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APPLICATIONS OF EQUATIONS OF MATHEMATICAL PHYSICS IN STUDYING TSUNAMI WAVES

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Abstract. In this paper we present equations of mathematical physics which have applications in studying tsunami waves. First we investigate an interesting system of non-linear PDE - the viscoelastic generalization of the Burger's equation. In the above mentioned system we are looking for travelling wave solutions and we are studying their profiles. Then we derive travelling wave solutions of the viscoelastic Burgers' equation. Cellular Nonlinear Networks (CNN) model of this equation is constructed. Using its computer realization new wave profiles of the travelling wave solutions are obtained.

1. Introduction

Most tsunami are caused by vertical movement along a break in the earth's crust. Other causes can include volcanic collapse, subsidence, as well as landslides. Contrary to popular imagination, a tsunami need be neither large nor destructive - classification is based on origin of the wave or wave period rather than on size.The thrust of a mathematical approach is to examine how a wave, once initiated, moves, evolves and eventually becomes such a destructive force of nature. We aim to describe how an initial disturbance gives rise to a tsunami wave.

An interesting phenomena in water channels is the appearance of waves with length much greater than the depth of the water. Korteweg and de Vries started

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the mathematical theory of this phenomenon and derived a model describing unidirectional propagation of waves of the free surface of a shallow layer of water. This is the well known KdV equation:

$$\begin{cases} u_t - 6uu_x + u_{xxx} = 0, & t > 0, \quad x \in \mathbf{R}, \\ u(0,x) = u_0(x), & \qquad\quad\; x \in \mathbf{R} \end{cases}$$

where u describes the free surface of the water; for a presentation of the physical derivation of the equation. The beautiful structure behind the KdV equation initiated a lot of mathematical investigations.

Recently, Camassa and Holm proposed a new model for the same phenomenon:

$$\begin{cases} u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & t > 0, \quad x \in \mathbf{R}, \\ u(0,x) = u_0(x), & \qquad\quad\; x \in \mathbf{R}. \end{cases}$$

The variable $u(t,x)$ in the Camassa-Holm (CH) equation represents the fluid velocity at time t in the x direction in appropriate nondimensional units (or, equivalently, the height of the water's free surface above a flat bottom). Unlike KdV, which is derived by asymptotic expansions in the equation of motion, CH is obtained by using an asymptotic expansions directly in the Hamiltonian for Euler's equations in the shallow water regime. The novelty of the Camassa and Holm's work was the physical derivation of CH equation and the discovery that the equation has solitary waves (solitons) that retain their individuality under interaction and eventually emerge with their original shapes and speeds.

As an alternative model to KdV, Benjamin, Bona and Mahoney (BBM) [12] proposed the so-called BBM-equation:

$$u_t + u_x + uu_x - u_{xxt} = 0, t > 0, x \in \mathbf{R}.$$

Numerical work of Bona, Pritchard and Scott shows that the solitary waves of the BBM-equation are not solitons.

Camassa-Holm considered [3,4] a third order nonlinear PDE of two variables modeling the propagation of unidirectional irrotational shallow water waves over a flat bad, as well as water waves moving over an underlying shear flow. As it is well known the motion of inviscid fluid with a constant density is described by the Euler's equations (system of nonlinear PDE). In the special case of the motion of a shallow water over a flat bottom the corresponding system was simplified by Green and Naghdi [12] and related to an appropriate two component first order Camassa-Holm system.

Let us denote by a_s the order of the free surface amplitude, by a_b the order of bottom topography variation, by h the characteristic water depth, by l_x the characteristic horizontal scale in the longitudinal direction, by l_y the characteristic horizontal scale in transverse direction. Then four parameters can be introduced: nonlinearity $\varepsilon = \frac{a_s}{h}$; shallowness $\mu = \frac{h^2}{l_x}$; topography $\beta = \frac{a_b}{h}$ and transversality $\gamma = \frac{l_x}{l_y}$. The assumptions on the size of ε, μ, β and γ correspond to asymptotic regimes, to which one can associate one or several asymptotic models. According to these parameters we have the following models:

- **Large surface and topography variations** $\varepsilon = O(1)$, $\beta = O(1)$. The Green-Naghdi equation (also called Serre [12] or fully nonlinear Boussinesq equations [12]) are the most general (but most complicated) of the models.

- **Small surface and large topography variations** $\varepsilon = O(\mu)$, $\beta = O(1)$. To this regime corresponds the Boussinesq-Peregrine model [12], which requires a small amplitude assumption (namely, $\varepsilon = O(\mu)$.

- **Medium surface and topography variations** $\varepsilon = O(\sqrt{\mu})$, $\beta = O(\sqrt{\mu})$. In this regime, the Green-Naghdi equations can be simplified into the medium amplitude Green-Naghdi equations.

- **Large surface and small topography variations** $\varepsilon = O(1)$, $\beta = O(\mu)$. Somehow symmetric to the Boussinesq-Peregrine model, the Green-Naghdi equations with almost flat bottom allow for large amplitude waves, but over small amplitude topography.

- **Small surface and topography variations** $\varepsilon = O(\mu)$, $\beta = O(\mu)$. This is the well-known long waves regime for which the Boussinesq system [12] can be derived.

- **Medium amplitude wave in 1d** $\varepsilon = O(\sqrt{\mu})$, $d = 1$. For well-prepared initial data the Camassa-Holm type equations can be derived.

-**Small amplitude waves in 1d** $\varepsilon = O(\mu)$, $d = 1$. When $\varepsilon = \mu$ (KdV regime) then the KdV/ BBM equations [12] can be derived and furnish an approximation of precision $O(\mu\sqrt{t})$ in general and $O(\mu)$ under a decay assumption on the initial data.

- **Small amplitude, weakly transverse 2d waves** $\varepsilon = \mu$, $d = 2$ and $\gamma = \sqrt{\varepsilon}$. The KP/KP-BBM [12] equations can be derived.

2. Travelling waves for viscoelastic generalization of the Burger's equation

In the paper [2] travelling waves and shocks in viscoelastic generalization of Burger's equation are studied. The wave profiles are given by using the numerical approach and by giving the corresponding computer visualizations. We propose in this section a purely mathematical approach to the same problem and give a qualitative picture of the behavior of these waves. To do this we use elementary facts from the theory of ODE.

In [2] the following generalization of the Burger's equation is investigated:

$$\left\{\begin{array}{l} u_t + uu_x = \sigma_x, \alpha, \beta = const > 0 \\ \sigma_t + u\sigma_x - \sigma u_x = \alpha u_x - \beta\sigma. \end{array}\right. \tag{1}$$

The function σ stands for the stress, u stands for velocity and (1) describes how the addition of viscoelasticity affects travelling wave solutions of Burger's equation. If there is no relaxation of stress then one takes $\beta = 0$.

Put now $u(x,t) = U(\xi)$, $\xi = x - ct$, $c = const$, $\sigma(x,t) = S(\xi)$ and substitute them in (1). Thus,

$$\left\{\begin{array}{l} -cU' + UU' = S' \\ -cS' + US' - SU' = \alpha U' - \beta S. \end{array}\right. \tag{2}$$

Integrating the first equation of (2) we get

$$S = \frac{1}{2}U^2 - cU + A, A = const$$

and therefore the second equation implies

$$c^2U' - cUU' + U(U'U - cU') - (\frac{1}{2}U^2 - cU + A)U' =$$

$$= \alpha U' - \beta(\frac{1}{2}U^2 - cU + A),$$

i.e.

$$U'[U(\frac{U}{2} - c) + c^2 - A - \alpha] = -\beta[U(\frac{U}{2} - c) + A]. \tag{3}$$

Let us assume that $U(-\infty) = u_l$, $S(-\infty) = 0$; $U(\infty) = u_r$, $S(\infty) = 0$, $u_r \neq u_l$. Then $S(\pm\infty) = 0$ implies that $0 = \frac{1}{2}u_l^2 - cu_l + A$, $0 = \frac{1}{2}u_r^2 - cu_r + A$ $\Rightarrow c = \frac{u_r+u_l}{2}$, $A = \frac{1}{2}u_l u_r$ and (3) takes the form

$$U'[(U - u_r)(U - u_l) + 2((\frac{u_r - u_l}{2})^2 - \alpha)] = -\beta(U - u_r)(U - u_l). \tag{4}$$

To simplify the things we denote $k = 2(\frac{(u_r-u_l)^2}{4} - \alpha)$ and consider several different cases having in mind that if $u_r < u_l \Rightarrow u_r < U < u_l \Rightarrow f(U) = (U - u_r)(U - u_l) < 0$, $f(u_r) = f(u_l) = 0$, $f(U) \in [-\frac{(u_l-u_r)^2}{4}, 0]$.

Moreover, $min_{U\in[u_r,u_l]} f(U) = f(\frac{u_r+u_l}{2}) = -\frac{(u_r-u_l)^2}{4}$. Thus,

1). $k < 0 \Rightarrow f(U) + k < 0$, $\forall U \in [u_r, u_l]$ (i.e. $\alpha > \frac{(u_r-u_l)^2}{4}$).

2). $k > 0$. Then $f(U) + k \in [-\frac{(u_l-u_r)^2}{4} + k, k]$, $\forall U \in [u_r, u_l]$ and we take $k > \frac{(u_l-u_r)^2}{4} \iff \alpha < \frac{1}{2}(\frac{u_r-u_l}{2})^2$.

3). $0 < k < \frac{(u_l-u_r)^2}{4} \iff \frac{(u_l-u_r)^2}{8} < \alpha < \frac{(u_l-u_r)^2}{4}$.

4). $k = 0 \Rightarrow (U' + \beta)(U - u_r)(U - u_l) = 0$; $k = \frac{(u_l-u_r)^2}{4}$.

In cases 1), 2) we have classical monotone solutions of (4), while in 3) the picture is rather interesting as then shock waves of the system (1) equipped with appropriate initial data can appear [2]. The corresponding solutions U of (4) are multivalued and therefore, no classical travelling wave solutions exist.

In this section we shall consider the case 3). Geometrically, the straight line $Z = -k$ is crossing twice the parabola $Z = f(U) = (U - u_r)(U - u_l)$ at the points $(u_{1,2}, -k)$, $u_1 \neq u_2$.

Certainly, we have: $f(u_1) + k = 0$, $f(u_2) + k = 0$, $u_r < u_1 < \frac{u_r+u_l}{2} < u_2 < u_l$,
$f(U) + k > 0$ for $U \in [u_r, u_1)$,
$f(U) + k > 0$ for $U \in (u_2, u_l]$ and
$U \in (u_1, u_2) \Rightarrow f(U) + k < 0$.

Moreover, if $k = \frac{(u_l-u_r)^2}{4}$ then $Z = -k$ is tangential to the parabola $Z = f(U)$ at its vertex. Then $\alpha = \frac{(u_r-u_l)^2}{8}$, $u_1 = u_2 = \frac{u_r+u_l}{2}$.

In the case 1) $U' < 0$ for $U \in (u_r, u_l)$ and therefore U is strictly monotonically decreasing function such that $U(-\infty) = u_l$, $U(+\infty) = u_r \Rightarrow u_l > u_r$. This is so called kink solution.

On the other hand in case 2) $U' > 0$ and therefore U is strictly monotonically increasing for each ξ, i.e. $U(-\infty) = u_l$, $U(\infty) = u_r \Rightarrow u_l < u_r$. $U = u_r$, $U = u_l$ are horizontal asymptotes of the solution. Certainly, such situation is impossible in our case.

In case 3) there exists a multivalued (three valued) continuous solution of (4) which is smooth up to two points of its graph where it possesses vertical tangents. This solution will be constructed in three steps. In the first step let $u_r < U_0 < u_1$ and $\xi_0 \in \mathbf{R}$. Then for $U \in (u_r, u_1) \Rightarrow U' > 0$. The right hand side of (4) is C^1 for $U \in [u_r, u_1)$ and consequently

$$(5) \qquad \xi - \xi_0 = -\int_{U_0}^{U} \frac{k + (\lambda - u_r)(\lambda - u_l)d\lambda}{(\lambda - u_r)(\lambda - u_l)\beta} \equiv F(U) \in C^1(u_r, u_1].$$

Evidently, $F'(U) > 0$ for $U \in [u_r, u_1)$, $F(U_0) = 0$, $lim_{U \to u_r} F(U) = -\infty$ as the integral is divergent at $U = u_r$, $lim_{U \to u_1} F(U) = F(u_1) > 0$, $F'(u_1) = 0$. Thus, $F : (u_r, u_1) \to (-\infty, F(u_1))$ is diffeormorphism, and homeomorphism $F : (u_r, u_1] \to (-\infty, F(u_1)]$, $\xi - \xi_0 = F(U) \Rightarrow \xi = \xi_0 + F(U) \in (-\infty, F(u_1) + \xi_0)$; if we put $\bar{\xi} = F(u_1) + \xi_0 > \xi_0$ then $U = F^{-1}(\xi - \xi_0)$, $\xi \in (-\infty, \bar{\xi})$, $U(\xi_0) = F^{-1}(0) = U_0$, $lim_{\xi \to -\infty} U(\xi) = u_r$, $lim_{\xi \to \bar{\xi}} U(\xi) = U(\bar{\xi}) = u_1$, $U'(\bar{\xi}) = +\infty$.

In the second step we construct a solution with $u_1 < U < u_2 \Rightarrow U' < 0$ and initial data $(\bar{\xi}, u_1)$. Then (5) takes the form

$$\xi - \bar{\xi} = -\int_{u_1}^{U} \frac{k + (\lambda - u_r)(\lambda - u_l)d\lambda}{(\lambda - u_r)(\lambda - u_l)\beta} = F_1(U) \in C^1[u_1, u_2] \tag{6}$$

as the underintegral function has not singularities. Thus, $\xi - \bar{\xi} = F_1(U) < 0$, $F_1(u_1) = 0$, $U(\bar{\xi}) = u_1$, $F_1'(u_1) = 0$, $F_1'(u_2) = 0$, i.e. if we put $\bar{\bar{\xi}} = \bar{\xi} + F_1(u_2)$ the mapping

$F_1 : [u_1, u_2] \to [F_1(u_2), F_1(u_1)]$ is a homeomorphism,

while $F_1 : (u_1, u_2) \to (F_1(u_2), F_1(u_1))$ is diffeomorphism

and $U'(\bar{\xi}) = -\infty$, $U'(\bar{\bar{\xi}}) = -\infty$, $\bar{\bar{\xi}} < \bar{\xi}$, $U(\bar{\bar{\xi}}) = u_2$, $U = F_1^{-1}(\xi - \bar{\xi})$.

The third step is standard as we construct a solution U, $U' > 0$ passing through $(\bar{\bar{\xi}}, u_2)$. Evidently, $U'(\bar{\bar{\xi}}) = +\infty$, $U(\xi) \in C^1(\bar{\bar{\xi}}, +\infty)$, $lim_{\xi \to \infty} U(\xi) = u_l$. (see Fig.1)

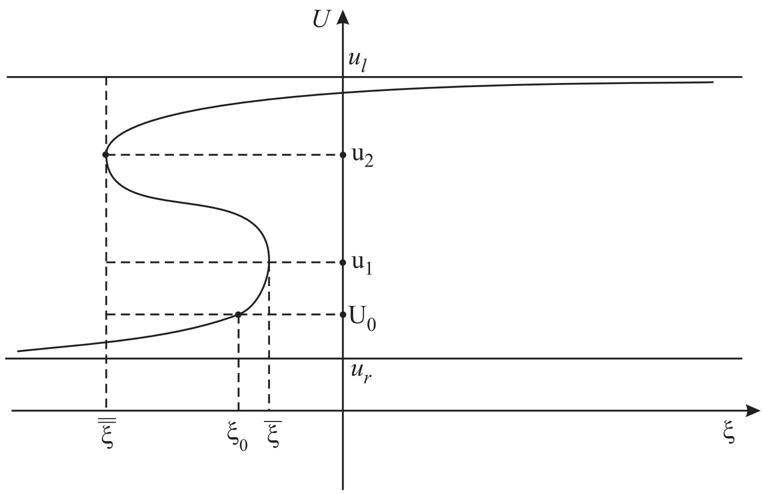


Figure 1 - Geometrically the solution in case 3).

It is interesting to point out that the conditions at $\pm\infty$ are not satisfied by U as $U(-\infty) = u_r$, $U(\infty) = u_l$. Moreover, U is triple valued for $\xi \in (\bar{\xi}, \bar{\bar{\xi}})$.

In case 4). $k = \frac{(u_l - u_r)^2}{4} \Rightarrow U' > 0$. The solution is again single valued and has a vertical tangent at only one point ($u_1 = u_2 = \frac{u_r + u_l}{2}$, $\bar{\xi} = \bar{\bar{\xi}}$) (then $\alpha = \frac{(u_r - u_l)^2}{8}$).

Again $U(-\infty) = u_r$, $U(+\infty) = u_l$. If $k = 0$, i.e. $\alpha = \frac{(u_r - u_l)^2}{4}$ the solution U is piecewise linear function (see Fig.2).

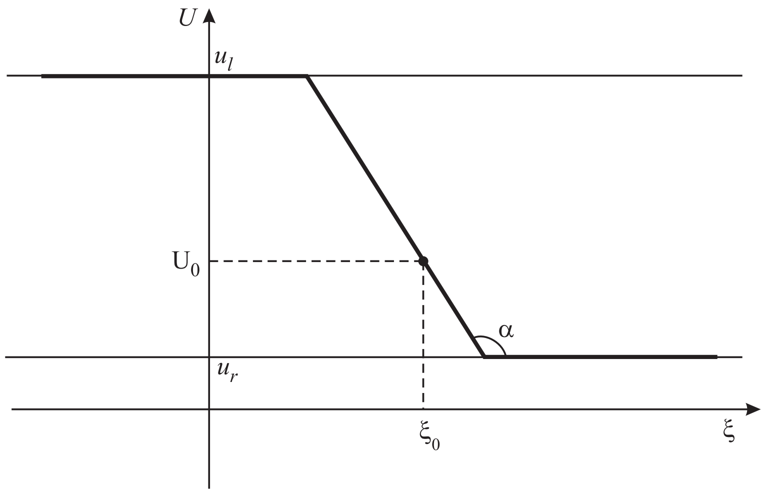


Figure 2 - Geometrically the solution in case 4).

3. Viscoelastic Burgers' equation

The simplest model that couples the nonlinear convective behavior of fluids with the dissipative viscous behavior is well-known Burgers' equation:

$$u_t + uu_x = \varepsilon u_{xx}. \tag{7}$$

It is introduced by Burgers [1] as a model for turbulence. Equation (7) and its inviscid counterpart

$$u_t + uu_x = 0, \tag{8}$$

are essential for their role in modelling a wide array of physical systems such as traffic flow, shallow water waves, and gas dynamics [1]. The equations also provide fundamental pedagogical examples for many important topics in nonlinear PDEs such as travelling waves, shock formation, similarity solutions, singular perturbation, and numerical methods for parabolic and hyperbolic equations [7].

In this section we consider how the addition of viscoelasticity affects travelling wave solutions of Burgers' equation (7). The equations we consider are:

$$u_t + uu_x = v_x, \tag{9}$$

$$v_t + uv_x - vu_x = au_x - bv. \tag{10}$$

The constitutive law (10) resembles a one-dimensional version of the upper convected Maxwell model [10,11,,12]. The relaxation time is $\tau = b^{-1}$, and $a = m\tau^{-1}$ could be interpreted as the elastic modulus of the material if there were no relaxation of stress ($b = 0$). In the other limit of instantaneous relaxation of stress ($\tau \to 0$), (10) reduces to $v = mu_x$, and the system (9)-(10) is equivalent to Burgers' equation (7) with fluid viscosity $m = \varepsilon$.

One of the simplest constitutive laws for viscoelastic materials is the Maxwell model. Consider a linear spring and dashpot in series, with spring constant k and damping coefficient m. The stress, v, in the element is

$$\tau\dot{v} + v = m\dot{\varepsilon}, \tag{11}$$

where ε is the strain in the element, and $\tau = k/m$ is the relaxation time. The linear Maxwell model for a continuum is

$$\tau v_t + v = 2mD, \tag{12}$$

where $2mD$ is the viscous stress. However, this is not a valid constitutive law because it is not frame invariant [11]. That is, the stress depends on the reference frame. Frame invariance is achieved by choosing an appropriate time derivative, akin to the material derivative for the velocity field. One frame invariant time derivative is the upper convected derivative, defined by

$$\bar{S} = S_t + u.\nabla S - \nabla uS - S\nabla u^T. \tag{13}$$

Replacing the partial time derivative in (12) with the upper convected derivative gives the upper convected Maxwell (UCM) equation

$$\tau\bar{v} + v = 2mD. \tag{14}$$

The ij component in (13) satisfies

$$\tau(\frac{\partial v_{ij}}{\partial t} + u_k \frac{\partial v_{ij}}{\partial x_k} - \frac{\partial u_i}{\partial x_k} v_{kj} - v_{ik} \frac{\partial u_j}{\partial x_k}) + \\ + v_{ij} = m(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), \tag{15}$$

where summation is over the repeated index k. Although there are many other frame invariant derivatives, in this paper we consider a one-dimensional reduction, in which case they yield identical reductions.

A one-dimensional version of the UCM equation is

$$\tau(v_t + uv_x - vu_x) + v = mu_x. \tag{16}$$

Equation (16) is equivalent to (10). This is seen by dividing through by the relaxation time τ to get

$$v_t + uv_x - vu_x = au_x - bv, \tag{17}$$

where

$$a = m\tau^{-1}, \tag{18}$$

$$b = \tau^{-1}. \tag{19}$$

The parameter a could be interpreted as the elastic modulus of the material if there were no relaxation of stress ($b = 0$). It is somewhat arbitrary whether the constitutive law is expressed in terms of the relaxation time (τ) and viscosity (m) or elastic modulus (a) and decay rate (b).

Remark.

We find that the solutions develop into travelling waves, however, with jump discontinuities in the wave profile. When solving equations with discontinuities care must be taken in order to capture the correct solution. There are several questions that arise: In the case of the double-shock solution, what determines the shock profile? What determines the shape of the solution between two shocks? Why is that we see a double-shock solution? To answer all these questions we shall apply RTD-based Cellular Nonlinear Networks approach.

4. RTD-based Cellular Neural Networks Realization

Many methods used in image processing and pattern recognition can be easily implemented in RTD-based Cellular Neural Networks (CNN), however, the mathematical analysis of the phenomena as wave propagation, spatial chaos properties, and its dynamical behavior are still not fully studied. In this paper we shall provide some new wave profiles in viscoelastic Burgers' CNN model. This investigation is motivated by the paper of Hsu and Yang [8], in which the resonant tunneling diode (RTD), a class of quantum effect devices, is presented for studying the wave propagation in CNNs. RTD-based CNN is an excellent candidate for both analog and digital nanoelectronics applications because of its structural simplicity, relative easy of fabrication, inherent high speed and design flexibility.

Cellular Neural Networks (CNNs)[5] are complex nonlinear dynamical systems, and therefore one can expect interesting phenomena like bifurcations and chaos to occur in such nets. It was shown that as the cell self-feedback coefficients are changed to a critical value, a CNN with opposite-sign template may change from stable to unstable. Namely speaking, this phenomenon arises as the loss of stability and the birth of a limit cycles.

We shall apply in this study one-dimensional original RTD-based CNN without input and threshold terms [5]. The dynamics of our RTD-based CNN model for the system of viscous Burger's equation (9), (10) will be the following:

$$
\begin{aligned}
\frac{du_j}{dt} &+ u_j A_1 * u_j = A_1 * v_j \\
\frac{dv_j}{dt} &+ u_j A_1 * v_j - v_j A_1 * u_j = \\
&= a A_1 * u_j - b v_j,
\end{aligned}
\tag{20}
$$

$1 \le j \le M$, where $A_1 = (1, -2, 1)$, is one-dimensional discretized Laplacian CNN template, $*$ is the convolution CNN operator [5].

We study here the structure of the travelling wave solutions of the RTD-based CNN model (20) of (9), (10) having the form:

$$
\begin{aligned}
u_j &= \Phi(j - ct), \\
v_j &= \Psi(j - ct)
\end{aligned}
\tag{21}
$$

Φ, Ψ being continuous functions. Let us substitute (21) in (20). Therefore we consider solutions $\Phi(s; c)$, $\Psi(s; c)$, $s = j - ct$ of:

$$
\begin{aligned}
-c\Phi^{'}(s;c) + G_1(\Phi(s;c)) &= 0,\\
-c\Psi^{'}(s;c) + G_2(\Psi(s;c)) &= 0
\end{aligned}
\tag{22}
$$

where $G_1(\Phi), G_2(\Psi) \in \mathbf{R}^1$. We consider travelling waves that correspond to heteroclinic connections between two equilibrium points with given velocity values at infinity. The equilibrium points of the system (22) correspond to all states with $\Psi = 0$, and thus we assume the following asymptotic boundary conditions:

$$
\begin{aligned}
lim_{s\to-\infty}\Phi(s;c) &= u_l,\\
lim_{s\to\infty}\Phi(s;c) &= u_r,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
lim_{s\to-\infty}\Psi(s;c) &= 0,\\
lim_{s\to\infty}\Psi(s;c) &= 0,
\end{aligned}
\tag{24}
$$

for some $c > 0$.

Below we propose the following result.

Theorem 1. *Suppose that $u_j(t) = \Phi(j - ct)$ and $v_j = \Psi(j - ct)$ are travelling wave solutions of the CNN model (20) of the system of viscoelastic Burgers' equations (9), (10). Then there exists $c = \frac{u_l+u_r}{2} > 0$ such that*

(i) for $a > \frac{d^2}{4}$, $d = u_l - u_r$ smooth travelling wave solution of (20) exists;

(ii) for $\frac{d^2}{8} < \alpha < \frac{d^2}{4}$ piecewise smooth travelling wave solution with two jump discontinuities exists;

(iii) for $a < \frac{d^2}{8}$ single shock wave solution exists.

Proof:

Without loss of generality we shall fix $b = 1$. The equilibrium points of the system (22) with the boundary conditions (23) and (24) are $E_1 = (u_l, 0)$ and $E_2 = (u_r, 0)$. We are looking for travelling wave solution of the RTD-base CNN model (20). It is a heteroclinic orbit connecting the two equilibrium points E_1 and E_2 (see Fig.3).

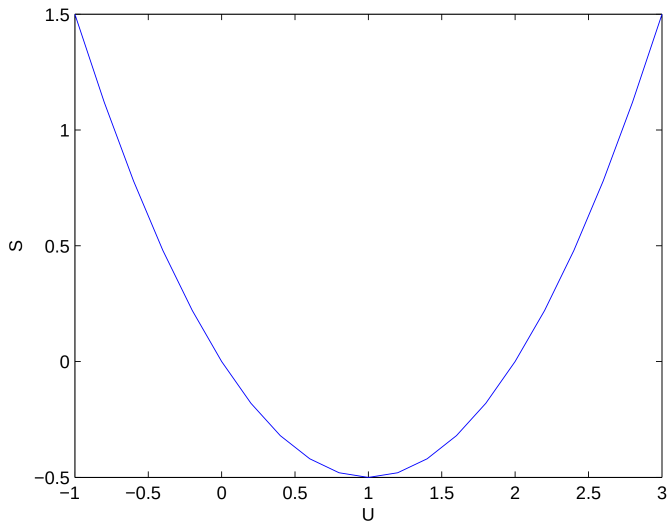


Figure 3- Heteroclinic orbit corresponding to travelling wave solution of (20).

After integrating system (22) and under the conditions (23), (24) we obtain:

$$\Psi(s;c) = \frac{\Phi(s;c)^2}{2} - c\Phi(s;c) + R, \tag{25}$$

where $c = \frac{u_l+u_r}{2}$ and $R = \frac{u_l u_r}{2}$. Substituting (19) in (16) we obtain for the wave profile $\Phi(s;c)$:

$$\Phi^{'}(s;c) = \frac{-b(\Phi(s;c)-u_l)(\Phi(s;c)-u_r)}{(\Phi(s;c)-u_l)(\Phi(s;c)-u_r)+2((\frac{u_l-u_r}{2})^2-a)}. \tag{26}$$

From (26) it is clear that we have the following possible cases: Travelling wave solution of (20) exists if and only if

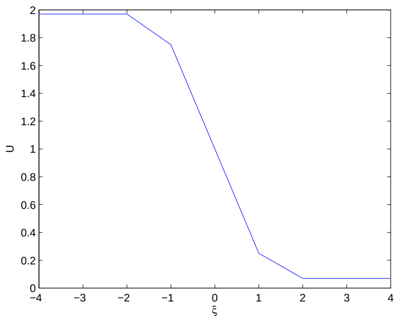
$$u_l > u_r \quad \text{and} \quad a > (\frac{u_l-u_r}{2})^2;$$

$$u_l < u_r \quad \text{and} \quad 2a < (\frac{u_l-u_r}{2})^2;$$

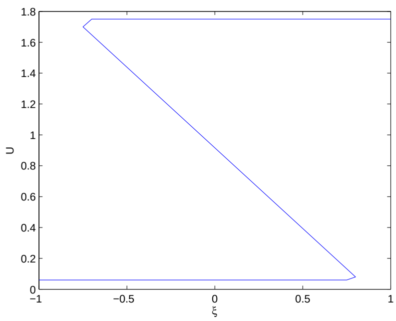
Equivalently, no travelling wave solutions exist if

$$(\frac{u_l-u_r}{8})^2 \leq a \leq (\frac{u_l-u_r}{4})^2.$$

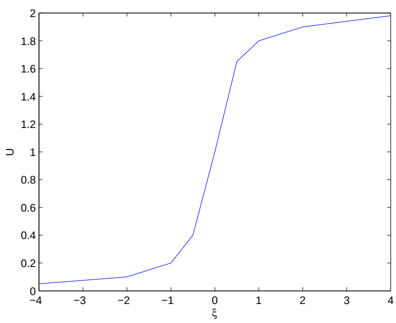
We shall give the simulation of the RTD-based CNN model (20) on Figure 4.



a). $u_l = 2,\ u_r = 0,\ b = 1,\ a = 1.2.$



b). $u_l = 2,\ u_r = 0,\ b = 1,\ a = 0.9.$



c). $u_l = 2,\ u_r = 0,\ b = 1,\ a = 0.25.$

Figure 4 - The wave profile of the RTD-based CNN model (20) for different values of the parameter sets.

Remark.

For the parameter values given in Figure 4, a travelling wave exists when $a > 1$ (a). As a approaches 1, the wave profile approaches the piecewise linear function. As a decreased further, the curve becomes multivalued and the asymptotic values are no longer satisfied (b).As a decreases even further, the solution returns to being single-valued but no longer yields a travelling wave solution with the given asymptotic limits (c). This transition occurs at $a = \frac{1}{2}(\frac{u_l - u_r}{2})^2$.

5. Conclusions

In this paper two models of tsunami waves are considered. Physical interpretation of different kinds of waves is presented.

First part of the paper is devoted to the travelling wave solutions of different type of equations of Mathematical Physics. We study viscoelastic generalization of the Burger's equation. In this systems we are looking for travelling wave solutions and we are studying their profiles. To do this we use several results from the classical Analysis of ODE that enable us to give the geometrical picture.

In this paper we study the wave profiles of the travelling wave solutions of viscoelastic Burgers' equation. We apply the RTD-based Cellular Neural Networks in the one-dimensional integer lattice. A circuit implementation of the RTD-based CNN can be found in [6]. It is also pointed out that the bistable RTD-based CNN exhibits good performance for a number of interesting image processing applications because of its high-speed processing and high cell density. The study of travelling wave solutions of partial differential equations and lattice dynamical systems has drawn considerable attention in the past decades.

Recall that $a = m/\tau$, where m and τ are the viscosity and relaxation time, respectively. From Theorem 1 the following conclusions can be made. For a fixed relaxation time τ, each of the three types of wave solutions is possible, depending on the size of the viscosity. For large enough viscosity $m > \frac{d^2}{4}\lambda$ the wave profile of the solutions is smooth. As $\frac{d^2}{4}\tau < m < \frac{d^2}{8}\tau$ is decreased the wave solutions becomes double-shock and then when $m] < \frac{d^2}{4}$ change to single shock wave.

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