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SPECTRAL FINITE DIFFERENCE ANALYSIS OF NATURAL CONVECTION IN A TWO-DIMENSIONAL ENCLOSURE OF A THREE-FINS TYPE

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Abstract. Spectral finite difference scheme is applied to two-dimensional natural laminar convection around a three-fins type enclosure, including a shape control parameter supplemented with a condition of doubly-connectedness. Streamlines, characteristics for the maximum stream function and the mean Nusselt number against a Grashof number are presented.

1. Introduction Fin type heat transfer is encountered at an element of latent thermal energy storage [1], which is a basically unsteady phenomenon, nevertheless natural convection in melting phase can be treated as quasi-steady state, since in such cases Stefan number Ste ($\equiv c_s \Delta T/(\Delta h), c_s$: specific heat in a liquid state, ΔT: temperature difference between surfaces, Δh: latent heat-of-fusion) is generally small for phase change materials. Treated is a steady-state two-dimensional laminar natural convection enclosed in doubly-connected region with a three-fins type, using a spectral finite difference scheme.

2. Analysis

2.1 General

Fluid is assumed to be substantially incompressible except for density, for which Boussinesq approximation applies. Treated is a steady-state natural convection enclosed in a two-dimensional configuration (in a vertical plane) possessing a

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three-fins type inside. For the Cartesian coordinate (x, y) (y : vertically upward), the following conformal mapping is introduced:

$$z = i\,w\left(1 + \frac{\epsilon}{w^3} - \frac{1}{5}\frac{\epsilon^2}{w^6}\right) , \tag{1}$$

$$z \equiv x + iy \ , \ w = \mathrm{e}^{\alpha + i\beta} \ , \ 0 \le \alpha \le \alpha_\infty \ , \ -\pi < \beta \le \pi \ ,$$

where ϵ is a real non-negative control parameter for a fin shape. Heated fin surface is given by $\alpha = 0$ and far away boundary is by $\alpha = \alpha_\infty$. The dimensionless coordinate z is based on the reference length L such that the height of the fin (the length between the location of the top and that at the bottom on the central axis) $/L \equiv 2\left(1 - \epsilon^2/5\right)$. For univalence of the mapping

$$0 \le \epsilon \le \epsilon_0 \ ,$$

for which

$$1 - 2\,\epsilon_0 \cos\theta + \frac{1}{5}\,{\epsilon_0}^2\left(1 - 12\cos^2\theta + 16\cos^4\theta\right) = 0 \ , \tag{2}$$

$$\frac{\sin 2\theta}{\sin\theta} \equiv 2\cos\theta \ , \ \text{(assuming } \sin\theta \ne 0) \ ,$$

$$\frac{\sin 5\theta}{\sin\theta} \equiv 1 - 12\cos^2\theta + 16\cos^4\theta \ .$$

The limit ϵ_0 is specified by the condition of the multiplicity of the roots

$$-2 + \frac{1}{5}\,\epsilon_0\left(-24\cos\theta + 64\cos^3\theta\right) = 0 \ . \tag{3}$$

Thus eliminating $\cos\theta$ between Eqs.(2) and (3), the resultant of Sylvester gives

$$\begin{vmatrix} 16 & 0 & -12 & -10/\epsilon_0 & 1+5/{\epsilon_0}^2 & 0 & 0 \\ 0 & 16 & 0 & -12 & -10/\epsilon_0 & 1+5/{\epsilon_0}^2 & 0 \\ 0 & 0 & 16 & 0 & -12 & -10/\epsilon_0 & 1+5/{\epsilon_0}^2 \\ 64 & 0 & -24 & -10/\epsilon_0 & 0 & 0 & 0 \\ 0 & 64 & 0 & -24 & -10/\epsilon_0 & 0 & 0 \\ 0 & 0 & 64 & 0 & -24 & -10/\epsilon_0 & 0 \\ 0 & 0 & 0 & 64 & 0 & -24 & -10/\epsilon_0 \end{vmatrix} = 0 \ , \tag{4}$$

which is a cubic equation with respect to ${\epsilon_0}^2$. The root $\epsilon_0(>0)$,satisfying $\cos\theta$ is real, is

$$\epsilon_0 = \sqrt{\frac{31}{4} + \frac{3\sqrt{226}}{2}\sin\varphi} \ , \tag{5}$$

$$\varphi = -\frac{1}{3}\sin^{-1}\left(\frac{29\times 197}{4\times 113\sqrt{226}}\right) . \tag{6}$$

Eventually $\epsilon_0 \approx 0.617$.

2.2 Basic equations

Basic governing equations under Boussinesq approximation with neglecting dissipation terms are

$$J\,\frac{\partial\zeta}{\partial t} + \frac{\partial(\zeta,\psi)}{\partial(\alpha,\beta)} = \frac{1}{\sqrt{Gr}}\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)\zeta + \frac{\partial(T,y)}{\partial(\alpha,\beta)} , \tag{7}$$

$$J\,\zeta + \left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)\psi = 0 , \tag{8}$$

$$J\,\frac{\partial T}{\partial t} + \frac{\partial(T,\psi)}{\partial(\alpha,\beta)} = \frac{1}{Pr\sqrt{Gr}}\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)T , \tag{9}$$

$$J \equiv \left|\frac{dz}{d(\alpha+i\beta)}\right|^2 , \tag{10}$$

where ζ: dimensionless vorticity, ψ: dimensionless stream function, T: dimensionless temperature, Pr: Prandtl number, Gr: Grashof number. Dimensionless temperature is based on the temperature difference between the assumed constant fin surface temperature and that at $\alpha = \alpha_\infty$. Reference velocity U is defined such that $UL/\nu = \sqrt{Gr}, \nu$: kinematic viscosity of the fluid. Dimensionless vorticity ζ and stream function ψ are based on velocity U and length L, and time t is based on (L/U).

2.3 Boundary conditions

Dynamic boundary conditions are given without loss of generality through no slip flow conditions by

$$\psi(\alpha=0,\beta)=0 \;,\; \frac{\partial}{\partial\alpha}\psi(\alpha=0,\beta)=0 \;,$$

$$\psi(\alpha=\alpha_\infty,\beta)=\text{const} \;,\; \frac{\partial}{\partial\alpha}\psi(\alpha=\alpha_\infty,\beta)=0 \;.$$

The following isothermal boundary conditions are introduced:

$$T(\alpha = 0, \beta) = 1 \ ,$$

$$T(\alpha = \alpha_\infty, \beta) = 0 \ .$$

2.4 Auxiliary condition

Doubly-connectedness of the configuration is

$$\oint \frac{\partial p}{\partial \beta}(\alpha = 0, \beta) \ d\beta = 0 \ \text{ or } \ \oint \frac{\partial p}{\partial \beta}(\alpha = \alpha_\infty, \beta) \ d\beta = 0 \ ,$$

where p is a static pressure (scalar quantity) appearing in the original dynamic equation of motion for substantially incompressible Newtonian fluids, which leads to (under the no-slip flow and iso-thermal boundary conditions)

$$\oint_{\alpha=0} \frac{\partial \zeta}{\partial \alpha} \ d\beta = 0 \ \text{ or } \ \oint_{\alpha=\alpha_\infty} \frac{\partial \zeta}{\partial \alpha} \ d\beta = 0 \ . \tag{11}$$

2.5 Spectral decomposition of variables

The following applies:

$$\begin{bmatrix} \psi \\ \zeta \\ T \end{bmatrix} = \sum_{n=1}^{\infty} \begin{bmatrix} \psi_{sn}(\alpha, t) \\ \zeta_{sn}(\alpha, t) \\ T_{sn}(\alpha, t) \end{bmatrix} \sin n\beta + \sum_{n=0}^{\infty} \begin{bmatrix} \psi_{cn}(\alpha, t) \\ \zeta_{cn}(\alpha, t) \\ T_{cn}(\alpha, t) \end{bmatrix} \cos n\beta \ . \tag{12}$$

2.6 Numerical solution of the governing equations

Spectral finite difference scheme [2] applies, where numerical integration can be executed semi-implicitly by truncating up to suitable Fourier terms, and steady-state solution is obtained as time tends to sufficiently large.

3. Numerical results

Mean Nusselt number Nu is defined as

$$Nu = -\frac{1}{L_0} \oint_{\alpha=0} \frac{\partial T}{\partial \alpha} \ d\beta \ ,$$

where L_0 is a perimeter along $\alpha = 0$, and $L_0 = \oint |dz| = 2\pi\left(1 + \epsilon^2\right)$. In case of $Pr = 2, \alpha_\infty = \ln 2$, streamlines $(\delta\psi = 0.01)$ at $Gr = 100, \epsilon = 0.5$ and those $(\delta\psi = 0.0005)$ at $Gr = 1, \epsilon = 0.3$ are shown in Figs. 1 and 2 respectively. In case of $Pr = 2, \alpha_\infty = \ln 2$, isotherms $(\delta T = 0.1)$ at $Gr = 3000, \epsilon = 0.5$ is shown in Fig.3 .

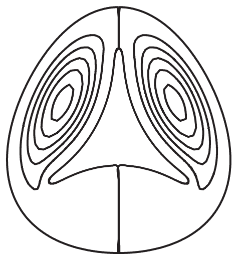

Fig. 1 Streamlines at $\epsilon = 0.5, Gr = 100$

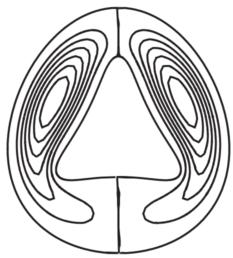

Fig. 2 Streamlines at $\epsilon = 0.3, Gr = 1$

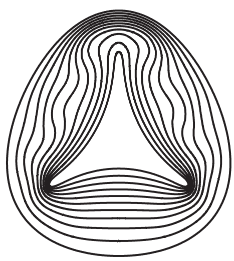

Fig. 3 Isotherms at $\epsilon = 0.5, Gr = 3000$

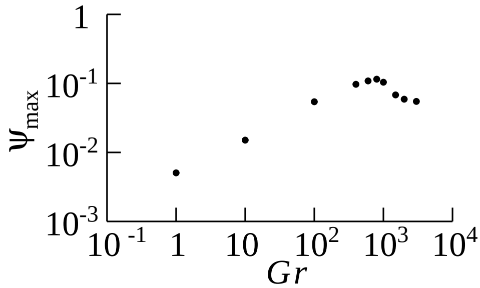

Fig. 4 $\psi_{\max}$-Characteristics

In case of $Pr = 2, \epsilon = 0.5$, maximum stream function $\psi_{\max}$ and Nu against Gr are shown in Figs. 4 and 5 respectively.

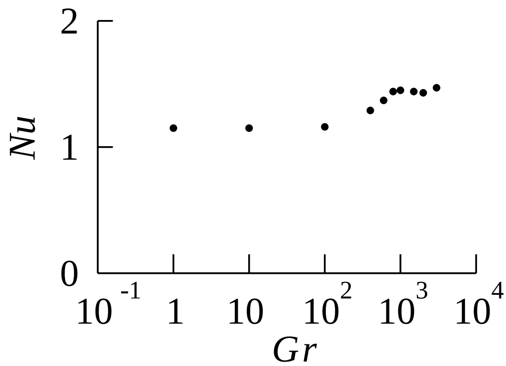

Fig. 5 Nu-Characteristics

As $\sqrt{Gr} \to 0$, for small non-negative ϵ under steady-state condition

$$T = 1 + T_0\,\alpha + \sum_{n=1}^{\infty} \epsilon^n T_n\,(\alpha, \beta)\ ,$$

$$T_0 \equiv -1/\alpha_\infty\ ,$$

$$\begin{bmatrix} \psi \\ \zeta \end{bmatrix} = \sum_{n=0}^{\infty} \epsilon^n \begin{bmatrix} \psi_n(\alpha,\beta) \\ \zeta_n(\alpha,\beta) \end{bmatrix} ,$$

$$\begin{aligned} \psi_0 &= \frac{1}{2} T_0 \sqrt{Gr} \left\{ \left(-\frac{r^3}{8} \ln r + \frac{3}{32} r^3 \right) \right. \\ &\quad \left. + Ar + \frac{B}{r} + Cr^3 + D \left(\frac{r}{2} \ln r - \frac{r}{4} \right) \right\} \sin\beta , \end{aligned}$$

where $r \equiv \mathrm{e}^{\alpha}$ and A, B, C, D are constants to be determined so that $\psi_0(\alpha = 0) = \psi_0(\alpha = \alpha_\infty) = (\partial/\partial\alpha)\psi_0(\alpha = 0) = (\partial/\partial\alpha)\psi_0(\alpha = \alpha_\infty) = 0$.

$$\zeta_0 = \frac{1}{2} T_0 \sqrt{Gr} \left(r \ln r + 8\, C\, r + \frac{D}{r} \right) \sin\beta ,$$

$$\psi_1 = \frac{1}{2} T_0 \sqrt{Gr} \left\{ g(\alpha) \sin 2\beta + h(\alpha) \sin 4\beta \right\} ,$$

$$g(\alpha) = \frac{\alpha + 16\, C}{4} + \frac{D}{2}\, \alpha\, \mathrm{e}^{-2\alpha} + a_1\, \mathrm{e}^{2\alpha} + b_1\, \mathrm{e}^{-2\alpha} + c_1\, \mathrm{e}^{4\alpha} + d_1 ,$$

$$h(\alpha) = -\frac{\alpha + 8\, C}{8} - \frac{D}{6}\, \mathrm{e}^{-2\alpha} + a_2\, \mathrm{e}^{4\alpha} + b_2\, \mathrm{e}^{-4\alpha} + c_2\, \mathrm{e}^{6\alpha} + d_2\, \mathrm{e}^{-2\alpha}$$

with $g(0) = g'(0) = g(\alpha_\infty) = g'(\alpha_\infty) = 0, h(0) = h'(0) = h(\alpha_\infty) = h'(\alpha_\infty) = 0$.

4. Discussion

The mean phase-change-surface-velocity v at $\alpha = \alpha_\infty$ is approximately estimated

(13) $$\frac{v}{U} \approx \frac{Ste}{Pr} \frac{\rho_l}{\rho_s} \frac{1}{\sqrt{Gr}} ,$$

where ρ_l: density of liquid state, ρ_s: density of solid state. For phase change material under thermal energy storage system such as n-Heptadecane, Ste is order of 0.1 and ρ_l/ρ_s is of unity, thus $v/U \ll 1$ if $Gr \gg 1$.

5. Conclusions

Natural convection in a two-dimensional enclosure of a three-fins type is successfully analyzed, using a spectral finite difference scheme, and characteristics are estimated.

References

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