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ON THE REGULARITY PROPERTIES OF THE PRESSURE FIELD ASSOCIATED TO A HOPF WEAK SOLUTION TO THE NAVIER-STOKES EQUATIONS

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Abstract. We give some new a priori estimates for the pressure field associated to a Hopf weak solution, under the minimal assumption that the initial data v_0 is in $L^2(\Omega)$. Then, such estimates are applied to obtain an existence theorem of suitable weak solutions on a bounded or exterior domain $\Omega \subset \mathbb{R}^3$, with the minimal assumption $v_0 \in L^2(\Omega)$.

1. Introduction

We consider the non-stationary Navier-Stokes equations with unit viscosity and zero body force

$$\begin{aligned} v_t - \Delta v + (v\cdot\nabla)v &= -\nabla\pi \qquad &\forall\ (x,t) \in \Omega \times (0,T), \\ \nabla\cdot v &= 0 \qquad &\forall\ (x,t) \in \Omega \times (0,T), \end{aligned} \tag{1}$$

where v and π represent the unknown velocity and pressure, respectively. In our notation $(v\cdot\nabla)v = (\nabla v)v$.
In addition to (1) we require the following initial and boundary conditions

$$\begin{aligned} v(x,t) &= 0 \qquad &\forall\ (x,t) \in \partial\Omega \times (0,T), \\ v(x,0) &= v_0(x) \qquad &\forall\ x \in \Omega, \end{aligned} \tag{2}$$

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If $n = 3$, the system (1)–(2) describes the motion of a Newtonian fluid with a nonslip boundary condition.

The initial data v_0 should satisfy the compatibility conditions $\nabla\cdot v_0 = 0$ in Ω and $v_0 \cdot \nu_{|\partial\Omega} = 0$, with $\nu(x)$ the outward pointing unit normal vector at $x \in \partial\Omega$, at least in weak form. Moreover, if the domain Ω is unbounded, we also assume the following condition at infinity

$$\lim_{|x|\to\infty} v(x,t) = 0 \qquad \forall\ t\ \in\ [0,T)\,.$$

For the Cauchy problem, the existence of weak solutions for the initial-boundary value problem (1) – (2) was proved by J. Leray in [10]; in particular, he introduced the first notion of weak solution for the Navier-Stokes system (cf. Definition 1).
In [8] E. Hopf proved the existence of weak solutions on any smooth enough domain $\Omega \subset R^n$, with $n \geq 2$; nevertheless, such solutions are slightly different to Leray's ones (cf. Definition 2).

Ever since, much effort has been made to establish results on the uniqueness and regularity of weak solutions; however, such questions remain mostly open so far. In particular, we are interested in regularity properties of the pressure field π associated to a Hopf weak solution. These properties are very important in studying the partial regularity theory of suitable weak solutions (cf. Definition 3) and they were deeply investigated (cf. e.g. [3, 21, 7]). Nevertheless, in case $\Omega \subset \mathbb{R}^n$ is a bounded or an exterior domain, the initial data v_0 is required to be in a suitable fractional Sobolev or Besov space.

In this paper we give some new a priori estimates for the pressure field associated to a Hopf weak solution, under the minimal assumption that the initial data v_0 is in $L^2(\Omega)$. Then, such estimates are applied to obtain an existence theorem of suitable weak solutions on a bounded or exterior domain $\Omega \subset \mathbb{R}^3$, with the minimal assumption $v_0 \in L^2(\Omega)$.
Thus, as far as we know, if $\Omega \subset \mathbb{R}^3$ is a bounded or an exterior domain, thanks to Theorem 2, $J(\Omega)$ is the largest class of initial data for which we can give an existence theorem of weak solutions which are both suitable weak solutions in $\Omega \times (0,\infty)$, and Leray weak solutions.

Weakening the hypotheses on the initial data isn't the main question about the Navier-Stokes system (1); nevertheless, the matter itself is interesting as it implies that the presence of the boundary doesn't upset the nature of the problem compared to the Cauchy one. Moreover, from Theorem 2 and [24, Theorem 2.1] there follows that suitable weak solutions are obtained for the same class of the initial data as Hopf weak ones.

The results presented in this paper are based on the Ph.D. Thesis [16] that the author defended at the University of Pisa, under the supervision of Prof. Vladimir Georgiev.
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1.1. Notations

Throughout this paper, we assume that Ω is a domain in $\mathbb{R}^n$, with $n \geq 2$, which satisfies one of the following conditions:

Assumption 1.

$(D1)$ $\Omega \equiv \mathbb{R}^n$;

$(D2)$ Ω *is a bounded domain in* $\mathbb{R}^n$;

$(D3)$ Ω *is an exterior domain in* $\mathbb{R}^n$.

Moreover, if Ω *satisfies condition* $(D2)$ *or* $(D3)$, *its bounded boundary* $\partial\Omega$ *is required to be (at least) of class* C^m, *where* m *is an even positive integer such that* $2m > n$.

For $1 \leq p \leq \infty$, let $L^p(\Omega)$ be the Lebesgue space of vector valued functions on Ω. The norm in $L^p(\Omega)$ is indicated by $\|\cdot\|_p$ and we use the notation $\langle u, v\rangle = \int_\Omega u \cdot v \, dx$ for any vector fields u, v for which the right hand side makes sense.
For $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, let $W^{m,p}(\Omega)$ be the Sobolev space of functions $u : \Omega \to \mathbb{R}^n$ in $L^p(\Omega)$ with distributional derivatives in $L^p(\Omega)$ up to order m included; the norm in $W^{m,p}(\Omega)$ is denoted by $\|\cdot\|_{W^{m,p}(\Omega)}$.
By $C_0^\infty(\Omega)$ we denote the space of all infinitely differentiable vector valued functions with compact support in Ω and, for $1 \leq p \leq \infty$ and $m \in \mathbb{N}$, $\overset{\circ}{W}{}^{m,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{m,p}(\Omega)}$.

By $\mathscr{C}_0(\Omega)$ we denote the class of all solenoidal vector fields $\varphi(x) \in C_0^\infty(\Omega)$; for $1 < p < \infty$, $J^p(\Omega)$ and $J^{1,p}(\Omega)$ are the closure of $\mathscr{C}_0(\Omega)$ in $L^p(\Omega)$ and $W^{1,p}(\Omega)$, respectively. If Ω satisfies condition (D2) or (D3), we can give the following characterization of the spaces $J(\Omega) \equiv J^2(\Omega)$ and $J^{1,2}(\Omega)$ (see theorems 1.4 and 1.6 in [25])

$$
\begin{aligned}
J(\Omega) &= \big\{ u \in L^2(\Omega) \; : \; \nabla \cdot u = 0 \,, \quad \gamma_\nu(u) = 0 \big\} \\
J^{1,2}(\Omega) &= \big\{ u \in \overset{\circ}{W}{}^{1,2}(\Omega) \; : \; \nabla \cdot u = 0 \,, \quad \gamma_0(u) = 0 \big\},
\end{aligned}
\tag{3}
$$

where γ_0 is the trace operator from $W^{1,2}(\Omega)$ into $W^{\frac{1}{2},2}(\partial\Omega)$, whereas γ_ν is a linear continuous operator from $E(\Omega) = \big\{u \in L^2(\Omega) \;:\; \nabla\cdot u \in L^2(\Omega)\big\}$ [1] into $W^{-\frac{1}{2},2}(\partial\Omega)$, such that $\gamma_\nu(u) = u\cdot\nu_{|\partial\Omega}$ for every vector field $u \in C^\infty(\overline{\Omega})$, with $\nu(x)$ the outward pointing unit normal vector at $x \in \partial\Omega$.

If $\Omega \subseteq \mathbb{R}^n$ is a domain satisfying condition $(D2)$ or $(D3)$, for $s \in (0,1)$ and $p \in [1,\infty)$, $W^{s,p}(\Omega)$ denotes the Slobodeckiĭ space of functions $v \in L^p(\Omega)$ for which the following norm

$$\|v\|_{W^{s,p}(\Omega)} = \left\{\|v\|_p^p + \int_\Omega\int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\right\}^{\frac{1}{p}}$$

is finite. Similarly, by $W^{s,p}(\partial\Omega)$ we denote the space of functions $v \in L^p(\partial\Omega)$ for which the norm

$$\|v\|_{W^{s,p}(\partial\Omega)} = \left\{\|v\|^p_{L^p(\partial\Omega)} + \int_{\partial\Omega}\int_{\partial\Omega} \frac{|v(x)-v(y)|^p}{|x-y|^{n-1+sp}}\,d\sigma_x\,d\sigma_y\right\}^{\frac{1}{p}}$$

is finite.

For $T \in (0,\infty)$ and for a given Banach space $\mathbb{X}$, with associated norm $\|\cdot\|_{\mathbb{X}}$, $L^p\big(0,T;\mathbb{X}\big)$ is the linear space of functions $f : (0,T) \to \mathbb{X}$ such that $\int_0^T \|u(\tau)\|^p_{\mathbb{X}}\,d\tau < \infty$, if $1 \le p < \infty$, or $\operatorname*{ess\,sup}_{\tau\in(0,T)} \|u(\tau)\|_{\mathbb{X}} < \infty$, if $p = \infty$.

If I is a real interval, we denote by $C\big(I;\mathbb{X}\big)$ the class of continuous functions from I to $\mathbb{X}$; for k a positive integer, we denote by $C^k\big(I;\mathbb{X}\big)$ the class of functions $f : I \to \mathbb{X}$ endowed with continuous derivatives (as functions into $\mathbb{X}$), up to the order k included.

For every $T \in (0,\infty)$, we set $\Omega_T = \Omega\times[0,T)$ and we define

$$\mathscr{C}_0(\Omega_T) = \{\varphi \in C_0^\infty\big(\Omega_T;\mathbb{R}^n\big) \;:\; \nabla\cdot\varphi = 0 \;\text{ in }\; \Omega_T\}\,.$$

By $\mathcal{C}(\Omega_T)$ we denote the class of vector fields $\varphi \in C\big([0,T];J^{1,2}(\Omega)\big)$ endowed with distributional partial derivative $\varphi_t \in L^2\big(0,T;J(\Omega)\big)$ and such that $\varphi(x,T) = 0$ for *a.e.* $x \in \Omega$.

In this work, we use the same symbol to denote functional spaces of scalar or vector valued functions. Moreover, the symbol c denotes a generic positive constant whose numerical value is not essential to our aims. It may assume several different values in a single computation.

[1] $E(\Omega)$ is a Hilbert space with respect to the inner product $\langle u,v\rangle_{E(\Omega)} = \langle u,v\rangle + \langle\nabla\cdot u,\nabla\cdot v\rangle$.

2. Weak solutions: definitions and properties

We give three different definitions of weak solutions of the initial-boundary value problem (1) – (2) and we collect some their properties which will be used afterwards.

Definition 1. *Let* $v_0 \in J(\Omega)$*. A vector field* $v : \Omega \times (0,\infty) \to \mathbb{R}^n$ *is said a* Leray weak solution *of problem* (1) – (2) *with initial data* v_0*, if it satisfies the following conditions for all* $T \in (0,\infty)$

1. $v \in L^\infty\big(0,T;J(\Omega)\big) \cap L^2\big(0,T;J^{1,2}(\Omega)\big)$ *;*
2. $\forall\, \varphi \in \mathscr{C}_0(\Omega_T)$

$$\int_0^T \big[\langle v,\varphi_t\rangle - \langle \nabla v,\nabla\varphi\rangle - \langle (v\cdot\nabla)v,\varphi\rangle\big]\,dt = -\langle v_0,\varphi_0\rangle\ ; \tag{4}$$

3. *there holds the following energy inequality*

$$\|v(t)\|_2^2 + 2\int_s^t \|\nabla v(\tau)\|_2^2\,d\tau \le \|v(s)\|_2^2 \tag{5}$$

for $s = 0$*, a.e.* $s > 0$ *and* $\forall\, t \ge s$*.*

Definition 2. *Let* $v_0 \in J(\Omega)$*. A vector field* $v : \Omega \times (0,\infty) \to \mathbb{R}^n$ *is said a* Hopf weak solution *of problem* (1) – (2) *with initial data* v_0*, if it satisfies, for all* $T \in (0,\infty)$*, conditions* 1, 2 *of Definition 1 and if the energy inequality* (5) *holds only for* $s = 0$ *and for all* $t \ge 0$*.*

If Ω is a domain in $\mathbb{R}^n$ (with $n = 2,3,4$) satisfying Assumption 1, for any initial data $v_0 \in J(\Omega)$ there exists at least a Leray weak solution of problem (1) – (2). Whereas, if Ω is an arbitrary domain in $\mathbb{R}^n$ (with $n \ge 2$), for any initial data $v_0 \in J(\Omega)$ there exists at least a Hopf weak solution (cf. [10, 8, 6, 17], see also [5, Section 3]).
Obviously, every Leray weak solution is a Hopf weak one too.

Remark. If v is a Hopf weak solution, by the energy inequality (5) we have

$$\|v\|_{L^\infty(0,\,\infty;\,J(\Omega))} \le \|v_0\|_2\,, \qquad \|\nabla v\|_{L^2(0,\,\infty;\,L^2(\Omega))} \le \tfrac{1}{2}\,\|v_0\|_2\,; \tag{6}$$

moreover, by Gagliardo-Nirenberg interpolation inequality, $v \in L^p\big(0,\infty;L^q(\Omega)\big)$ for every pair of exponents (p,q) such that

$$\frac{n}{q} + \frac{2}{p} = \frac{n}{2} \quad\text{and}\quad \begin{cases} q \in [2,q^*]\,, \ \text{with } \frac{1}{q^*} = \frac{1}{2} - \frac{1}{n}\,, & \text{if } n \ge 3 \\ q \in [2,\infty)\,, & \text{if } n = 2\,; \end{cases} \tag{7}$$

and there holds the following estimate

$$\|v\|_{L^p(0,\,\infty;\,L^q(\Omega))} \le c\,\|v_0\|_2\,, \tag{8}$$

where the positive constant c does not depend on v.

The following result due to Hopf ([8]) is relevant for our purposes (for its proof, see Lemma 1 in [18], Theorem 4 in [19] or Lemma 2.1 in [5]).

Lemma 1. *Let v be a Hopf weak solution of initial-boundary value problem (1) – (2), with initial data $v_0 \in J(\Omega)$. Then, for every $T \in (0,\infty)$, v can be redefined on a subset of $[0,T]$ having zero Lebesgue measure, in such a way that*

1. *$v(\cdot,t) \in J(\Omega) \qquad \forall\ t \in [0,T]$;*
2. *there holds the following relation*

$$\int_0^s \big[\langle v,\varphi_t\rangle - \langle \nabla v,\nabla\varphi\rangle - \langle (v\cdot\nabla)v,\varphi\rangle\big]\,dt = \langle v(s),\varphi(s)\rangle - \langle v_0,\varphi_0\rangle \tag{9}$$

for every $s \in [0,T]$ and for every $\varphi \in \mathscr{C}_0(\Omega_T)$.

Remark. The so redefined Hopf weak solution v is weakly continuous in $J(\Omega)$ as a function of time; thus, from the energy inequality (5) (with $s = 0$) we deduce

$$\lim_{t\to 0^+} \|v(t) - v_0\|_2 = 0.$$

In what follows we will regularly assume that all Hopf weak solutions (and then Leray weak ones too) under discussion have been redefined according to the previous Lemma.

The following Proposition, originally presented in [9], is proved in [13, Section 2.2] (see also [16]).

Proposition 1. *Let $\Omega \subseteq \mathbb{R}^n$, with $n \ge 2$, be a domain satisfying Assumption 1; let $T \in (0,\infty)$ and $\varphi(x,t) \in \mathcal{C}(\Omega_T)$, then, $\forall\ \varepsilon > 0$ there exists $\widetilde{\varphi}(x,t) \in \mathscr{C}_0(\Omega_T)$ such that*

$$\max_{t\in[0,T]} \|\varphi(t) - \widetilde{\varphi}(t)\|_{W^{1,2}(\Omega)} + \int_0^T \|\varphi_t(t) - \widetilde{\varphi}_t(t)\|_2^2\,dt < \varepsilon \tag{10}$$

Remark. Concerning the space to which test functions φ belong, in case Ω is a domain in $\mathbb{R}^n$ with $n = 2, 3, 4$, using Proposition 1 and a limit process, we can extend the weak formulation of the Navier-Stokes equations (4) or (9) to "less regular" test functions $\varphi \in \mathcal{C}(\Omega_T)$.
If $n > 4$, we should consider test functions $\varphi \in \mathcal{C}(\Omega_T) \cap C\big([0,T]; J^n(\Omega)\big)$. We need this further property to assure the summability and the convergence of the nonlinear term.

Definition 3. *Let $v_0 \in J(\Omega)$ and $T \in (0, \infty]$. A pair (v, π), having as first component a vector field $v : \Omega \times (0,T) \to \mathbb{R}^n$ and as second component a scalar function $\pi : \Omega \times (0,T) \to \mathbb{R}$, is said a* suitable weak solution *of problem* (1) – (2)*, in $\Omega \times (0,T)$, with initial data v_0, if the following conditions are satisfied*

1. $v \in L^\infty\big(0,T; J(\Omega)\big) \cap L^2\big(0,T; J^{1,2}(\Omega)\big)$;
2. *the energy inequality* (5) *holds, at least, for $s = 0$ and for all $t \in (0,T)$;*
3. $\forall\, \phi \in C_0^\infty\big(\Omega_T; \mathbb{R}^n\big)$

$$\int_0^T \big[\langle v, \phi_t\rangle - \langle \nabla v, \nabla \phi\rangle - \langle (v\cdot\nabla)\, v, \phi\rangle\big]\, dt = -\int_0^T \langle \pi, \nabla\cdot\phi\rangle\, dt - \langle v_0, \phi_0\rangle\,; \tag{11}$$

4. *for every non-negative, scalar valued function $\sigma \in C_0^\infty\big(\Omega_T; \mathbb{R}\big)$ there holds the following generalized energy inequality*

$$\begin{aligned}\int_\Omega |v(t)|^2\sigma(t)\, dx + 2\int_s^t\!\!\int_\Omega |\nabla v|^2\sigma\, dx\, d\tau &\le \int_\Omega |v(s)|^2\sigma(s)\, dx \\ + \int_s^t\!\!\int_\Omega |v|^2(\sigma_\tau + \Delta\sigma)\, dx\, d\tau &+ \int_s^t\!\!\int_\Omega (|v|^2 + 2\pi) v\cdot\nabla\sigma\, dx\, d\tau\end{aligned} \tag{12}$$

for $s = 0$, a.e. $s \in (0,T)$ and $\forall\, t \in (s,T)$.

Definition 4. *A point $(x,t) \in \Omega \times (0,T)$ is called* singular *for a solution v of system* (1) *iff the vector field v is not essentially bounded [i.e. $v \notin L^\infty(I_{(x,t)})$] on any neighborhood $I_{(x,t)}$ of (x,t).*

Let $\mathscr{P}^1$ denote a measure on $\mathbb{R}^3_x \times \mathbb{R}_t$ analogous to one-dimensional Hausdorff measure $\mathscr{H}^1$, but defined using parabolic cylinders instead of Euclidean balls (cf. [3, Section 2D]). For a suitable weak solution (v, π), there holds the following local partial regularity result (cf. [3, Theorem B] and [11]).

Theorem. Let Ω be an arbitrary domain in $\mathbb{R}^3$ and let $T \in (0,\infty]$; for any suitable weak solution (v,π) of problem (1)–(2) in $\Omega \times (0,T)$, with $\pi \in L^{\frac{3}{2}}\big(\Omega\times(0,T)\big)$, the associated set $\mathcal{S}$ of possible singular points satisfies $\mathscr{P}^1(\mathcal{S}) = 0$.

In the previous theorem, the hypothesis $\pi \in L^{\frac{3}{2}}\big(\Omega\times(0,T)\big)$ can be weakened to $\pi \in L^{\frac{5}{4}}\big(0,T;L^{\frac{5}{4}}_{\mathrm{loc}}(\Omega)\big)$ (cf. [3, Section 2C] and [24]).

3. The nonstationary Stokes problem

This section is concerned with the following initial-boundary value problem

$$\begin{aligned} v_t(x,t) - \Delta v(x,t) &= -\nabla\pi(x,t) + f(x,t) && \forall\ (x,t) \in \Omega\times(0,T)\,, \\ \nabla\cdot v(x,t) &= 0 && \forall\ (x,t) \in \Omega\times(0,T)\,, \end{aligned} \tag{13a}$$

$$\begin{aligned} v(x,t) &= 0 && \forall\ (x,t) \in \partial\Omega\times(0,T)\,, \\ v(x,0) &= v_0(x) && \forall\ x \in \Omega, \end{aligned} \tag{13b}$$

with $T \in (0,\infty]$ and $\Omega \subseteq \mathbb{R}^n$ a domain satisfying Assumption 1.

As for problem (1)–(2), the initial data v_0 should satisfy the compatibility conditions $\nabla\cdot v_0 = 0$ in Ω and $v_0\cdot\nu_{|\partial\Omega} = 0$, with $\nu(x)$ the outward pointing unit normal vector at $x \in \partial\Omega$, at least in weak form. Furthermore, if the domain Ω in unbounded, we also assume the condition at infinity

$$\lim_{|x|\to\infty} v(x,t) = 0 \qquad \forall\ t \in [0,T)\,.$$

Remark 1. If $\Omega \subseteq \mathbb{R}^n$, with $n \geq 2$, is a domain satisfying Assumption 1, by Theorem 1.1 in [25, Ch. 3] (see also [9, Ch. 4 Theorem 3]), for any initial data $v_0(x) \in J(\Omega)$, the Stokes problem (13) with $f(x,t) \equiv 0$ has a unique "weak" solution v such that, for every $T \in (0,\infty)$,

$$v \in C\big([0,T];J(\Omega)\big) \cap L^2\big(0,T;J^{1,2}(\Omega)\big)$$

$$\int_0^s \big[\langle v,\varphi_t\rangle - \langle\nabla v,\nabla\varphi\rangle\big]\,dt = \langle v(s),\varphi(s)\rangle - \langle v_0,\varphi_0\rangle \tag{14}$$

$$\text{for every } s \in [0,T] \text{ and for every } \varphi \in \mathscr{C}_0(\Omega_T).$$

The following Proposition concerns some properties of the pressure field π associated to the weak solution of problem (13); they will turn out useful to our purposes.

Proposition 2. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, be a domain satisfying Assumption 1. Let v be the unique "weak" solution of the Stokes problem* (13) *with $f(x,t) \equiv 0$ and initial data $v_0(x) \in J(\Omega)$; then there exists a distribution $\pi : \Omega \times (0,\infty) \to \mathbb{R}$ such that*

1. *for every $T \in (0,\infty)$ and for every $\eta \in (0,T)$,*

$$\nabla\pi \in L^p\big(0,T;L^q(\Omega)\big) \cap L^\infty\big(\eta,T;L^q(\Omega)\big) \tag{15a}$$

for every pair (p,q) such that $1 < q < 2$ and $1 \leq p < \frac{4q}{5q-2}$,

$$\pi \in L^r\big(0,T;L^2(\Omega')\big) \cap L^\infty\big(\eta,T;L^2(\Omega')\big) \qquad \forall\ 1 \leq r < \frac{4}{3}, \tag{15b}$$

where

(16) *$\Omega' \subset \Omega$ is an arbitrary bounded domain, if there holds condition $(D1)$; $\Omega' \subset \Omega$ is a bounded domain such that* $\operatorname{dist}\big(\Omega \setminus \Omega', \partial\Omega\big) > 0$, *if Ω satisfies condition $(D3)$, while $\Omega' \equiv \Omega$ if there holds condition $(D2)$;*

2. *for every $T \in (0,\infty)$,*

$$\begin{aligned}\int_0^s \big[\langle v,\phi_t\rangle - \langle \nabla v, \nabla\phi\rangle\big]\,dt = -\int_0^s \big[\langle \pi, \nabla\cdot\phi\rangle\big]\,dt \\ + \langle v(s),\phi(s)\rangle - \langle v_0,\phi_0\rangle\end{aligned} \tag{17}$$

for every $s \in [0,T]$ and for every $\phi \in C_0^\infty(\Omega_T;\mathbb{R}^n)$;

Proof. By density of $\mathscr{C}_0(\Omega)$ in $J(\Omega)$, there exists a sequence $\{v_0^n\} \subset \mathscr{C}_0(\Omega)$ converging to v_0 in $J(\Omega)$. From [7, Theorem 2.8], [14, Theorem 1.4] and Section 2 in [9, Chapter 4]) it follows that for every $n \in \mathbb{N}$ there exists a unique solution $(v^n, \nabla\pi^n)$ of problem (13) with $f(x,t) \equiv 0$ and initial data $v_0^n \in \mathscr{C}_0(\Omega)$, satisfying the following properties

$$v \in C\big([0,T];W^{2,2}(\Omega) \cap J^{1,2}(\Omega)\big) \cap\ L^p\big(0,T;J^{1,q}(\Omega) \cap W^{2,q}(\Omega)\big)$$

$$\frac{\partial v}{\partial t} \in C\big([0,T];J(\Omega)\big) \cap\ L^p\big(0,T;J^q(\Omega)\big)$$

$$\nabla\pi \in C\big([0,T];L^2(\Omega)\big) \cap\ L^p\big(0,T;L^q(\Omega)\big)$$

for every $T \in (0,\infty)$ and for all $p,q \in (1,\infty)$.

For any $T \in (0,\infty)$, we can multiply both sides of $(13a)_1$ by an arbitrary $\phi \in C_0^\infty(\Omega_T;\mathbb{R}^n)$ and integrate the product over $\Omega \times (0,s)$. Then, integrating by parts with respect to x and t, we obtain the following relation

$$\int_0^s \big[\langle v^n, \phi_t\rangle - \langle \nabla v^n, \nabla\phi\rangle\big]\,dt = -\int_0^s \big[\langle \pi^n, \nabla\cdot\phi\rangle\big]\,dt + \langle v^n(s), \phi(s)\rangle - \langle v_0^n, \phi_0\rangle \tag{18}$$

$$\text{for every } s \in [0,T].$$

By the linearity property of system (13a), for every n, $m \in \mathbb{N}$ the difference $(v^n - v^m, \nabla\pi^n - \nabla\pi^m)$ is the unique solution of problem (13) with $f(x,t) \equiv 0$ and initial data $v_0^n - v_0^m$. Considering the summability properties of $(v^n - v^m, \nabla\pi^n - \nabla\pi^m)$, for every $s \in (0,\infty)$, we can multiply both sides of $(13a)_1$ by $v^n - v^m$ in $L^2\big(\Omega \times (0,s)\big)$. Integrating by parts with respect to x and t and reminding that the vector field $v^n - v^m$ is solenoidal, we obtain the following identity

$$\|v^n(s) - v^m(s)\|_2^2 + 2\int_0^s \|\nabla(v^n(\tau) - v^m(\tau))\|_2^2\,d\tau = \|v_0^n - v_0^m\|_2^2, \tag{19}$$

$$\text{for every } s \in [0,\infty),$$

from which we derive that $\{v^n\}$ is a Cauchy sequence in $C\big([0,T]; J(\Omega)\big) \cap L^2\big(0,T; J^{1,2}(\Omega)\big)$, for every $T \in (0,\infty)$. Then, there exists its limit $\bar v \in C\big([0,T]; J(\Omega)\big) \cap L^2\big(0,T; J^{1,2}(\Omega)\big)$, for every $T \in (0,\infty)$, which is a "weak" solution of problem (13) with $f(x,t) \equiv 0$ and initial data v_0. By the uniqueness of the "weak" solution of the Stokes problem, it follows that $\bar v \equiv v$ *a.e.* in $\Omega \times (0,\infty)$ and, then, for every $T \in (0,\infty)$

$$\begin{aligned} v^n(x,t) &\to v(x,t) && \text{in } J(\Omega), \text{ uniformly in } [0,T] \\ \nabla v^n(x,t) &\to \nabla v(x,t) && \text{in } L^2\big(0,T; L^2(\Omega)\big)\,. \end{aligned} \tag{20}$$

Now, let us apply the divergence operator to $(13a)_1$, in the distribution sense, and let us multiply both sides of $(13a)_1$ by $\nu(x)$, the outward pointing unit normal vector at $x \in \partial\Omega$, in $W^{-\frac{1}{2},2}(\partial\Omega)$. As $v^n - v^m$ is a solenoidal vector field which satisfies homogeneous boundary conditions, we have that, for every n, $m \in \mathbb{N}$, for all $\bar t \in (0,\infty)$, the function $\pi^n(x,\bar t) - \pi^m(x,\bar t)$ is a weak solution of the following Neumann problem

$$\begin{aligned} &\Delta\big(\pi^n(x,\bar t) - \pi^m(x,\bar t)\big) = 0 && \forall\ x \in \Omega\,, \\ &\frac{\partial}{\partial\nu}\big(\pi^n(x,\bar t) - \pi^m(x,\bar t)\big) = \nu(x)\cdot\Delta\big(v^n(x,\bar t) - v^m(x,\bar t)\big) && \forall\ x \in \partial\Omega\,, \end{aligned} \tag{21}$$

If $\Omega \subseteq \mathbb{R}^n$ is an exterior domain, we also assume the following condition at infinity

$$\lim_{|x|\to\infty} \big(\pi^n(x,\bar t) - \pi^m(x,\bar t)\big) = 0$$

For every $\bar t \in (0,\infty)$, let $a^{(n,m,\bar t)}(x) = (v^n(x,\bar t) - v^m(x,\bar t))$; as $a^{(n,m,\bar t)}$ is a divergence free vector field, we have

$$\nu(x)\cdot\Delta\big(v^n(x,\bar t) - v^m(x,\bar t)\big) = \nu(x)\cdot\big(\Delta a^{(n,m,\bar t)}(x) - \nabla(\nabla\cdot a^{(n,m,\bar t)}(x))\big)$$

$$= \sum_{i,j=1}^{n} \nu_i(x)\frac{\partial}{\partial x_j}\Big(\frac{\partial a_i^{(n,m,\bar t)}}{\partial x_j}(x) - \frac{\partial a_j^{(n,m,\bar t)}}{\partial x_i}(x)\Big)$$

$$= \frac{1}{2}\sum_{i,j=1}^{n}\Big(\nu_i(x)\frac{\partial}{\partial x_j} - \nu_j(x)\frac{\partial}{\partial x_i}\Big)\Big(\frac{\partial a_i^{(n,m,\bar t)}}{\partial x_j}(x) - \frac{\partial a_j^{(n,m,\bar t)}}{\partial x_i}(x)\Big)$$

By virtue of estimate (2.18) in [14, Lemma 2.3] (see also [22, Lemma 2.1]), using the trace embedding theorem and an interpolation inequality, we obtain that for every $q \in (1,2)$ and $\varepsilon \in \big(0, \frac{1}{q} - \frac{1}{2}\big)$

$$\|\nabla\big(\pi^n(\bar t) - \pi^m(\bar t)\big)\|_q \le c\left\{\sum_{i,j=1}^{n}\int_{\partial\Omega}\int_{\partial\Omega}\frac{\Big|\frac{\partial a_i^{(n,m,\bar t)}}{\partial x_j}(x) - \frac{\partial a_i^{(n,m,\bar t)}}{\partial x_j}(y)\Big|^q}{|x-y|^{n-2+q}}\,d\sigma_x\,d\sigma_y\right\}^{\frac{1}{q}}$$

$$\le c\,\|\nabla a^{(n,m,\bar t)}\|_{W^{1-\frac{1}{q},q}(\partial\Omega)} \le c(\varepsilon)\,\|\nabla a^{(n,m,\bar t)}\|_{W^{1-\frac{1}{q}+\varepsilon,2}(\partial\Omega)}$$

$$\le c(\varepsilon)\,\|\nabla a^{(n,m,\bar t)}\|_{W^{\frac{3}{2}-\frac{1}{q}+\varepsilon,2}(\Omega)} \le c(\varepsilon)\,\|\nabla a^{(n,m,\bar t)}\|_2^{\frac{1}{q}-\frac{1}{2}-\varepsilon}\,\|\nabla a^{(n,m,\bar t)}\|_{W^{1,2}(\Omega)}^{\frac{3}{2}-\frac{1}{q}+\varepsilon}$$

$$\le c(\varepsilon)\Big(\|\nabla a^{(n,m,\bar t)}\|_2 + \|\nabla a^{(n,m,\bar t)}\|_2^{\frac{1}{q}-\frac{1}{2}-\varepsilon}\,\|\mathrm{D}^2 a^{(n,m,\bar t)}\|_2^{\frac{3}{2}-\frac{1}{q}+\varepsilon}\Big)$$

with the constant $c(\varepsilon)$ blowing up as $\varepsilon \to 0^+$; $\mathrm{D}^2 a^{(n,m,\bar t)}$ denotes the matrix of the second order spatial derivatives of $a^{(n,m,\bar t)}$.

Since $v_0^n - v_0^m \in \mathscr{C}_0(\Omega)$, using the decay estimates (4.3) in [14, Theorem 4.1] (which also hold if Ω is bounded), we get

$$\|\nabla\big(\pi^n(t) - \pi^m(t)\big)\|_q \le c(\varepsilon,T)\,t^{-(\frac{5}{4}-\frac{1}{2q}+\frac{\varepsilon}{2})}\,\|v_0^n - v_0^m\|_2 \qquad \forall\ T\in(0,\infty)\ \text{ and for any } t\in(0,T), \tag{22}$$

and, therefore, $\{\nabla\pi^n\}$ is a Cauchy sequence in $L^p\big(0,T;L^q(\Omega)\big)\cap L^\infty\big(\eta,T;L^q(\Omega)\big)$, for every pair (p,q) such that $1<q<2$ and $1\le p<\frac{4q}{5q-2}$, for every $T\in(0,\infty)$ and for every $\eta\in(0,T)$.

In a similar way, using estimate (2.17) in [14, Lemma 2.3], we obtain that for every $\varepsilon \in \left(0, \frac{1}{2}\right)$

$$\begin{aligned}\|\pi^n(\bar{t}) - \pi^m(\bar{t})\|_{L^2(\Omega')} \leq & c\,\|\nabla a^{(n,m,\bar{t})}\|_{W^{\varepsilon,2}(\partial\Omega)} \leq c\,\|\nabla a^{(n,m,\bar{t})}\|_{W^{\frac{1}{2}+\varepsilon,2}(\Omega)} \\ \leq & c\,\|\nabla a^{(n,m,\bar{t})}\|_2^{\frac{1}{2}-\varepsilon}\,\|\nabla a^{(n,m,\bar{t})}\|_{W^{1,2}(\Omega)}^{\frac{1}{2}+\varepsilon} \\ \leq & c\left(\|\nabla a^{(n,m,\bar{t})}\|_2 + \|\nabla a^{(n,m,\bar{t})}\|_2^{\frac{1}{2}-\varepsilon}\,\|\mathrm{D}^2 a^{(n,m,\bar{t})}\|_2^{\frac{1}{2}+\varepsilon}\right)\end{aligned}$$

where Ω' is any domain satisfying (16).
Then, by the decay estimates (4.3) in [14, Theorem 4.1] we have

$$\begin{aligned}&\|\left(\pi^n(t) - \pi^m(t)\right)\|_2 \leq c(T)\left(t^{-\frac{1}{2}} + t^{-\left(\frac{3}{4}+\frac{\varepsilon}{2}\right)}\right)\|v_0^n - v_0^m\|_2 \\ &\qquad\qquad \forall\ T \in (0,\infty)\ \text{ and for any } t \in (0,T)\,,\end{aligned} \tag{23}$$

from which we obtain that $\{\pi^n\}$ is a Cauchy sequence in $L^r\big(0,T;L^2(\Omega')\big) \cap L^\infty\big(\eta,T;L^2(\Omega')\big)$, for every $1 \leq r < \frac{4}{3}$, for every $T \in (0,\infty)$, for every $\eta \in (0,T)$ and for any domain Ω' satisfying (16).

Therefore, there exists a function $\pi : \Omega \times (0,\infty) \to \mathbb{R}$, enjoying summability properties (15), such that, for every $T \in (0,\infty)$ and for every $\eta \in (0,T)$,

$$\nabla\pi^n(x,t) \to \nabla\pi(x,t) \quad \text{in } L^p\big(0,T;L^q(\Omega)\big) \cap L^\infty\big(\eta,T;L^q(\Omega)\big)\,,$$

$$\text{for every pair } (p,q) \text{ such that } 1 < q < 2 \text{ and } 1 \leq p < \tfrac{4q}{5q-2},$$

$$\pi^n(x,t) \to \pi(x,t) \quad \text{in } L^r\big(0,T;L^2(\Omega')\big) \cap L^\infty\big(\eta,T;L^2(\Omega')\big) \quad \forall\ 1 \leq r < \frac{4}{3},$$

where Ω' is any domain satisfying (16).

Then, if we let $n \to \infty$ in relation (18), recalling (20), we deduce the weak formulation (17). □

Remark 2. Of course, the pressure field π is only determined up to a function $\pi_0 : (0,\infty) \to \mathbb{R}$.
If $\Omega \subseteq \mathbb{R}^n$, with $n \geq 3$, is a domain satisfying Assumption 1 and $\nabla\pi \in L^r\big(0,T;L^s(\Omega)\big)$ for some $T \in (0,\infty]$, $r \in [1,\infty]$ and $s \in (1,n)$, by [6, Lemma 1.3] (see also [17, Lemma 3.2] or [4, Theorem II.6.1]), $\pi_0(t)$ can be chosen so that $\widetilde{\pi}(x,t) \equiv \pi(x,t) + \pi_0(t)$ is in $L^r\big(0,T;L^\alpha(\Omega)\big)$ with $\alpha = \frac{ns}{n-s}$.

We point out that the chosen function $\pi_0(t)$ is independent of s, i.e. if $\nabla\pi \in L^r\big(0,T;L^s(\Omega)\big)$ for some $T\in(0,\infty]$, $r\in[1,\infty]$ and for every $1<s_1\le s\le s_2<n$ then $\widetilde{\pi}\in L^r\big(0,T;L^\alpha(\Omega)\big)$ for all $\alpha\in\big[\frac{ns_1}{n-s_1},\frac{ns_2}{n-s_2}\big]$.
Hereafter, for the sake of simplicity, the tilde mark is omitted in $\widetilde{\pi}$.
In the case $\Omega\equiv\mathbb{R}^n$, in Proposition 2 we can choose $\pi(x,t)=0$ almost everywhere.

4. The pressure field π associated to a Hopf weak solution

Theorem 1. *Let $\Omega\subseteq\mathbb{R}^n$, with $n\ge 3$, be a domain satisfying Assumption 1 and let v be a Hopf weak solution of problem* (1) – (2) *with initial data $v_0\in J(\Omega)$; then, there exists a scalar field $\pi:\Omega\times(0,\infty)\to\mathbb{R}$ such that,*

1. *if Ω satisfies condition $(D2)$ or $(D3)$, for every $T\in(0,\infty)$, for every $\eta\in(0,T)$ and for any domain Ω' satisfying* (16),

   i. $\nabla\pi\in L^p\big(0,T;L^q(\Omega)\big)$, *for every pair of exponents (p,q) such that*

   $$\frac{n}{q}+\frac{2}{p}=n+1 \quad \textit{and} \quad \frac{2(n-1)}{2n-3}<q<\frac{n}{n-1}\,;$$

   $\nabla\pi\in L^r\big(0,T;L^q(\Omega)\big)\cap L^p\big(\eta,T;L^q(\Omega)\big)$, *for every tern of exponents (r,p,q) such that*

   $$1<q\le\frac{2(n-1)}{2n-3}\quad,\quad 1\le r<\frac{4q}{5q-2}\quad \textit{and}\quad \frac{n}{q}+\frac{2}{p}=n+1\,;$$

   ii. $\pi\in L^p\big(0,T;L^{\tilde q}(\Omega)\big)$, *for every pair of exponents $(p,\tilde q)$ such that*

   $$\frac{n}{\tilde q}+\frac{2}{p}=n \quad \textit{and}\quad \frac{2n(n-1)}{(2n-1)(n-2)}<\tilde q<\frac{n}{n-2}\,;$$

   $\pi\in L^r\big(0,T;L^{\tilde q}(\Omega)\big)\cap L^p\big(\eta,T;L^{\tilde q}(\Omega)\big)$, *for every tern of exponents $(r,p,\tilde q)$ such that*

   $$\frac{n}{n-1}<\tilde q\le\frac{2n(n-1)}{(2n-1)(n-2)}\quad,\quad 1\le r<\frac{4\tilde q n}{5\tilde q n-2(n+\tilde q)}\quad \textit{and}\quad \frac{n}{\tilde q}+\frac{2}{p}=n\,;$$

   iii. $$\pi\in L^r\big(0,T;L^{\frac{2n}{2n-3}}(\Omega')\big)\qquad \forall\ 1\le r<\frac{4}{3}\,;$$

   iv. *if $n\ge 4$,* $\pi\in L^p\big(0,T;L^{\tilde q}(\Omega')\big)$ *for every pair of exponents $(p,\tilde q)$ such that*

   $$\frac{n}{\tilde q}+\frac{2}{p}=n \quad \textit{and}\quad \frac{2n}{2n-3}<\tilde q\le\frac{2n(n-1)}{(2n-1)(n-2)}\,;$$

2. if $\Omega \equiv \mathbb{R}^n$,

i. $\nabla\pi \in L^p\big(0,\infty; L^q(\mathbb{R}^n)\big)$, *for every pair of exponents* (p,q) *such that*

$$\frac{n}{q} + \frac{2}{p} = n+1 \quad \text{and} \quad 1 < q < \frac{n}{n-1}\,; \tag{24}$$

ii. $\pi \in L^p\big(0,\infty; L^{\tilde{q}}(\mathbb{R}^n)\big)$, *for every pair of exponents* $(p,\tilde{q})$ *such that*

$$\frac{n}{\tilde{q}} + \frac{2}{p} = n \quad \text{and} \quad 1 < \tilde{q} \le \frac{n}{n-2}\,;$$

3. $\forall\ T \in (0,\infty)$, *the Hopf weak solution* v *and the scalar field* π *satisfy the following relation*

$$\begin{aligned} \int_0^s \big[\langle v, \phi_t\rangle - \langle \nabla v, \nabla\phi\rangle - \langle (v\cdot\nabla)\,v, \phi\rangle\big]\,dt &= -\int_0^s \langle \pi, \nabla\cdot\phi\rangle\,dt \\ &\quad + \langle v(s), \phi(s)\rangle - \langle v_0, \phi_0\rangle \\ \forall\ s\in[0,T]\,,\quad \forall\ \phi &\in C_0^\infty(\Omega_T;\mathbb{R}^n)\,. \end{aligned} \tag{25}$$

Remark. In [7, Theorem 3.1] it was proved that if $\Omega \subseteq \mathbb{R}^n$ is a domain satisfying condition $(D2)$ or $(D3)$ and if the initial data v_0 is in $B^{2-\frac{2}{p}}_{q,p}(\Omega)\cap J^q(\Omega)$, for some pair of exponents (p,q) satisfying condition (24), then $\nabla\pi$ is in $L^p\big(0,\infty; L^q(\Omega)\big)$. Even if this result is better than ours, however we require *less regularity* for the initial data, i.e. we require only $v_0 \in J(\Omega)$. That is very important to understand if it's possible to deduce partial regularity properties of weak solutions under the only hypotheses which assure their existence.

Proof. Let v be a Hopf weak solution with initial data $v_0 \in J(\Omega)$; by the Hölder inequality and estimates (6), (8) we have

$$\|(v\cdot\nabla)\,v\|_{L^p(0,\,\infty;\,L^q(\Omega))} \le \|v\|_{L^s(0,\,\infty;\,L^r(\Omega))}\,\|\nabla v\|_{L^2(0,\,\infty;\,L^2(\Omega)} \le c\,\|v_0\|_2^2$$

with $\frac{1}{q} = \frac{1}{r} + \frac{1}{2}$ and $\frac{1}{p} = \frac{1}{s} + \frac{1}{2}$, for every pair of exponents (s,r) satisfying condition (7). Then $(v\cdot\nabla)\,v \in L^p\big(0,\infty; L^q(\Omega)\big)$ for every pair of exponents (p,q) satisfying condition (24).

Let $(u_1, \nabla\pi_1)$ be the unique solution of the Stokes problem (13) with $v_0(x) \equiv 0$ and $f = -(v\cdot\nabla)\,v$; we have (cf. [7, Theorem 2.8], [14, Lemma 4.2], [15])

– as $\frac{n}{n-1} \le \frac{n}{2}$ for $n \ge 3$, $u_1 \in L^p\big(0,T; W^{2,q}(\Omega)\cap J^{1,q}(\Omega)\big)$, for every $T\in(0,\infty)$, and $\nabla\pi_1 \in L^p\big(0,\infty; L^q(\Omega)\big)$ for every pair of exponents (p,q) satisfying condition (24);

- u_1, π_1 satisfy the following relations, with $f = -(v\cdot\nabla)v$ and for every $T \in (0,\infty)$

$$\int_0^s \big[\langle v, \phi_t\rangle - \langle \nabla v, \nabla \phi\rangle\big]\, dt = -\int_0^s \big[\langle \pi, \nabla\cdot\phi\rangle + \langle f, \phi\rangle\big]\, dt + \langle v(s), \phi(s)\rangle \tag{26}$$
$$\forall\ s \in [0,T]\,, \quad \forall\ \phi \in C_0^\infty(\Omega_T;\mathbb{R}^n)\,,$$

$$\int_0^s \big[\langle v, \varphi_t\rangle - \langle \nabla v, \nabla \varphi\rangle\big]\, dt = -\int_0^s \langle f, \varphi\rangle\, dt + \langle v(s), \varphi(s)\rangle \tag{27}$$
$$\forall\ s \in [0,T]\,, \quad \forall\ \varphi \in \mathscr{C}_0(\Omega_T)\,.$$

Moreover, since $\frac{n}{n-1} < n$ for $n \geq 3$, by Remark 2, we deduce

$$\begin{gathered}\pi_1 \in L^p\big(0,\infty; L^{\tilde q}(\Omega)\big) \text{ for every pair of exponents } (p,\tilde q) \text{ such that} \\ \tfrac{n}{\tilde q} + \tfrac{2}{p} = n \quad \text{and} \quad \tfrac{n}{n-1} < \tilde q < \tfrac{n}{n-2}\,.\end{gathered} \tag{28}$$

By Remark 1, there exists a unique "weak" solution u_2 of the Stokes problem (13) with $f(x,t) \equiv 0$ and initial data v_0, which satisfies relation (14), for every $T \in (0,\infty)$. By Proposition 2, there exists a function π_2 such that

- $\pi_2 = 0$ if Ω satisfies condition $(D1)$;

- π_2 enjoys summability properties (15), if Ω is a domain satisfying condition $(D2)$ or $(D3)$.

Moreover u_2, π_2 satisfy relation (17), for every $T \in (0,\infty)$.

We set $\pi = \pi_1 + \pi_2$.

Let $\Omega \equiv \mathbb{R}^n$. In such a case $\pi \equiv \pi_1$ and, therefore, property 2-i holds.
Following [3, Section 2c], we can apply the divergence operator to the equation $(1)_1$, in the distribution sense. As v is a solenoidal vector field, we have that, for almost every $\bar t \in (0,\infty)$, the function $\pi(\cdot,\bar t)$ is a weak solution of the following elliptic equation

$$\Delta\pi(x,\bar t) = -\sum_{h,k=1}^{n} \frac{\partial^2}{\partial x_k \partial x_k}\big(u_h u_k\big)(x,\bar t) \qquad \forall\ x \in \mathbb{R}^n\,. \tag{29}$$

The previous equation can be solved explicitly; there follows that the function $\pi(\cdot,\bar t)$ is a sum of singular integral transforms applied to the functions $u_h u_k$. By the Calderón – Zygmund Theorem (cf. Theorems 2 – 4 in [23, Chapter II]), for almost every $\bar{t} \in (0,\infty)$, we have

$$\|\pi(\bar{t})\|_{L^q(\mathbb{R}^n)} \le c\,\|v(\bar{t})\|^2_{L^{2q}(\mathbb{R}^n)} \qquad \forall\ 1 < \tilde{q} \le \frac{n}{n-2}\,.$$

Thus, property 2-ii follows from estimate (8).

If Ω is a domain satisfying condition $(D2)$ or $(D3)$, since

$$\tfrac{2(n-1)}{2n-3} < q < \tfrac{n}{n-1} \qquad \Longrightarrow \qquad \tfrac{n+1}{2} - \tfrac{n}{2q} > \tfrac{5}{4} - \tfrac{1}{2q}\,,$$
$$1 < q \le \tfrac{2(n-1)}{2n-3} \qquad \Longrightarrow \qquad \tfrac{n+1}{2} - \tfrac{n}{2q} \le \tfrac{5}{4} - \tfrac{1}{2q}\,,$$

from the summability properties of $\nabla\pi_1$, $\nabla\pi_2$, there follows that $\nabla\pi$ satisfies summability properties 1-i.
As $\frac{n}{n-1} < n$, by Remark 2, we deduce properties 1-ii from properties 1-i.
By (28), $\pi_1 \in L^{\frac{4}{3}}\big(0,T;L^{\frac{2n}{2n-3}}(\Omega)\big)$; since $\frac{2n}{2n-3} \le 2$ for $n \ge 3$, by (15b) we obtain property 1-iii.
As $\frac{2n(n-1)}{(2n-1)(n-2)} < 2$ for $n \ge 4$ and

$$\tfrac{2n}{2n-3} < \tilde{q} \le \tfrac{2n(n-1)}{(2n-1)(n-2)} \qquad \Longrightarrow \qquad \tfrac{1}{p} = \tfrac{n}{2}\Big(1 - \tfrac{1}{\tilde{q}}\Big) > \tfrac{3}{4}\,,$$

by (28) and (15b) we obtain property 1-iv.

Adding relation (27), for u_1, and relation (14), for u_2, we deduce that the vector field $u(x,t) = u_1(x,t) + u_2(x,t)$ satisfy the following identity

$$\int_0^s \big[\langle u\,,\,\varphi_t\rangle - \langle \nabla u\,,\,\nabla\varphi\rangle\big]\,dt = \int_0^s \langle (v\cdot\nabla)\,v\,,\,\varphi\rangle\,dt + \langle u(s)\,,\,\varphi(s)\rangle - \langle v_0\,,\,\varphi_0\rangle \tag{30}$$
$$\forall\ T \in (0,\infty)\,,\quad \forall\ s \in [0,T]\,,\quad \forall\ \varphi \in \mathscr{C}_0(\Omega_T)\,.$$

Subtracting identity (30) from identity (9), written for the Hopf weak solution v, we obtain

$$\int_0^s \big[\langle (v-u)\,,\,\varphi_t\rangle - \langle \nabla(v-u)\,,\,\nabla\varphi\rangle\big]\,dt = \langle (v(s) - u(s))\,,\,\varphi(s)\rangle \tag{31}$$
$$\forall\ T \in (0,\infty)\,,\quad \forall\ s \in [0,T]\,,\quad \forall\ \varphi \in \mathscr{C}_0(\Omega_T)\,.$$

Using Proposition 1 and a limit process, we can extend this last relation to "less regular" test functions $\varphi \in \mathcal{C}(\Omega_T)$.

Let $\psi : \Omega \times (0,\infty) \to \mathbb{R}^n$ be the unique solution of the Stokes problem (13) with $f(x,t) \equiv 0$ and arbitrary smooth initial data $\psi_0(x) \in \mathscr{C}_0(\Omega)$. By Theorem 1

in [9, Ch. 4] (see also Proposition 1.2 in [25, Ch. 3]), $\psi \in C\big([0,T];J^{1,2}(\Omega)\big)$ with $\psi_t \in L^2\big(0,T;J(\Omega)\big)$, for every $T \in (0,\infty)$.
For any $\bar{s} \in (0,\infty)$, $\psi(x,\bar{s}-t)$ is the (unique) backward in time solution of the adjoint Stokes problem on $\Omega \times (0,\bar{s})$.

Let us set

$$\widetilde{\psi}(x,t) = \begin{cases} \psi(x,\bar{s}-t)\,, & \forall\ (x,t) \in \Omega \times [0,\bar{s}] \\ \psi_0(x)\,, & \forall\ (x,t) \in \Omega \times (\bar{s},\infty)\,; \end{cases}$$

Of course, for every $T \in (\bar{s},\infty)$, $\widetilde{\psi} \in C\big([0,T];J^{1,2}(\Omega)\big)$ with $\widetilde{\psi}_t \in L^2\big(0,T;J(\Omega)\big)$.

Let $\theta : \mathbb{R} \to [0,1]$ be a smooth, decreasing, non negative function such that $\theta(t) = 1$, for $t \le 1$, and $\theta(t) = 0$, for $t \ge 2$.
For some $T \in (\bar{s},\infty)$, we set

$$\widetilde{\varphi}(x,t) = \widetilde{\psi}(x,t)\,\theta\Big(\frac{t+T-2\bar{s}}{T-\bar{s}}\Big) \qquad \forall\ (x,t)\ \in\ \Omega \times [0,\infty)$$

As $\widetilde{\varphi}(x,T) = \widetilde{\psi}(x,T)\,\theta(2) \equiv 0$ for *a.e.* $x \in \Omega$, taking the regularity properties of $\widetilde{\psi}(x,t)$ and $\theta(t)$ into account, we may conclude that $\widetilde{\varphi} \in \mathcal{C}(\Omega_T)$; moreover, since $\widetilde{\varphi}(x,t) \equiv \psi(x,\bar{s}-t)$ for *a.e.* $x \in \Omega$ and $\forall\ t \in [0,\bar{s}]$, $\widetilde{\varphi}$ also satisfies the adjoint Stokes problem on $\Omega \times (0,\bar{s})$.
Therefore, substituting $\widetilde{\varphi}$ in (31) and using Green's identity, we obtain

$$\langle (v(\bar{s}) - u(\bar{s}))\,,\,\psi_0\rangle = 0$$

Since $\bar{s} \in (0,\infty)$ and $\psi_0(x) \in \mathscr{C}_0(\Omega)$ are been arbitrarily chosen, by the density of $\mathscr{C}_0(\Omega)$ in $J(\Omega)$, we may conclude that $v(x,t) = u_1(x,t) + u_2(x,t)$ for *a.e.* $x \in \Omega$ and $\forall\ t \in (0,\infty)$.

Finally, adding relation (26), for (u_1,π_1) with $f = -(v\cdot\nabla)v$, and relation (17), for (u_2,π_2), we obtain identity (25) with $\pi \equiv \pi_1 + \pi_2$. $\square$

Remark. When $\Omega \subseteq \mathbb{R}^n$ is an exterior domain, the summability properties given in Theorem 1 imply a certain decay of π, for $|x| \to \infty$. As pointed out in [21, 7], an important consequence of that is the regularity of suitable weak solutions for large x, when $\Omega \subseteq \mathbb{R}^3$ is an exterior domain. Such result is proved in [3] in case $\Omega \equiv \mathbb{R}^3$, in [12] for exterior domains.

Analogously in the Appendix of [3], using Proposition 2 and Theorem 1, we can prove the following

Theorem 2. *Let $\Omega \subseteq \mathbb{R}^3$ be a domain satisfying condition $(D2)$ or $(D3)$; for every $v_0 \in J(\Omega)$, there exists a suitable weak solution (v,π) of problem* (1)–(2), *in $\Omega \times (0,\infty)$. Moreover, v is a Leray weak solution and, for every $T \in (0,\infty)$, for every $\eta \in (0,T)$ and for any domain Ω' satisfying (16), we have*

$$\begin{aligned} &\nabla\pi \in L^p\big(0,T;L^q(\Omega)\big), \text{ for every pair } (p,q) \text{ such that} \\ &\qquad \tfrac{3}{q}+\tfrac{2}{p}=4 \quad \text{and} \quad \tfrac{4}{3}<q<\tfrac{3}{2}\,; \\ &\nabla\pi \in L^r\big(0,T;L^q(\Omega)\big)\cap L^p\big(\eta,T;L^q(\Omega)\big), \text{ for every tern of exponents} \\ &(r,p,q) \text{ such that} \\ &\qquad 1<q\le\tfrac{4}{3} \quad , \quad 1\le r<\tfrac{4q}{5q-2} \quad \text{and} \quad \tfrac{3}{q}+\tfrac{2}{p}=4\,; \end{aligned} \tag{32}$$

$$\begin{aligned} &\pi \in L^p\big(0,T;L^{\tilde q}(\Omega)\big), \text{ for every pair } (p,\tilde q) \text{ such that} \\ &\qquad \tfrac{3}{\tilde q}+\tfrac{2}{p}=3 \quad \text{and} \quad \tfrac{12}{5}<\tilde q<3\,; \\ &\pi \in L^r\big(0,T;L^{\tilde q}(\Omega)\big)\cap L^p\big(\eta,T;L^{\tilde q}(\Omega)\big), \text{ for every tern of exponents} \\ &(r,p,\tilde q) \text{ such that} \\ &\qquad \tfrac{3}{2}<\tilde q\le\tfrac{12}{5} \quad , \quad 1\le r<\tfrac{12\tilde q}{13\tilde q-6} \quad \text{and} \quad \tfrac{3}{\tilde q}+\tfrac{2}{p}=3\,; \\ &\pi \in L^r\big(0,T;L^2(\Omega')\big) \qquad \forall\ 1\le r<\tfrac{4}{3}. \end{aligned} \tag{33}$$

Remark. In [24, Theorem 2.1] it was shown that, if $\Omega \subset \mathbb{R}^3$ satisfies condition $(D2)$ or $(D3)$, for each $v_0 \in J(\Omega)$ there exists at least a suitable weak solution (v,π), of problem (1)−(2), in $\Omega \times (0,\infty)$, such that $\pi \in L^{\frac{5}{4}}_{\rm loc}\big(\Omega \times (0,\infty)\big)$ and the generalized energy inequality (12) is satisfied for every non-negative $\sigma \in C_0^\infty\big(\Omega \times (0,\infty);\mathbb{R}\big)$.
In Theorem 2, for any initial data $v_0 \in J(\Omega)$, we can prove the existence of a suitable weak solution (v,π) with a more regular associated pressure field π (in fact, it satisfies summability properties (32)−(33)); moreover, there hold both the usual energy inequality (5) and the generalized one (12) for every non-negative $\sigma \in C_0^\infty\big(\Omega \times [0,\infty);\mathbb{R}\big)$ (i.e. $\sigma(x,0)$ doesn't need to be zero).

So, as far as we know, if $\Omega \subset \mathbb{R}^3$ is a bounded or an exterior domain, thanks to Theorem 2, $J(\Omega)$ is the largest class of initial data for which we can give an existence theorem of weak solutions which are both suitable weak solutions, in $\Omega \times (0,\infty)$, and Leray weak solutions.

Proof. Let $v_0 \in J(\Omega)$; by density of $\mathscr{C}_0(\Omega)$ in $J(\Omega)$, there exists a sequence $\{v_0^k\} \subset \mathscr{C}_0(\Omega)$ converging to v_0 in $J(\Omega)$; of course, there exists $c>0$ such that

$$\|v_0^k\|_2 \le c\,\|v_0\|_2 \qquad \forall\ k \in \mathbb{N}\,. \tag{34}$$

For every $k \in \mathbb{N}$, let us consider the following initial-boundary value problem

$$\text{(35a)} \qquad \begin{aligned} \partial_t v^k - \Delta v^k + (F^k \cdot \nabla) v^k &= -\nabla \pi^k && \text{in } \Omega \times (0,T)\,, \\ \nabla \cdot v^k &= 0 && \text{in } \Omega \times (0,T), \end{aligned}$$

$$\text{(35b)} \qquad \begin{aligned} v^k(x,t) &= 0 && \text{on } \partial\Omega \times (0,T), \\ v^k(x,0) &= v_0^k(x) && \text{in } \Omega, \end{aligned}$$

where, for almost every $t \in (0,T)$,

$$F^k(x,t) = \int_{\mathbb{R}^3} j_{\frac{1}{k}}(|x-y|)\, \widetilde{v}^k(y,t)\, dy\,,$$

is the regularized function, in the sense of Friederichs, in the space variables of

$$\widetilde{v}^k(x,t) = \begin{cases} v^k(x,t) & \text{in } \Omega\,, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

with $j_{\frac{1}{k}}(s)$ an even, nonnegative, real-valued function belonging to $C_0^\infty(\mathbb{R})$, such that $j_{\frac{1}{k}}(s) = 0$ if $|s| \geq \frac{1}{k}$ and $\int_{\mathbb{R}} j_{\frac{1}{k}}(s)\, ds = 1$.

If the domain Ω is unbounded, we also assume the condition at infinity

$$\lim_{|x|\to\infty} v^k(x,t) = 0 \qquad \forall\ t \in [0,T)\,.$$

Let $k \in \mathbb{N}$ and $T \in (0,\infty)$ be fixed; using the Faedo-Galerkin method, we can construct a weak solution $v^k : \Omega \times (0,T) \to \mathbb{R}^n$ to problem (35), in $\Omega \times (0,T)$, such that $v^k \in L^\infty\big(0,T; J(\Omega)\big) \cap L^2\big(0,T; J^{1,2}(\Omega)\big)$. Since the kernel $j_{\frac{1}{k}}(s) \in C_0^\infty(\mathbb{R})$, by Young's inequality for convolution, we get $F^k \in L^\infty\big(0,T; L^q(\Omega)\big)$ for every $q \in [2,\infty]$ [2]. Since $v_0^k \in \mathscr{C}_0(\Omega)$, from Theorem 4.2 in [22] there follows that there exists a pressure field $\pi^k : \Omega \times (0,T) \to \mathbb{R}$, associated to v^k, such that

- $(v^k, \nabla\pi^k)$ is the unique solution to problem (35), in $\Omega \times (0,T)$;
- system (35a) is satisfied almost everywhere in $\Omega \times (0,T)$;
- there hold

$$\text{(36)} \qquad \begin{aligned} v(x,t) &\in L^q\big(0,T; W^{2,q}(\Omega) \cap J^{1,q}(\Omega)\big)\,, \\ v_t(x,t) &\in L^q\big(0,T; J^q(\Omega)\big)\,, \\ \nabla\pi(x,t) &\in L^q\big(0,T; L^q(\Omega)\big)\,, \\ \pi(x,t) &\in L^q\big(0,T; L^q(\Omega')\big)\,, \end{aligned}$$

  for every $q \in (1,\infty)$ and for any domain Ω' satisfying (16).

[2]Of course, the sequence $\{F^k\}$ is not bounded in $L^\infty\big(0,T; L^q(\Omega)\big)$, for $q \in (2,\infty]$.

Then, for $0 \leq s \leq t \leq T$, we can multiply both sides of $(35a)_1$ by v^k in $L^2\big(\Omega \times (s,t)\big)$ (respectively by $v^k\sigma$, with non-negative $\sigma \in C_0^\infty\big(\Omega\times[0,T);\mathbb{R}\big)$). Integrating by parts with respect to x and t, we obtain the following usual and generalized energy equalities:

$$\|v^k(t)\|_2^2 + 2\int_s^t \|\nabla v^k(\tau)\|_2^2\, d\tau = \|v^k(s)\|_2^2, \tag{37}$$

$$\begin{aligned} &\int_\Omega |v^k(t)|^2\sigma(t)\, dx + 2\int_s^t\int_\Omega |\nabla v^k|^2\sigma\, dx\, d\tau = \int_\Omega |v^k(s)|^2\sigma(s)\, dx \\ &+\int_s^t\int_\Omega |v^k|^2(\sigma_\tau + \Delta\sigma)\, dx\, d\tau + \int_s^t\int_\Omega \big[|v^k|^2\, F^k\cdot\nabla\sigma + 2\pi^k\, v^k\cdot\nabla\sigma\big]\, dx\, d\tau \end{aligned} \tag{38}$$

for all $s,t \in \big[0,T\,\big]$ with $s \leq t$ and for every non-negative, scalar valued function $\sigma \in C_0^\infty\big(\Omega\times[0,T);\mathbb{R}\big)$.

From the energy equality (37) and estimate (34), we deduce

$$\begin{aligned} \|v^k\|_{L^\infty(0,\,T;\,J(\Omega))} &\leq \|v_0^k\|_2 \leq c\,\|v_0\|_2\,, \\ \|\nabla v^k\|_{L^2(0,\,T;\,L^2(\Omega))} &\leq \tfrac{1}{2}\,\|v_0^k\|_2 \leq c\,\|v_0\|_2\,, \end{aligned} \tag{39}$$

where the constant c is independent of k; from (39) we can also obtain a uniform estimate with respect to k, like (8).

Let $(u_1^k, \nabla\pi_1^k)$ the unique solution of problem (13) with external force $f = -(F^k\cdot\nabla)v^k$ and initial data $u_1^k(x,0) = 0$ and $(u_2^k, \nabla\pi_2^k)$ the unique solution of problem (13) with external force $f = 0$ and initial data $u_1^k(x,0) = v_0^k$ (analogously in the proof of Theorem 1).

Using Hölder's inequality and the properties of Friederichs mollifiers (cf. [1, Section 2.29]), by estimates (39) we obtain

$$\begin{aligned} \|(F^k\cdot\nabla)v^k\|_{L^{\frac{4}{3}}(0,\,T;\,L^{\frac{6}{5}}(\Omega))} &\leq \|F^k\|_{L^4(0,\,T;\,L^3(\Omega))}\|\nabla v^k\|_{L^2(0,\,T;\,L^2(\Omega))} \\ &\leq \|v^k\|_{L^4(0,\,T;\,L^3(\Omega))}\|\nabla v^k\|_{L^2(0,\,T;\,L^2(\Omega))} \leq c\,\|v_0\|_2^2\,. \end{aligned}$$

So, by estimate (4.12) in [14, Lemma 4.2] or estimate (2.22) in [7, Theorem 2.8], we obtain $\|\nabla\pi_1^k\|_{L^{\frac{4}{3}}(0,\,T;\,L^{\frac{6}{5}}(\Omega))} \leq c\,\|v_0\|_2^2$.

By estimate (22) in Proposition 2, we get $\|\nabla\pi_2^k\|_{L^{\frac{7}{6}}(0,\,T;\,L^{\frac{6}{5}}(\Omega))} \leq c(T)\,\|v_0\|_2$.

Then, for $\pi^k \equiv \pi_1^k + \pi_2^k$ there holds the following estimate

$$\|\nabla\pi^k\|_{L^{\frac{7}{6}}(0,\,T;\,L^{\frac{6}{5}}(\Omega))} \leq c(T)\,\big(\|v_0\|_2 + \|v_0\|_2^2\big)\,, \tag{40}$$

from which, by virtue of Remark 2, we get

$$\|\pi^k\|_{L^{\frac{7}{6}}(0,\,T;\,L^2(\Omega))} \le c(T)\left(\|v_0\|_2 + \|v_0\|_2^2\right), \tag{41}$$

where the constant $c(T)$ is independent of k.

Then, from estimates (39), (41), we may deduce the existence of subsequences of $\{v^k\}$, $\{\pi^k\}$ — again denoted by $\{v^k\}$, $\{\pi^k\}$ respectively, for simplicity — and of functions $v \in L^\infty\big(0,T;J(\Omega)\big) \cap L^2\big(0,T;J^{1,2}(\Omega)\big)$, $\pi \in L^{\frac{7}{6}}\big(0,T;L^2(\Omega)\big)$ such that

$$\begin{aligned} v^k &\to v && \text{weakly in } L^2\big(0,T;J^{1,2}(\Omega)\big) \\ v^k &\to v && \text{weak-star in } L^\infty\big(0,T;J(\Omega)\big) \\ \pi^k &\to \pi && \text{weakly in } L^{\frac{7}{6}}\big(0,T;L^2(\Omega)\big) \end{aligned}$$

Moreover, by Friederichs Lemma, we obtain that

$$v^k \to v \qquad \text{strongly in } L^2\big(0,T;L^2(\Omega')\big)$$

for any bounded domain $\Omega' \subseteq \Omega$.

Considering the summability properties (36) of $(v^k, \nabla\pi^k)$, for $0 \le t \le T$, we can multiply both sides of $(35\text{a})_1$ by $\varphi \in \mathcal{C}(\Omega_T)$ in $L^2\big(\Omega \times (0,t)\big)$. Integrating by parts with respect to x and t, we obtain

$$\int_0^t \big[\langle v^k, \varphi_t\rangle - \langle \nabla v^k, \nabla\varphi\rangle - \langle (F^k \cdot \nabla)\, v^k, \varphi\rangle\big]\, dt = \langle v^k(t), \varphi(t)\rangle - \langle v_0^k, \varphi_0\rangle$$

If we let $k \to \infty$, we obtain that v satisfies the weak formulation (9) of the Navier-Stokes equations, with test functions $\varphi \in \mathcal{C}(\Omega_T)$.
Analogously in [6] and in the Appendix of [3] (see also [12, 24]), if we let $k \to \infty$ in (37) and (38), we deduce for v both the usual (5) and the generalized (12) energy inequalities.

Finally, since v is in particular a Hopf weak solution, from Theorem 1 we can deduce the summability properties (32) – (33). □

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