

$$(73)\ \varphi\left(\frac{\sqrt{-\partial_x^2+W}}{M}\right)(x,y)-\varphi\left(\frac{\sqrt{-\partial_x^2}}{M}\right)(x,y)=cI_3(x,y;M)+cI_4(x,y;M),$$

where

$$I_3(x,y;M)=\int_{\mathbb{R}}\varphi\left(\frac{\tau}{M}\right)m_+(y,\tau)R_+(\tau)m_+(x,\tau)e^{\mathrm{i}\tau(x+y)}d\tau,$$

$$I_4(x,y;M)=\int_{\mathbb{R}}\varphi\left(\frac{\tau}{M}\right)[m_+(y,\tau)m_+(x,-\tau)-1]e^{-\mathrm{i}\tau(x-y)}d\tau$$

each of the integrals I_3 and I_4 can be treated as in the (Case A) by using the asymptotic expansions of Lemmas 3 and 6, so we get

$$|I_3(x,y;M)|\leq\frac{C}{\langle M^{1/3}(x-y)\rangle^2}.$$

and

$$|I_4(x,y;M)|\leq\frac{C}{\langle M^{1/3}(x+y)\rangle^2}.$$

In the (Case C) we follow the argument used in the (Case B), but this time we replace (65) by the relation (70) and derive (60).

This completes the proof of the Theorem.

□

Now we can prove the Bernstein inequality (9).

Corollary 1. *Suppose the condition (1) is fulfilled, the operator H has no point spectrum and 0 is not a resonance for H. Assume further $1\leq p\leq q\leq\infty$ and $\varphi(\tau)$ is a smooth function with compact support separated from the origin. Then there exists a constant $C>0$ so that:*

a) the Bernstein inequality

$$\left\|\varphi\left(\frac{\sqrt{-\partial_x^2+W}}{M}\right)f\right\|_{L^q(\mathbb{R})}\leq CM^{1/p-1/q}\|f\|_{L^p(\mathbb{R})}\tag{74}$$

holds for any $M>0$ and any $f\in S(\mathbb{R})$;

b) if $0<M\leq 1$, then we have a stronger estimate

$$\left\|\varphi\left(\frac{\sqrt{-\partial_x^2+W}}{M}\right)f\right\|_{L^q(\mathbb{R})}\leq CM^{1+1/p-1/q}\|f\|_{L^p(\mathbb{R})}.\tag{75}$$

Proof. First consider the case $0 < M \leq 1$. Then we have the estimates (59) so we can write

$$\left|\varphi\left(\frac{\sqrt{-\partial_x^2+W}}{M}\right)(x,y)\right| \leq \frac{CM^2}{\langle M(x-y)\rangle^2} + \frac{CM^2}{\langle M(x+y)\rangle^2}$$

so we can use the pointwise estimates

$$\left|\varphi\left(\frac{\sqrt{-\partial_x^2+W}}{M}\right)f(x)\right| \leq CMK_+^M(|f|)(x) + CMK_-^M(|f|)(x),$$

where

$$K_\pm^M(f)(x) = M\int_{\mathbb{R}} \langle M(x\pm y)\rangle^{-2} f(y)dy$$

The Young inequality implies that we have

$$\|K_\pm^M(f)\|_{L_x^q} \leq CM^{1/p-1/q}\|f\|_{L_x^p}$$

so (75) is fulfilled for $0 < M \leq 1$.

For $M \geq 1$ we quote (60) and start with

$$\left|\varphi\left(\frac{\sqrt{-\partial_x^2+W}}{M}, M\right)(x,y)\right| \leq \frac{C}{\langle M^{1/3}(x-y)\rangle^2} + \frac{C}{\langle M^{1/3}(x+y)\rangle^2}.$$

and we can use the pointwise estimates

$$\left|\varphi\left(\frac{\sqrt{-\partial_x^2+W}}{M}\right)f(x)\right| \leq C\widetilde{K_-^M}(|f|)(x) + C\widetilde{K_+^M}(|f|)(x),$$

where

$$\widetilde{K_\pm^M}(f)(x) = \int_{\mathbb{R}} \langle M^{1/3}(x\pm y)\rangle^{-2} f(y)dy.$$

The Young inequality implies

$$\|\widetilde{K_\pm^M}(f)\|_{L_x^q} \leq CM^{(1/p-1/q-1)/3}\|f\|_{L_x^p} \leq CM^{1/p-1/q}\|f\|_{L_x^p},$$

since $M \geq 1$.

Hence (9) is fulfilled for $M \geq 1$ and this completes the proof.

□

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