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Pliska Stud. Math. Bulgar. **23** (2014), 49–56

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EXISTENCE RESULTS FOR SOME VARIATIONAL INEQUALITIES INVOLVING NON-NEGATIVE, NON-COERCITIVE BILINEAR FORMS

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Abstract. In the present paper the existence of solutions to variational inequalities for semi-coercive bilinear forms is studied. The result generalizes a result by Lions-Stampacchia and is close to an abstract result by Fichera.

In the present paper we study the existence of solutions u of variational inequalities

$$a(u, v - u) \geq (f, v - u)$$

for all $v \in K$, where $a(u, v)$ is semi-coercive continuous bilinear form on a Hilbert space H and K is a closed convex subset in H. In this direction central place occupy the results in [3] (Theorem 2.I) and [6] (Theorem 5.1) (or [4], Ch. III, theorem 2.3). Both give sufficient conditions for the existence of solutions involvig the kernel of the bilinear form, the set of the so called unbounded directions of the convex set and the right hand side f. In [6] is considered convex set containing the origin, whereas in [3] more general convex set is considered, as well as more elements in the right hand-side, but some projections of the convex set are assumed closed. (Another paper treating similar problems and in particular in more detail the relations between sufficient and necessary conditions is [1].)

2000 *Mathematics Subject Classification:* Primary: 49J40; Secondary: 35J85.
Key words: variational inequalities, semi-coercive bilinear forms, recession cone.

The result proposed in the present paper is in a sence intermediate. It generalizes the result in [6] and the proof is closer to the one given there. On the other hand the formulations are along the lines of [3], but are more concize using the notion of recession cone. Although the proposed sufficient conditions do not include all the right hand-sides of [3], no conditions of closedness are imposed. The results in [6] are obtained as a corollary. It seems that the proof we propose gives more insight into the nature of the conditions imposed on f.

Notations. Let H be a real Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $a(u,v)$ be a continuous bilinear form on H. Let a be nonnegative, i.e. $a(u,u)\geq 0$. Let N be the kernel of a, i.e.

$$N=\{u : a(u,u)=0\}.$$

The bilinear form defines an operator

$$L : H \longrightarrow H$$

according to

$$(Lu,v)=a(u,v) \qquad \forall v\in H.$$

The operator L thus defined is monotone and furthermore

$$N=\ker(L+L^*)$$

where L^* is the adjoint of L and

$$\ker L=\ker L^*\subset \ker(L+L^*).$$

Let K be a closed convex nonempty subset of H. Recession cone (or the set of unbounded directions, or assymptotic cone) is the (possibly empty) set

$$K_\infty=\bigcap_{\lambda>0}\lambda(K-k_0)$$

where $k_0\in K$. The elementary facts we need about recession cones are collected in the following

Lemma 1. *If $u_n\in K$, $\|u_n\|\to\infty$ and*

$$w=\lim\frac{u_n}{\|u_n\|}$$

then $w\in K_\infty$.

If $w\in K_\infty$ then for every $v_0\in K$ and $t\geq 0$ we have

$$v_0+tw\in K.$$

Let M be a closed subspace of H with $N \subset M$, let $Q : H \longrightarrow H$ be the orthogonal projection of H on M and let $P = I - Q$ where I is the identity of H.

Theorem 1. *Let a be as above and let furthermore*
(i) M is finite dimensional
(ii) there is a positive constant α such that

$$\alpha \|Pu\|^2 \leq (Lu, u).$$

Let $f \in H$ be such that

$$(f, w) < 0 \qquad \forall w \in \ker L \cap K_\infty \ (w \neq 0) \tag{1}$$

and for every $w \in \ker(L + L^*) \cap K_\infty$, $w \notin \ker L$, $(w \neq 0)$ there exists a $v_w \in K$ such that

$$(f, w) + (w, L^* v_w) < 0. \tag{2}$$

Then the variational inequality

$$(Lu, u - v) \leq (f, u - v) \qquad \forall v \in K \tag{3}$$

has a solution.

Remark. In many applications and in particular in [3], $M = N$. This more general case is needed in order to obtain the result in [6] as a direct corollary.

Proof. Let for $R > 0$

$$B_R = \{u : \|Qu\| \leq R\}$$

and

$$K_R = K \cup B_R.$$

Obviously K_R is closed, convex and is nonempty for all sufficiently large R. The operator L is coercitive on K_R. Indeed from (ii) it follows that

$$\alpha \|u\|^2 \leq (Lu, u) + \alpha \|Qu\|^2$$

whence

$$\frac{(Lu, u - k)}{\|u\|} \geq \alpha \|u\| - \frac{(Lu, k)}{\|u\|} - \frac{\alpha \|Qu\|^2}{\|u\|} \geq \alpha \|u\| - c\|k\| - \frac{\alpha R^2}{\|u\|} \to \infty$$

for $\|u\| \to \infty$ $(u \in K_R)$. Then the variational inequality

$$(Lu, u - v) \leq (f, u - v) \qquad \forall v \in K_R \tag{4}$$

has a solution, say u_R for every $f \in H$ (cf. for instance [5], ch II, Th 8.2). As it is well known this can be interpreted in term of subdifferentials of convex sets, i.e.

$$-(Lu_R - f) \in \partial I_{K_R}(u_R) \tag{5}$$

Since B_R has nonempty interior we have ([2, Proposition 5.7]),

$$\partial I_{K_R}(u_R) = \partial I_{B_R}(u_R) + \partial I_K(u_R)$$

for R sufficiently large. It is easily seen that

$$\partial I_{B_R}(u_R) = \{\lambda Q u_R : \lambda \geq 0\}$$

hence (5) becomes

$$Lu_R + \lambda_R Q u_R + \nu_R = f \tag{6}$$

for some $\lambda_R \geq 0$ and $\nu_R \in \partial I_K(u_R)$ or

$$(Lu_R + \lambda_R Q u_R, u_R - v) \leq (f, u_R - v) \qquad \forall v \in K. \tag{7}$$

Let for every R we have $\|Qu_R\| = R$. Then the family λ_R is bounded. Indeed, let $R' < R''$, $u' = u_{R'}$, $u'' = u_{R''}$, $\lambda' = \lambda_{R'}$, $\lambda'' = \lambda_{R''}$. From (7) we obtain

$$\begin{aligned}(Lu' + \lambda' Q u', u' - u'') &\leq (f, u' - u'')\\ (Lu'' + \lambda'' Q u'', u'' - u') &\leq (f, u'' - u')\end{aligned}$$

whence adding

$$(Lu'' - Lu', u'' - u') + (\lambda'' Q u'' - \lambda' Q u', u'' - u') \leq 0$$

and since L is monotone

$$(\lambda'' Q u'' - \lambda' Q u', u'' - u') \leq 0$$

or

$$\begin{aligned}\lambda''(Qu'', u'' - u') &\leq \lambda'(Qu', u'' - u')\\ = -\lambda'(Qu'' - Qu', u'' - u') &+ \lambda'(Qu'', u'' - u')\\ = -\lambda'\|Qu'' - Qu'\|^2 &+ \lambda'(Qu'', u'' - u')\\ &\leq \lambda'(Qu'', u'' - u').\end{aligned}$$

This implies

$$(\lambda'' - \lambda')(Qu'', u'' - u') \leq 0$$

and since

$$(Qu'', u'' - u') = (Qu'', u'') - (Qu', u'') = (Qu'', Qu'') - (Qu', Qu'')$$

$$\geq \|Qu''\|^2 - \|Qu'\|\|Qu''\| = R''(R'' - R') > 0$$

we get $\lambda'' \leq \lambda'$. This means that the family λ_R is bounded.

For an arbitrary fixed $v \in K$ now (7) can be rewritten as

$$(Lu_R + \lambda_R Qu_R, u_R) \leq (Lu_R, v) + \lambda_R(Qu_R, v) + (f, u_R) - (f, v) \tag{8}$$

whence

$$(Lu_R, u_R) + \lambda_R(Qu_R, u_R) \leq \|L\|\,\|u_R\|\,\|v\| + \lambda_R\|Qu_R\|\,\|v\| + \|f\|\,\|u_R\| + \|f\|\,\|v\| \tag{9}$$

and denoting by C various constants (since the family λ_R is bounded) we obtain

$$\alpha\|Pu_R\|^2 \leq C(\|Pu_R\| + \|Qu_R\| + 1)$$

or

$$\|Pu_R\|^2 \leq C(\|Qu_R\| + 1) \tag{10}$$

Let now $w_R = u_R/R$. Then

$$\|Qw_R\| = 1.$$

From (10) it follows that

$$\|Pw_R\| \to 0 \tag{11}$$

and in particular $\|Pw_R\|$ is bounded, hence

$$\|w_R\| \leq C.$$

From (9) and $(Lu, u) \geq 0$ it follows

$$\lambda_R R^2 \leq C(R + \lambda_R R + 1)$$

and since λ_R is bounded

$$\lambda_R R \leq C. \tag{12}$$

Now we can choose a sequence $R_n \to \infty$, such that for $\lambda_n = \lambda_{R_n}$ and $w_n = w_{R_n}$ we have

$$\lambda_n \to 0$$

$$w_n \to w \qquad \text{weakly in } H.$$

From (11) we get

$$\lim_{n\to\infty} \|Pw_n\| = 0,$$

Since M is finite-dimensional we obviously have

$$\lim_{n\to\infty} Qw_n = Qw.$$

All this imply

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} Pw_n + \lim_{n\to\infty} Qw_n = Qw,$$

i.e. the convergence is strong. Moreover we have $Qw = w$, i.e. $w \in M$. From (10) it is easy to see that $\|u_n\|/R_n \to 1$ since

$$\frac{R_n - \sqrt{C(R_n+1)}}{R_n} \leq \frac{\|Qu_n\| - \|Pu_n\|}{R_n} \leq \frac{\|u_n\|}{R_n} \\ \leq \frac{\|Qu_n\| + \|Pu_n\|}{R_n} \leq \frac{R_n + \sqrt{C(R_n+1)}}{R_n}$$

whence

$$\lim_{n\to\infty} \frac{u_n}{\|u_n\|} = \lim_{n\to\infty} \frac{u_n}{R_n}\frac{R_n}{\|u_n\|} = w$$

so $w \in K_\infty$. From (7) it follows for arbitrary $v_0 \in K$

$$(Lw_n + \lambda_n Qw_n, w_n - \frac{v_0}{R_n}) \leq R_n^{-1}(f, w_n - \frac{v_0}{R_n}) \tag{13}$$

or

$$(Lw, w) \leq 0.$$

Together with $\|w\| = 1$ this implies $w \in \ker(L + L^*)\backslash\{0\}$.

On the other hand (13) gives

$$(Lw_n + \lambda_n Qw_n, w_n) \leq R_n^{-1}(f, w_n) + R_n^{-1}(Lw_n + \lambda_n Qw_n, v_0) + R_n^{-2}(f, v_0).$$

Since

$$0 \leq (Lw_n + \lambda_n Q w_n, w_n)$$

passing to limit gives

$$0 \leq (f, w) + (Lw, v_0) \qquad \forall v_0 \in K.$$

Since $\|w\| = 1$ this contradicts (1) or (2). $\Box$

Remark. The following condition is sufficient for (2) to hold. For every $w \in \ker(L + L^*) \cap K_\infty$, $w \notin \ker L$, $(w \neq 0)$ there exists a $k_w \in K_\infty$ such that

$$(Lw, k_w) < 0.$$

Indeed, in this case we have

$$(f, w) + (Lw, v + tk_w) \to -\infty$$

as $t \to \infty$ for arbitrary $v \in K$.

Now we give the theorem of Lions-Stampacchia ([6], Theorem 5.1).

Theorem 2. *We assume that the norm $\|\cdot\|$ is equivalent to $p_0(\cdot) + p_1(\cdot)$, where $p_0(\cdot)$ is a norm with respect to which H is a pre-Hilbert space, and $p_1(\cdot)$ is a semi-norm on H. the space $M = \{v \in H | p_1(v) = 0\}$ has finite dimension, there exists a constant $c_1 > 0$ such that*

$$\inf_{\zeta \in M} p_0(v - \zeta) \leq c_1 p_1(v) \quad \forall v \in H.$$

Let $a(u, u)$ be a continuous bilinear form on H which is semi-coercive, i.e.

$$a(v, v) \geq \alpha (p_1(v))^2$$

for some $\alpha > 0$.

Let K be a closed convex set containing 0 and let $f \in H'$ be such that $f = f_0 + f_1$ with $f_i \in H'$, $i = 0, 1$ satisfying if $M \cap K \neq \{0\}$ the following conditions

$$|\langle f_1, v \rangle| \leq c_2 p_1(v) \quad \forall v \in H \tag{14}$$

$$\langle f_0, \zeta \rangle < 0 \quad \forall \zeta \in M \cap K, \ \zeta \neq 0. \tag{15}$$

Then the variational inequality

$$a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in K$$

has a solution.

P r o o f. Let $P : H \to M^{\perp}$ be the orthogonal projection on the orthogonal complement of M. We have

$$\|Pv\| = \inf_{\zeta \in M} \|v - \zeta\|.$$

Let ζ_0 be the element for which the infimum $\inf_{\zeta \in M} p_0(v - \zeta)$ is attained. Since the norm is equivalent to $p_0 + p_1$, we have

$$\|Pv\| \le \|v - \zeta_0\| \le c_0(p_0(v-\zeta_0) + p_1(v-\zeta_0)) \le c_0(c_1 p_1(v) + p_1(v) + p_1(\zeta_0)) \le C p_1(v)$$

whence

$$a(v, v) \ge \frac{\alpha}{C}\|Pv\|^2$$

and we can apply the theorem. Indeed, since $0 \in K$ (14) and (15) imply (1) and (2) with v_w taken to be 0 for every $w \in \ker(L + L^*) \cap K_\infty$, $(w \ne 0)$. $\square$

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