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SHARP HARDY INEQUALITIES IN A BALL

Alexander Fabricant, Nikolai Kutev and Tsviatko Rangelov

Abstract. Several Hardy–type inequalities in a sectorial area and in a ball are obtained. Sharpness of the inequalities is shown. An application to the lower bound of the first eigenvalue for the p–Laplacian in bounded domains is given.

1. Introduction

The well known Hardy inequality, proved in Hardy [1, 2] states

$$\int_0^\infty |u'(x)|^p x^\alpha dx \geq \left(\frac{p-1-\alpha}{p}\right)^p \int_0^\infty x^{-p+\alpha}|u(x)|^p dx \tag{1}$$

where $1 < p < \infty$, $\alpha < p-1$ and $u(x)$ is absolutely continuous on $[0,\infty)$, $u(0) = 0$. In the multidimensional case (1) is generalized by Neĉas [3] for Lipschitz domain $\Omega \subset R^n$, $n \geq 2$, i.e.,

$$\int_\Omega d(x)^\alpha |\nabla u(x)|^p dx \geq C_\Omega \int_\Omega d(x)^{\alpha-p}|u(x)|^p dx, \tag{2}$$

for every $u \in C_0^\infty(\Omega)$, $\alpha < p-1$, $p > 1$ and $d(x) = \text{dist}(x, \partial\Omega)$. The constant C_Ω in (2) is optimal, i.e., there is no greater constant $C'_\Omega > C_\Omega$ for which (2) holds for all $u \in W_0^{1,p}(\Omega)$. However, even in the one - dimensional case where

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$C_\Omega = \left(\dfrac{p-1-\alpha}{p}\right)^p$, inequality (2) is not sharp (see Hardy [1, 2]). This means that there is no a nontrivial function $u \in W_0^{1,p}(\Omega)$ such that (2) becomes an equality.

The other direction of generalization of (1) is an inequality with a kernel, singular in an internal point of Ω

$$\int_\Omega |\nabla u(x)|^2 dx \geq C_\Omega \int_\Omega \frac{|u(x)|^2}{|x|^2} dx \tag{3}$$

where $u \in C_0^\infty(\Omega)$, $\Omega \subseteq R^n$, $0 \in \Omega$, $n \geq 3$. The optimal constant $C_\Omega = \left(\dfrac{n-2}{2}\right)^2$ is obtained in Leray [4] for $\Omega = R^n$, see also Peral and Vazquez [5]. Let us mention that in all these papers the constant in Hardy inequality is optimal, i.e. there is no greater constant $C'_\Omega > C_\Omega$ for which (3) holds. However the inequality (3) is not sharp. That is why Brezis and Marcus [6], resp. Brezis and Vazquez [7] state the question on the existence of an additional positive term such that the improved inequalities (2), resp. (3) still hold for the optimal constant C_Ω.

Recently, the so–called improved Hardy inequalities are intensively investigated, see e.g. Barbatis et al. [8], Dávila and Dupaigne [9], Filippas and Tertikas [10], Hoffmann-Ostenhof et al. [11], Tidblom [12], Vázquez and Zuazua [13], Filippas et al. [14], Marcus and Shafrir [15], Kinnunen and Korte [16] and references therein. However, these improved Hardy inequalities are not sharp.

The aim of this paper is to present several improved Hardy inequalities which are sharp and with optimal constant C_Ω. We consider the model cases of a ball and a sectorial area which are basecs for further generalizations. Moreover, in the applications for estimates from below of the first eigenvalue $\lambda_p(\Omega)$ of the p–Laplacian in section 5., the Hardy inequality in a ball is important. The reason is the Faber–Krahn theorem which says that the infinimum of $\lambda_p(\Omega)$ in all domains Ω with a fixed volume is attained exactly in the ball with this volume.

The paper is organized in the following way. In section 2. the main results are formulated. In section 3. some auxiliary results of Hardy inequalities in a sectorial area are proved while section 4. deals with the proofs of the main results. In section 5. an application for estimates from below of the first eigenvalue $\lambda_p(\Omega)$ of the p–Laplacian is given.

2. Main results

Let $p > 1$, $p' = \dfrac{p}{p-1}$, $n \geq 2$ and $m = \dfrac{p-n}{p-1} = \dfrac{p-1}{p}p'$. Denote the ball centered at zero with radius δ as $B_\delta = \{x \in R^n, |x| < \delta\}$. In order to formulate the main results let us first define the sets of functions.

$$M_1(0,R)=\begin{cases} u: \int_{B_R}\left|\frac{<x,\nabla u>}{|x|}\right|^p dx<\infty \text{ and} \\ \left|R^m-\hat{R}^m\right|^{1-p}\int_{\partial B_{\hat{R}}}|u|^p d\sigma\to 0,\quad \hat{R}\to R-0,\quad m\neq 0, \\ \left|\ln\frac{R}{\hat{R}}\right|^{1-n}\int_{\partial B_{\hat{R}}}|u|^n d\sigma\to 0,\quad \hat{R}\to R-0,\quad m=0 \end{cases}$$

$$M_2(0,R)=\begin{cases} u: \int_{B_R}\left|\frac{<x,\nabla u>}{|x|}\right|^p dx<\infty,\quad \int_{B_R}\frac{|u|^p}{|x|^{(n-1)p'}}dx<\infty, \text{ and} \\ \delta^{1-p}\int_{\partial B_\delta}|u|^p d\sigma\to 0,\quad \delta\to +0,\quad m>0, \\ \delta^{1-n}\int_{\partial B_\delta}|u|^p d\sigma\to 0,\quad \delta\to +0,\quad m\leq 0. \end{cases}$$

$$M(0,R)=\begin{cases} u\in W_0^{1,p}(B_R), m>0, \\ \hat{\rho}^{1-p}\int_{\partial B_{\hat{\rho}}}|u|^p dx\to 0,\quad \text{for } \hat{\rho}\to 0+, \\ \left(R^m-\hat{R}^m\right)^{1-p}\int_{\partial B_{\hat{R}}}|u|^p dx\to 0,\quad \hat{R}\to R-0. \end{cases}$$

Our aim is to prove the following theorems.

Theorem 1. *For functions $u\in M_1(0,R)$ we get the inequalities:*

i) for $m>0$:

(4)

$$\left(\int_{B_R}\left|\frac{<x,\nabla u>}{|x|}\right|^p dx\right)^{\frac{1}{p}}\geq\frac{p-n}{p}\left(\int_{B_R}\frac{|u|^p}{|x|^{(n-1)p'}\,|R^m-|x|^m|^p}dx\right)^{\frac{1}{p}}$$

$$+\ \frac{1}{p}R^{n-p}limsup_{r\to 0}\left[r^{1-n}\int_{\partial B_r}|u|^p dS\right]\left(\int_{B_R}\frac{|u|^p}{|x|^{(n-1)p'}\,|R^m-|x|^m|^p}dx\right)^{-\frac{1}{p'}}.$$

For the functions $u(x)=\left(\frac{R^m-|x|^m}{m}\right)^k$, $k>\frac{1}{p'}$, (4) becomes an equality.

ii) for $m < 0$:

(5)

$$\left(\int_{B_R}\left|\frac{< x, \nabla u >}{|x|}\right|^p dx\right)^{\frac{1}{p}} \geq \left|\frac{p-n}{p}\right| \left(\int_{B_R} \frac{|u|^p}{|x|^{(n-1)p'}|R^m - |x|^m|^p} dx\right)^{\frac{1}{p}}$$

$$+ \quad \frac{1}{p} R^{n-p} limsup_{r\to 0}\left[r^{1-p}\int_{\partial B_r}|u|^p dS\right]\left(\int_{B_R}\frac{|u|^p}{|x|^{(n-1)p'}|R^m - |x|^m|^p}dx\right)^{-\frac{1}{p'}}.$$

The constant $\left|\frac{p-n}{p}\right|$ *is optimal one.*

iii) for $m = 0$:

(6)

$$\left(\int_{B_R}\left|\frac{< x, \nabla u >}{|x|}\right|^n dx\right)^{\frac{1}{n}} \geq \frac{n-1}{n}\left(\int_{B_R}\frac{|u|^n}{|x|^n\left|\ln\frac{R}{|x|}\right|^n}dx\right)^{\frac{1}{n}}$$

$$+ \quad \frac{1}{n} limsup_{r\to 0}\left[\left(r\ln\frac{R}{r}\right)^{1-n}\int_{\partial B_r}|u|^n dS\right]\left(\int_{B_R}\frac{|u|^n}{|x|^n\left|\ln\frac{R}{|x|}\right|^n}dx\right)^{\frac{1-n}{n}}.$$

The constant $\frac{n-1}{n}$ *is optimal one.*

Theorem 2. *For functions $u \in M_2(0,R)$ we get the inequalities:*

i) for $m > 0$:

(7)

$$\left(\int_{B_R}\left|\frac{< x, \nabla u >}{|x|}\right|^p dx\right)^{\frac{1}{p}} \geq \frac{p-n}{p}\left(\int_{B_R}\frac{|u|^p}{|x|^p}dx\right)^{\frac{1}{p}}$$

$$+ \quad \frac{1}{p}R^{1-p}\int_{\partial B_R}|u|^p dS\left(\int_{B_R}\frac{|u|^p}{|x|^p}dx\right)^{-\frac{1}{p'}}.$$

For the functions $u(x) = |x|^{km}$, $k > \frac{1}{p'}$ inequality (7) is sharp.

ii) for $m < 0$:

$$\left(\int_{B_R} \frac{|u|^p}{|x|^p} dx\right)^{\frac{1}{p'}} \left(\int_{B_R} \left|\frac{< x, \nabla u >}{|x|}\right|^p dx\right)^{\frac{1}{p}} \geq \frac{1}{p} R^{1-p} \int_{\partial B_R} |u|^p dS. \tag{8}$$

For the functions $u(x) = e^{-q|x|^m}$, $q > 0$ inequality (9) is sharp.

iii) for $m = 0$:

$$\left(\int_{B_R} \frac{|u|^n}{|x|^n} dx\right)^{\frac{n-1}{n}} \left(\int_{B_R} \left|\frac{< x, \nabla u >}{|x|}\right|^n dx\right)^{\frac{1}{n}} \geq \frac{1}{n} R^{1-n} \int_{\partial B_R} |u|^n dS. \tag{9}$$

For the function $u(x) = |x|^q$, $q > 0$ inequality (9) is sharp.

Theorem 3. *If $m > 0$ and $r = 2^{-1/m} R$, then for $u \in M(0, R)$ we get the inequality*

$$L(u) \geq \left(\frac{1}{p}\right)^p \sum_1^2 \frac{[K_0^j + (p-1)K_1^j]^p}{(K_1^j)^{p-1}}, \tag{10}$$

where

$$\begin{aligned}
L(u) &= L^1(u) + L^2(u), \quad L^1(u) = \int_{B_r} \left|\frac{< x, \nabla u >}{|x|}\right|^p dx, \\
L^2(u) &= \int_{B_R \setminus B_r} \left|\frac{< x, \nabla u >}{|x|}\right|^p dx, \\
K_1^1(u) &= m^p \int_{B_r} \frac{|u|^p}{|x|^p} dx, \quad K_1^2(u) = m^p \int_{B_R \setminus B_r} \frac{|u|^p}{|x|^{(n-1)p'} (R^m - |x|^m)^p} dx \\
K_0^1(u) &= m^{p-1} r^{1-p} \int_{\partial B_r} |u|^p d\sigma, \quad K_0^2(u) = m^{p-1} r^{1-n} (R^m - r^m)^{1-p} \int_{\partial B_r} |u|^p d\sigma.
\end{aligned}$$

The inequality (10) becomes an equality for the functions

$$u(x) = \begin{cases} \left(\frac{|x|^m}{m}\right)^k, & x \in B_r, \\ \left(\frac{R^m - |x|^m}{m}\right)^k & x \in B_R \setminus B_r. \end{cases} \tag{11}$$

with $k > 1/p'$.

3. Preliminaries

In this section we consider sectorial area $B_R \backslash B_r$, $0 < r < R$ and prove the following proposition and corollaries.

Proposition 1. *Suppose that the vector function* $f = (f_1, \dots, f_n)$, $f \neq 0$, $f_i \in C^{0,1}(B_R \backslash B_r)$ *satisfies the identity*

$$-divf - (p-1)|f|^{p'} = 0, \quad \text{for } x \in B_R \backslash B_r. \tag{12}$$

Then

$$L(u) \geq \left(\frac{1}{p}\right)^p \frac{[K_0(u) + (p-1)K_1(u)]^p}{(K_1(u))^{p-1}}, \tag{13}$$

with

$$L(u) = \int_{B_R \backslash B_r} \left| \frac{\langle f, \nabla u \rangle}{|f|} \right|^p dx, \ \ K_1(u) = \int_{B_R \backslash B_r} |f|^{p'} |u|^p dx, \ \ K_0(u) = \int_{\partial(B_R \backslash B_r)} \langle f, \nu \rangle |u|^p dx,$$

where ν *is the outward normal to* $B_R \backslash B_r$ *and* $\langle , \rangle$ *is a scalar product in* R^n.

Proof. Since

$$\int_{B_R \backslash B_r} \langle f, \nabla |u|^p \rangle dx = p \int_{B_R \backslash B_r} |u|^{p-2} u \langle f, \nabla u \rangle dx, \tag{14}$$

then applying Hölder inequality on the *rhs* of (14) with $\dfrac{\langle f, \nabla u \rangle}{|f|}$ and $|f||u|^{p-2}u$ as factors of the integrand we get

$$\begin{aligned} \int_{B_R \backslash B_r} \langle f, \nabla |u|^p \rangle dx \ &\leq \ p \left(\int_{B_R \backslash B_r} \left| \frac{\langle f, \nabla u \rangle}{|f|} \right|^p dx \right)^{1/p} \\ &\times \ \left(\int_{B_R \backslash B_r} |f|^{p'} |u|^p dx \right)^{1/p'}. \end{aligned} \tag{15}$$

Rising to p power both sides of (15) it follows that

$$\int_{B_R \backslash B_r} \left| \frac{\langle f, \nabla u \rangle}{|f|} \right|^p dx \geq \frac{\left| \dfrac{1}{p} \displaystyle\int_{B_R \backslash B_r} \langle f, \nabla |u|^p \rangle dx \right|^p}{\left(\displaystyle\int_{B_R \backslash B_r} |f|^{p'} |u|^p dx \right)^{p-1}}. \tag{16}$$

Integrating by parts the numerator of the *rhs* of (16) we get

$$\begin{aligned}\frac{1}{p}\int_{B_R\setminus B_r}\langle f,\nabla|u|^p\rangle dx &= \frac{1}{p}\int_{\partial B_R\cup\partial B_r}\langle f,\nu\rangle|u|^p dS-\frac{1}{p}\int_{B_R\setminus B_r}\mathrm{div}f|u|^p dx\\ &=\frac{1}{p}\int_{\partial B_R\cup\partial B_r}\langle f,\nu\rangle|u|^p dS+\left(\frac{p-1}{p}\right)\int_{B_R\setminus B_r}|f|^{p'}|u|^p dx.\end{aligned}\tag{17}$$

From (15) and (16) we obtain (13) due to the equality (12). □

The idea of the proof of Proposition 1 is similar as in Boggio [17] (for $p=2$), Flekinger et al. [18] and Barbatis et al. [19]. In contrast with these works in our case we consider functions not necessary zero on the whole boundary $\partial(B_R\setminus B_r)$ and due to this there is an additional boundary term $K_0(u)$ in (13).

The following corollaries of Proposition 1 hold.

Corollary 1. *For $u\in M_1(r,R)$ the inequality (13) has the form:*

i) for $m\neq 0$

$$\left(\int_{B_R\setminus B_r}\left|\frac{<x,\nabla u>}{|x|}\right|^p\right)^{\frac{1}{p}}\geq\left|\frac{n-p}{p}\right|\left(\int_{B_R\setminus B_r}\frac{|u|^p}{|x|^{(n-1)p'}|R^m-|x|^m|^p}dx\right)^{\frac{1}{p}}$$
$$+\ \frac{1}{p}r^{1-n}\left|R^m-r^m\right|^{1-p}\int_{\partial B_r}|u|^p dS\left(\int_{B_R\setminus B_r}\frac{|u|^p}{|x|^{(n-1)p'}\left|R^m-|x|^m\right|^p}dx\right)^{-\frac{1}{p'}}.\tag{18}$$

For functions $u(x)=\left(\frac{R^m-|x|^m}{m}\right)^k$, $k>\frac{1}{p'}$ (18) becomes an equality.

ii) for $m=0$, i. e. $p=n$:

$$\left(\int_{B_R\setminus B_r}\left|\frac{<x,\nabla u>}{|x|}\right|^n dx\right)^{\frac{1}{n}}\geq\frac{n-1}{n}\left(\int_{B_R\setminus B_r}\frac{|u|^n}{|x|^n\left|\ln\frac{R}{|x|}\right|^n}dx\right)^{\frac{1}{n}}$$
$$+\ \frac{1}{n}\left(r\ln\frac{R}{r}\right)^{1-n}\int_{\partial B_r}|u|^n dS\left(\int_{B_R\setminus B_r}\frac{|u|^n}{|x|^n\left|\ln\frac{R}{|x|}\right|^n}dx\right)^{\frac{1-n}{n}}.\tag{19}$$

For functions $u(x)=\left(\ln\frac{R}{|x|}\right)^k$, $k>\frac{1}{p'}$ (19) becomes an equality.

P r o o f. Let us define vector function $f(x)$ in $B_R\backslash B_r$ as:

$$f(x) = \begin{cases} -|x|^{-n}x\left(\dfrac{R^m - |x|^m}{m}\right)^{1-p}, & m \neq 0 \\ |x|^{-n}x\left(\ln\dfrac{R}{|x|}\right)^{1-n}, & m = 0. \end{cases}$$

Note that for the outward normal ν to $B_R\backslash B_r$ it holds

$$\nu|_{\partial B_R} = \frac{x}{|x|}|_{\partial B_R}, \quad \nu|_{\partial B_r} = -\frac{x}{|x|}|_{\partial B_r}. \tag{20}$$

Since vector function $f(x)$ satisfies (12) then applying Proposition 1 and using (20) we obtain inequalities (18), (19). □

Corollary 2. *For $u \in M_2(r,R)$ the inequality (13) has the form:*

i) for $m \neq 0$

(21)

$$\left(\int_{B_R\backslash B_r}\left|\frac{< x, \nabla u >}{|x|}\right|^p dx\right)^{\frac{1}{p}} \geq \left|\frac{p-n}{p}\right|\left(\int_{B_R\backslash B_r}\frac{|u|^p}{|x|^{(n-1)p'}|r^m - |x|^m|^p}dx\right)^{\frac{1}{p}}$$

$$+ \frac{1}{p}R^{1-n}|R^m - r^m|^{1-p}\int_{\partial B_R}|u|^p dS\left(\int_{B_R\backslash B_r}\frac{|u|^p}{|x|^{(n-1)p'}|r^m - |x|^m|^p}dx\right)^{-\frac{1}{p'}}.$$

For function $u(x) = \left(\dfrac{|x|^m - r^m}{m}\right)^k$, $k > \dfrac{1}{p'}$ (21) becomes an equality.

ii) for $m = 0$, i. e. $p = n$:

(22)

$$\left(\int_{B_R\backslash B_r}\left|\frac{< x, \nabla u >}{|x|}\right|^n dx\right)^{\frac{1}{n}} \geq \frac{n-1}{n}\left(\int_{B_R\backslash B_r}\frac{|u|^n}{|x|^n\left|\ln\dfrac{|x|}{r}\right|^n}dx\right)^{\frac{1}{n}}$$

$$+ \frac{1}{n}\left(R\ln\frac{R}{r}\right)^{1-n}\int_{\partial B_R}|u|^n dS\left(\int_{B_R\backslash B_r}\frac{|u|^n}{|x|^n\left|\ln\dfrac{|x|}{r}\right|^n}dx\right)^{\frac{1-n}{n}}.$$

For function $u(x) = \left(\ln\dfrac{|x|}{r}\right)^k$, $k > \dfrac{n-1}{n}$ (22) becomes an equality.

P r o o f. Let us define vector function $f(x)$ in $B_R\backslash B_r$ as:

$$f(x) = \begin{cases} |x|^{-n}x\left(\dfrac{|x|^m - r^m}{m}\right)^{1-p}, & m \neq 0 \\ |x|^{-n}x\left(\ln\dfrac{|x|}{r}\right)^{1-n}, & m = 0. \end{cases}$$

Since vector function $f(x)$ satisfies (12) then applying Proposition 1 and using (20) we obtain inequalities (21), (22). □

4. Proof of the main results

P r o o f. [of Theorem 1] Applying Proposition 1, Corollary 1, after the limit $r \to 0$ we obtain (4), (5) and (6) for functions $u \in M_1(0,R)$. It is easy to check that (4) is sharp and the constants in (4) - (6) are optimal. Indeed, in order to prove that constants $\left|\dfrac{p-n}{p}\right|$ and $\left|\dfrac{n-1}{n}\right|$ respectively are optimal in (4), (5), respectively (6) it is enough by means of the functions $u_\varepsilon(x) = |x|^{\frac{m}{p'}(1-\varepsilon)}\left(\dfrac{R^m - |x|^m}{m}\right)^{\frac{m}{p'}(1+\varepsilon)}$, and $u_\varepsilon(x) = |x|^{\frac{n-1}{n}(1-\varepsilon)}\left(\ln\dfrac{R}{|x|}\right)^{\frac{n-1}{n}(1+\varepsilon)}$ respectively, for $0 < \varepsilon < 1$ to check that

$$\int_{B_R}\left|\frac{< x, \nabla u >}{|x|}\right|^p dx \Big/ \left|\frac{p-n}{p}\right|^p \int_{B_R}\frac{|u|^p}{|x|^{(n-1)p'}\,|R^m - |x|^m|^p}dx \in (1, (1+\varepsilon)^p)$$

$$\int_{B_R}\left|\frac{< x, \nabla u >}{|x|}\right|^n dx \Big/ \left|\frac{n-1}{n}\right|^n \int_{B_R}\frac{|u|^n}{|x|^n\left|\ln\dfrac{R}{|x|}\right|^n}dx \in (1, (1+\varepsilon)^n).$$

For $\varepsilon \to 0$ it follows that the constants $\left|\dfrac{p-n}{p}\right|^p$ and $\left|\dfrac{n-1}{n}\right|^n$ respectively are optimal. □

P r o o f. [of Theorem 2]

Applying Proposition 1, Corollary 2, after the limit $r \to 0$ we obtain (7), (8) and (9) for functions $u \in M_2(0,R)$. It is easy to check that these inequalities are sharp and the constants are optimal. □

P r o o f. [of Theorem 3] Applying Theorem 2 in B_r and Corollary 1 in $B_R\backslash B_r$ we obtain

$$L^1(u) \geq K^1(u), \quad \text{in } B_r, \quad \text{and } L^2(u) \geq K^2(u), \quad \text{in } B_R\backslash B_r, \tag{23}$$

where $K^j(u) = K_1^j(u) + K_0^j(u)$.

Since $r = 2^{-1/m}R$, then $r^p = r^{(n-1)p'}(R^m - r^m)^p$ and we have continuous kernel for K_1^j on ∂B_r.

Adding inequalities in (23) we get inequality (10).

It is easy to check that (10) is an equality for functions $u \in M(0,R)$, defined in (11). This follows from the sharpness of Theorem 2 and Corollary 1. $\square$
Theorem 3 gives an example of sharp Hardy inequality in a ball $B_R \subset R^n$ for functions $u \in M(0,R) \subset W_0^{1,p}(B_R)$, $p > n$.

5. Applications

One of the applications of the Hardy inequality are the embedding theorems. The second one is the estimate from below of the first eigenvalue for the p–Laplacian

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda_p(\Omega)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{24}$$

where $\Omega \subset R^n$ is a bounded domain, $p > 1$.

The first eigenvalue $\lambda_p(\Omega)$ is defined as

$$\lambda_p(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^p dx}{\displaystyle\int_\Omega |u|^p dx}. \tag{25}$$

and $\lambda_p(\Omega)$ is simple, i.e., the first eigenfunction $\psi(x)$ is unique up to multiplication with nonzero constant C. Moreover, ψ is positive in Ω, $\psi \in W_0^{1,p}(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0,1)$ (see e. g. Belloni and Kawohl [20] and the references therein).

Let us recall Faber–Krahn theorem, see Kawohl and Fridman [21].

Theorem 4 (Faber–Krahn) *Among the domains of given n–dimensional volume the ball B_R with the same volume as Ω minimizes every $\lambda_p(\Omega)$, in other words*

$$\lambda_p(\Omega) \geq \lambda_p(B_R). \tag{26}$$

We will illustrate an application of the Hardy inequalities (4) - (6) for the estimates of the first eigenvalue $\lambda_p(\Omega)$ (see also Fabricant et al. [22]).

From (4) - (6) for $u \in W_0^{1,p}(B_R)$ ignoring the boundary terms we have the Hardy inequalities

$$\int_{B_R} \left| \frac{< x, \nabla u >}{|x|} \right|^p dx \geq \left| \frac{p-n}{p} \right|^p \int_{B_R} \frac{|u|^p}{|x|^{n-m} \left| R^m - |x|^m \right|^p} dx, \quad \text{for } p \neq n, \tag{27}$$

$$\int_{B_R}\left|\frac{< x,\nabla u >}{|x|}\right|^n dx \geq \left(\frac{n-1}{n}\right)^n \int_{B_R}\frac{|u|^n}{|x|^n\left|\ln\dfrac{R}{|x|}\right|^n}dx, \quad \text{for } p=n. \tag{28}$$

If $|x| = \rho \in [0, R)$ then for every $x \in B_R$ and $p \neq n$ we get the estimate

$$\int_{B_R}\frac{|u|^p}{|x|^{n-m}\,|R^m - |x|^m|^p}dx \geq \inf_{\rho\in(0,R)}\left(|x|^{n-m}|R^m-\rho^m|^p\right)^{-1}\int_{B_R}|u|^p dx.$$

Hence from (25) and the identity $m - n = (m-1)p$ we have

$$\begin{aligned} \lambda_p(B_R) &\geq \left|\frac{p-n}{p}\right|^p \inf_{\rho\in(0,R)}\left[\rho^{1-m}|R^m-\rho^m|\right]^{-p} \\ &= \left|\frac{p-n}{p}\right|^p \left[\sup_{\rho\in(0,R)}\left(\rho^{1-m}|R^m-\rho^m|\right)\right]^{-p}. \end{aligned} \tag{29}$$

because $1 - m = \dfrac{n-1}{p-1} > 0$.

The function $z(\rho) = \rho^{1-m}|R^m - \rho^m|$ attains its maximum in the interval $(0, R)$ at the point $\rho_0 = R(1-m)^{1/m} = R\left(\dfrac{n-1}{p-1}\right)^{1/m}$ and

$$z(\rho_0) = R\left(\frac{n-1}{p-1}\right)^{\frac{n-1}{p-n}}\left|\frac{p-n}{p-1}\right|. \tag{30}$$

Hence from (29) and (30) we get

$$\begin{aligned} \lambda_p(B_R) &\geq \left(\frac{p-1}{p}\right)^p\left(\frac{n-1}{p-1}\right)^{-\frac{(n-1)p}{p-n}} R^{-p} \\ &= \left(\frac{1}{Rp}\right)^p\left[\frac{(n-1)^{n-1}}{(p-1)^{p-1}}\right]^{\frac{p}{n-p}}. \end{aligned} \tag{31}$$

As a consequence of Fabrt–Krahn Theorem 4 we have the following result.

Theorem 5. *For every $p \neq n$, $p > 1$, $n \geq 2$ and every bounded domain $\Omega \subset R^n$ the estimate*

$$\lambda_p(\Omega) \geq \left(\frac{\omega_n}{|\Omega|}\right)^{p/n}\left(\frac{1}{p}\right)^p\left[\frac{(n-1)^{n-1}}{(p-1)^{p-1}}\right]^{\dfrac{p}{n-p}}. \tag{32}$$

holds, where ω_n, $|\Omega|$ are resp., the volume of the unite ball in R^n and Ω.

Analogously, from (28) we obtain

Theorem 6. *For $p = n$, $n \geq 2$ and every bounded domain $\Omega \subset R^n$ the estimate*

$$\lambda_n(\Omega) \geq \frac{\omega_n}{|\Omega|} \left(\frac{n-1}{n} \right)^n e^n. \tag{33}$$

holds.

In the case $m > 0$, i.e. $p > n$ from Theorem 1 by inequality (4) we get the following better estimate than (32).

Theorem 7. *For every bounded domain $\Omega \subset R^n$, $n \geq 2$, $p > n$ the estimate*

$$\lambda_p(\Omega) \geq \left(\frac{\omega_n}{|\Omega| p} \right)^p \left[n \left((p-n)^{p-n} \frac{(n-1)^{n-1}}{(p-1)^{p-1}} \right)^{\frac{p-1}{p-n}} + \left(\frac{(p-1)^{p-1}}{(n-1)^{n-1}} \right)^{\frac{1}{p-n}} \right] \tag{34}$$

holds, where ω_n, $|\Omega|$ are resp., the volume of the unite ball in R^n and Ω.

Let us recall that estimates from below for $\lambda_p(B_R)$ are developed numerically by Biezuner et al. [23] and analytically by Lefton and Wei [24] with Cheeger's constant, by Sobolev's constant in Lindqvist [25], or via different Hardy inequalities in Tidblom [12] - see Fabricant et al. [22] and the references therein for more details.

It is shown in Fabricant et al. [22] that the estimate

$$\lambda_p(B_R) \geq \frac{1}{(Rp)^p} \left(\frac{(p-1)^{p-1}}{(n-1)^{n-1}} \right)^{\frac{p}{p-n}} \tag{35}$$

obtained by means of Hardy inequality in Fabricant et al. [26] with double singular kernel is better then the other ones for $p > n > 2$. Since (34) is better then (35) it is clear that for the time being the analytical estimate (34) for the first eigenvalue $\lambda_p(B_R)$ from below is the best one for $p > n \geq 2$.

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Alexander Fabricant, Nikolai Kutev, Tsviatko Rangelov
Institute of Mathematics and Informatics
Acad. G. Bonchev Str. Bl. 8
1113 Sofia, Bulgaria
e-mail: rangelov@math.bas.bg