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SUBCRITICAL MARKOV BRANCHING PROCESSES WITH NON-HOMOGENEOUS POISSON IMMIGRATION*

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The paper proposes an extension of Sevastyanov (1957) model based on a Markov branching process allowing an immigration component in the moments of a homogeneous Poisson process. Now Markov branching processes are also considered but assuming a time-nonhomogeneous Poisson immigration. These processes could be interpreted as models in cell proliferation kinetics with stem cell immigration. Limit theorems are proved in the subcritical case and new effects are obtained due to the non-homogeneity.

1. Introduction

The first model of branching process with immigration was proposed by Sevastyanov (1957). He investigated a Markov branching process admitted an immigration component in the moments of a homogeneous Poisson process.

One of the goals of this paper concerns modeling of renewing cell population. The considered models are based on Markov branching processes allowing immigration in the moments of a time-inhomogeneous Poisson component. For a comprehensive review of branching processes and their biological applications, the reader is referred to Harris (1963), Sevastyanov (1971), Athreya and Ney (1972), Jagers (1975), Yakovlev and Yanev (1989), Kimmel and Axelrod (2002), Haccou et al. (2005) and Ahsanullah and Yanev (2008). Some problems of biological

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models using branching processes with nonhomogeneous Poisson immigration are considered in Yakovlev and Yanev (2006, 2007), and Hyrien and Yanev (2010, 2012).

Now we consider the asymptotic behaviour of subcritical Markov branching processes with non-homogeneous Poisson immigration. Note that a supercritical case was investigated in Hyrien et al. (2013). The critical case was studied in Mitov and Yanev (2013) for more general situation of Sevastyanov branching processes allowing non-homogeneous Poisson immigration.

The paper is organized as follows. In Section 2 the biological background and motivation are proposed which gives the general ideas for constructing the corresponding models in Section 3. The basic equations for the p.g.f. and the moments are considered also in Section 3. The asymptotic behaviour for the means, variances and covariances of the subcritical processes with immigration is investigated in Section 4 and limit theorems are also proved. Two general cases for the immigration rate are considered, $r(t) = re^{\rho t}$, $\rho \in \mathbb{R}$, and $r(t) = rt^{\theta}$, $\theta \in \mathbb{R}$.

Note that $\rho = 0$ and $\theta = 0$ correspond to the case of a homogeneous Poisson immigration where Sevastyanov (1957) obtained a stationary limiting distribution in the subcritical case. Now new effects are obtained due to the non-homogeneity. Thus for $\rho < 0$ and $\theta < 0$ conditional limiting distributions are obtained (Theorem 1 and Theorem 4) under the condition of non-extinction. In Theorem 4 the obtained distribution is just the same as in the classical Bienaymé-Galton-Watson branching process but as a contrast the asymptotic behaviour of the probability of non-extinction is quite different ($\sim Ct^{\theta}$). In the cases $\rho > 0$ and $\theta > 0$ we proved LLN (Theorem 2 and Theorem 5) and also CLT (Theorem 3 and Theorem 6). Finally in the case $r(t) \to r > 0$ the classical Sevastyanov result is confirmed (Theorem 7).

2. Biological background and motivation

Continuous-time branching processes have been used to quantify the development of cell populations in cell kinetics studies. For example, when studying tissue development during embryogenesis, it is reasonable to set the initial number $N_0 = 0$ if the experiment begins before the first cell of the tissue has been generated. As time increases, cells will begin populating the tissue of interest once precursor cells have started differentiating. We refer to these cells as immigrants and describe their influx using a non-homogeneous Poisson process with arrival rate $r(t)$. Upon arrival, these immigrants are assumed to be of age zero. Upon completion of its life-span, every cell of the population either divides into two new cells, or it exits the population (due to cell differentiation or cell death). These events occur with probability p and $q = 1 - p$, respectively. The lifespan of any cell is described by a non-negative random variable τ with cumulative distribution function (c.d.f.) $G(x) = \mathbf{P}\{\eta \leq x\}$ that satisfies $G(0) = 0$. Cells are assumed to evolve independently of each other. The work presented in this paper was motivated by this example, and we investigate properties of a more general class of Markov branching processes with non-homogeneous Poisson immigration.

3. Models and Equations

We consider cell populations (*in vivo)* whose proliferation kinetics can be described as follows. The process begins with immigration of stem cells which appear at the moments of immigration as *progenitors* at zero age. Then every cell has a life-time c.d.f. $G(t) = P(\eta \leq t) = 1 - e^{-t/\mu}, t \geq 0$, and at the end of its mitotic cycle η produces an offspring ξ with a p.g.f. $h(s) = E\{s^{\xi}\}$, $|s| \leq 1$. We assume that all new born cells are at zero age and continue their evolutions independently and in the same way. Therefore the development of this population can be described in the framework of a Markov branching process with immigration.

The offspring moments

$$m = E\{\xi\} = \left.\frac{dh(s)}{ds}\right|_{s=1} \quad \text{and } m_2 = E\{\xi(\xi-1)\} = \left.\frac{d^2h(s)}{ds^2}\right|_{s=1}$$

play further an important role as well as the life-span mean $\mu = \int_0^\infty x dG(x)$, assuming that all these characteristics are finite.

The models with an offspring p.g.f. $h(s) = 1 - p + ps^2$, $m = 2p = m_2$, are very interesting from biological point of view. It means that at the end of the mitotic cycle every cell can die with probability $1 - p$ or it can divide in two cells with probability p. This example may be treated more carefully but now we will investigate the general case.

Let us first consider the process without immigration $Z(t)$ (which denotes the number of cells at the moment t) and introduce the corresponding p.g.f. $F(t; s) = E\{s^{Z(t)}|Z(0) = 1\}$. Under the assumptions, it is not difficult to realize that $\{Z(t), t \geq 0\}$ can be considered as Markov branching process well determined by the following nonlinear differential equation:

$$\frac{\partial}{\partial t} F(t; s) = f(F(t; s)), \quad F(0; s) = s, \tag{1}$$

where $f(s) = [h(s) - s]/\mu$ (see e.g. Harris, 1963).

Note that the Malthusian parameter α is determined as usually from the equation $m\int_0^\infty e^{-\alpha x}dG(x) = 1$ and in the Markov case $\alpha = f'(1) = [m-1]/\mu$. Introduce also $\beta = f''(1) = m_2/\mu$.

Further on we will consider only the subcritical case $\alpha < 0$.

For the moments one has (see e.g. Harris, 1963):

$$(2) \quad A(t) = \frac{\partial}{\partial s}F(t;s)|_{s=1} = E\{Z(t)|Z(0) = 1\} = e^{\alpha t},$$

$$B(t) = \frac{\partial^2}{\partial s^2}F(t;s)|_{s=1} = E\{Z(t)(Z(t)-1)|Z(0) = 1\} = \beta e^{\alpha t}(e^{\alpha t} - 1)/\alpha,$$

$$(3) \quad V(t) = Var\{Z(t)|Z(0) = 1\} = (\beta/\alpha - 1)e^{\alpha t}(e^{\alpha t} - 1).$$

Let us now describe the process with immigration. First we will assume that $0 = S_0 < S_1 < S_2 < S_3 < \cdots$ are the time-points of the immigration which form a **non-homogeneous Poisson process** $\Pi(t)$ with a rate $r(t)$, i.e. the cumulative rate is $R(t) = \int_0^t r(u)du$, $r(t) \geq 0$, and $\Pi(t) \in Po(R(t))$. Let $U_i = S_i - S_{i-1}$ be the inter-arrival times. Then $S_k = \sum_{i=1}^{k} U_i$, $k = 1, 2, ...$

We will assume also that at every point S_k there is an independent immigration component I_k of cells at zero age, where $\{ I_k\}$ are i.i.d. r.v's with a p.g.f. $g(s) = E\{s^{I_k}\} = \sum_{i=0}^{\infty} g_i s^i$, $|s| \leq 1$. Let $\gamma = E\{I_k\} = \left.\frac{dg(s)}{ds}\right|_{s=1}$ be the immigration mean and introduce the second factorial moment $\gamma_2 = \left.\frac{d^2g(s)}{ds^2}\right|_{s=1} = E\{I_k(I_k - 1)\}$.

Let now $Y(t)$ be the number of cells at the moment t in the process with immigration, where the cell evolution is determined by a (G, h) - Markov branching processes defined above. Then the considered process admits the following representation

$$(4) \qquad Y(t) = \sum_{k=1}^{\Pi(t)} Z^{I_k}(t - S_k) \text{ if } \Pi(t) > 0 \text{ and } Y(t) = 0 \text{ if } \Pi(t) = 0,$$

where $Z^{I_k}(t)$ are i.i.d. branching processes with a given evolution of the cells as $Z(t)$ but started with a random number of ancestors I_k. We assume that $Y(0) = 0$, but in fact, the process $Y(t)$ begins from the first non-zero immigrants.

Introduce the p.g.f. $\Psi(t;s) = E\{s^{Y(t)}|Y(0) = 0\}$. Using (4) Yakovlev and Yanev (2007, Theorem 1) obtained that

$$\Psi(t;s) = \exp\left\{-\int_0^t r(t-u)[1 - g(F(u;s))]du\right\}, \Psi(0,s) = 1, \tag{5}$$

where in our case the p.g.f. $F(u;s)$ satisfies the equation (1). One has to point out that $\{Y(t), t \geq 0,\}$ is a time non-homogeneous Markov process.

Remark that if $\{U_i\}$ are i.i.d. r.v. with c.d.f. $G_0(x) = P(U_i \leq x) = 1 - e^{-rx}$, $x \geq 0$, then $\Pi(t)$ reduces to an ordinary Poisson process with a cumulative rate $R(t) = rt$ and we obtain the first model with immigration proposed and investigated by Sevastyanov (1957).

Introduce the moments of the process with immigration

$$M(t) = E\{Y(t)|Y(0) = 0\} = \frac{\partial}{\partial s}\Psi(t;s)\Big|_{s=1},$$

$$M_2(t) = E\{Y(t)(Y(t)-1)|Y(0) = 0\} = \frac{\partial^2}{\partial s^2}\Psi(t;s)\Big|_{s=1},$$

$$W(t) = Var\{Y(t)|Y(0) = 0\} = M_2(t) + M(t)[1 - M(t)].$$

Then from (5) it is not difficult to obtain that

$$M(t) = \gamma\int_0^t r(t-u)A(u)du, \tag{6}$$

$$M_2(t) = \gamma\int_0^t r(t-u)B(u)du + \left[\gamma\int_0^t r(t-u)A(u)du\right]^2 + \gamma_2\int_0^t r(t-u)A^2(u)du,$$

$$W(t) = \int_0^t r(t-u)\left[\gamma V(u) + (\gamma+\gamma_2)A^2(u)\right]du. \tag{7}$$

To derive also equations for the covariances we have to consider first the joint p.g.f. $F(s_1, s_2; t, \tau) = E\{s_1^{Z(t)}s_2^{Z(t+\tau)}|Z(0) = 1\}, \tau \geq 0$.

Conditioning on the evolution of the initial cell and applying the law of the total probability one can obtain the equation:

$$F(s_1, s_2; t, \tau) = \int_0^t h(F(s_1, s_2; t-u, \tau))dG(u) \tag{8}$$

$$+s_1 \int_t^{t+\tau} h(F(1, s_2; t, \tau - v))dG(v) + s_1 s_2 (1 - G(t+\tau)),$$

with the initial condition $F(s_1, s_2; 0, 0) = s_1 s_2$ (see also Harris (1963)).

Let us now introduce the joint p.g.f. for the process with immigration $Y(t)$ defined by (4)

$$\Psi(s_1, s_2; t, \tau) = E\{s_1^{Y(t)} s_2^{Y(t+\tau)} | Y(0) = 0)\}, \tau \geq 0.$$

Similarly to (5) one can obtain that

$$\Psi(s_1, s_2; t, \tau) = \exp\left\{ - \int_0^t r(u)[1 - g(F(s_1, s_2; t-u, \tau))]du \right. \tag{9}$$

$$\left. - \int_t^{t+\tau} r(v)[1 - g(F(1, s_2; t, \tau - v))]dv \right\}.$$

For the proof one has to consider definition (4) and to follow the method developed in Theorem 1 (Yakovlev and Yanev(2007)) for (5).

Introduce the moments

$$A(t, \tau) = E\{Z(t)Z(t+\tau) \mid Z(0) = 1\} = \left. \frac{\partial^2}{\partial s_1 \partial s_2} F(s_1, s_2; t, \tau) \right|_{s_1 = s_2 = 1},$$

$$M(t, \tau) = E\{Y(t)Y(t+\tau) \mid Y(0) = 0\} = \left. \frac{\partial^2}{\partial s_1 \partial s_2} \Psi(s_1, s_2; t, \tau) \right|_{s_1 = s_2 = 1}.$$

Then from (8) and (9) the following equations hold

$$(10)\qquad A(t,\tau) = m\int_0^t A(t-u,\tau)dG(u) + m_2\int_0^t A(t-u)A(t+\tau-u)dG(u)$$

$$+m\int_t^{t+\tau} A(t+\tau-u)dG(u) + 1 - G(t+\tau),$$

$$M(t,\tau) = \int_0^t r(u)[\gamma A(t-u,\tau) + \gamma_2 A^2(t-u)]du$$

$$+\gamma^2\left[\int_0^t r(u)A(t-u)du\right]^2,$$

$$(11)\qquad C(t,\tau) = Cov\{Y(t), Y(t+\tau)\} = \frac{\partial^2}{\partial s_1 \partial s_2}\log\Psi(s_1,s_2;t,\tau)\Big|_{s_1=s_2=1}$$

$$= \int_0^t r(u)[\gamma A(t-u,\tau) + \gamma_2 A(t-u)A(t+\tau-u)]du$$

with initial conditions $A(0,\tau) = A(\tau)$ and $M(0,\tau) = 0 = C(0,\tau)$.

4. Limit theorems

Recall that we consider the subcritical case $\alpha < 0$. From (6) one has

$$M(t) = \gamma e^{\alpha t}\widehat{r}_t(\alpha),$$

where $\widehat{r}_t(\alpha) = \int_0^t e^{-\alpha u} r(u)du.$

If we assume first that

$$\lim_{t\to\infty}\widehat{r}_t(\alpha) = \widehat{r}(\alpha) < \infty.$$

then

$$(12)\qquad M(t) \sim \gamma\widehat{r}(\alpha)e^{\alpha t} \to 0, \quad t\to\infty.$$

Remark 1. The relation (12) is fulfilled if, for example, the intensity has the form $r(t) = O(e^{\rho t}), \rho < \alpha$.

Let us now consider more carefully the case

$$r(t) = re^{\rho t}, \quad r > 0.$$

Then from (6) and (2) one has

$$M(t) = \gamma rte^{\alpha t} \to 0, \text{ for } \rho = \alpha,$$

and

$$M(t) = \gamma r(e^{\rho t} - e^{\alpha t})/(\rho - \alpha), \text{ for } \rho \neq \alpha.$$

Therefore

$$M(t) \sim \gamma re^{\alpha t}/(\alpha - \rho) \to 0, \text{ if } \rho < \alpha,$$

and

$$M(t) \sim \gamma re^{\rho t}/(\rho - \alpha), \text{ if } \rho > \alpha. \tag{13}$$

Note that in the case $\rho > \alpha$ one has that $M(t) \to 0$ for $\rho < 0$, $M(t) \to \infty$ for $\rho > 0$ and $M(t) \to \gamma r/(-\alpha)$ for $\rho = 0$ (homogeneous Poisson immigration).

Let us now assume that for some $r > 0$

$$r(t) = rt^\theta, \ 0 < \theta < \infty, \text{ or } r(t) = r(t+1)^\theta, \ -\infty < \theta < 0. \tag{14}$$

Then it is not difficult to obtain from (6) and (14) that

$$M(t) \sim \gamma rt^\theta/(-\alpha), \quad t \to \infty. \tag{15}$$

Therefore $M(t) \to 0$ for $\theta < 0$ and $M(t) \to \infty$ for $\theta > 0$. Note that $\theta = 0$ implies the homogeneous Poisson case $r(t) \equiv r$ and $M(t) \to \gamma r/(-\alpha), t \to \infty$.

Remark 2. If $\lim_{t\to\infty} M(t) = 0$ then $Y(t) \to 0$ *in probability* when $t \to \infty$, and one can conjecture conditional limit theorems, i.e. to check when $\lim_{t\to\infty} \mathbf{P}\{Y(t) = k|Y(0) > 0\} = P_k^*$, $\sum_{k=1}^{\infty} P_k^* = 1$ (like in the subcritical case without immigration). On the other hand, when $M(t) \to \infty$ one can consider the asymptotic behaviour of $Y(t)/M(t)$ (like in the supercritical case).

Theorem 1. *Let* $r(t) = re^{\rho t}$, $r > 0$, $\rho < 0$. *Assume that* $\gamma < \infty$ *and*

$$0 < -\log K = \int_0^1 \{[\alpha x + f(1-x)]/xf(1-x)\}dx < \infty. \tag{16}$$

Then

$$\lim_{t\to\infty} \mathbf{P}\{Y(t) = k | Y(0) > 0\} = P_k^*, \quad \sum_{k=1}^{\infty} P_k^* = 1. \tag{17}$$

P r o o f. Introduce the conditional p.g.f.

$$\Psi^*(t; s) = E\{s^{Y(t)} | Y(t) > 0\} = 1 - \frac{1 - \Psi(t; s)}{1 - \Psi(t; 0)}. \tag{18}$$

Note first that $\Psi(t; 0) = \exp\{-re^{\rho t} J(t)\}$, where

$$J(t) = \int_0^t e^{-\rho u}[1 - g(F(u; 0))]du\}.$$

As $u \to \infty$ one has

$$1 - g(F(u; 0)) \sim \gamma[1 - F(u; 0)] \sim \gamma K e^{\alpha u},$$

because of the condition (16) (see also Sevastyanov, 1971). Now it is not difficult to check that for $t \to \infty$

$$\begin{aligned} &J(t) \to C_1 && \text{if} \quad \rho > \alpha; \\ &J(t) \sim C_2 t && \text{if} \quad \rho = \alpha; \\ &J(t) \sim C_3 e^{(\alpha-\rho)t} && \text{if} \quad \rho < \alpha, \end{aligned}$$

where C_i, $i = 1, 2, 3$, are some positive constants.

Therefore

$$1 - \Psi(t; 0) \sim \begin{cases} rC_1 e^{\rho t} & \text{if} \quad \rho > \alpha \\ rC_2 t e^{\alpha t} & \text{if} \quad \rho = \alpha; \\ rC_3 e^{\alpha t} & \text{if} \quad \rho < \alpha. \end{cases} \tag{19}$$

It is known also that (see Sevastyanov, 1971)

$$\lim_{u\to\infty} \{[1 - F(u; s)]/[1 - F(u; 0)]\} = \exp\left\{\alpha \int_0^s dx/f(x)\right\}. \tag{20}$$

Hence as $u \to \infty$

$$1 - g(F(u;s)) \sim \gamma[1 - F(u;s)] \sim \gamma K e^{\alpha u} \exp\left\{\alpha \int_0^s dx/f(x)\right\}.$$

Then for $J(t;s) = \int_0^t e^{-\rho u}[1 - g(F(u;s))]du\}$ one obtains as $t \to \infty$

$$\begin{aligned} J(t;s) &\to C_1(s) && \text{if} \quad \rho > \alpha; \\ J(t;s) &\sim C_2(s)t && \text{if} \quad \rho = \alpha; \\ J(t;s) &\sim C_3(s)e^{(\alpha-\rho)t} && \text{if} \quad \rho < \alpha, \end{aligned}$$

where $C_i(s)$, $i = 1, 2, 3$, are some positive functions.

Since $\Psi(t;s) = \exp\{-re^{\rho t}J(t;s)\}$ then as $t \to \infty$

$$1 - \Psi(t;s) \sim \begin{cases} rC_1(s)e^{\rho t} & \text{if} \quad \rho > \alpha, \\ rC_2(s)te^{\alpha t} & \text{if} \quad \rho = \alpha, \\ rC_3(s)e^{\alpha t} & \text{if} \quad \rho < \alpha. \end{cases}$$

Now applying also (18) and (19) one obtains

$$\lim_{t\to\infty} \Psi^*(t;s) = \Psi^*(s), \tag{21}$$

where

$$\Psi^*(s) = \begin{cases} 1 - C_1(s)/C_1 & \text{if} \quad \rho > \alpha, \\ 1 - C_2(s)/C_2 & \text{if} \quad \rho = \alpha, \\ 1 - C_3(s)/C_3 & \text{if} \quad \rho < \alpha. \end{cases}$$

Then (17) follows from (21) by the continuity theorem for p.g.f.'s. □

Theorem 2. *Let* $r(t) = re^{\rho t}$, $r > 0$, $\rho > 0$, *and* $\beta < \infty, \gamma_2 < \infty$. *Then as* $t \to \infty$

$$\zeta(t) = Y(t)/M(t) \to 1, \ \textit{a.s. and in } L_2.$$

Proof. For the convergence in L_2 it will be sufficient to show that as $t \to \infty$,

$$\Delta(t,\tau) = E\{\zeta(t+\tau) - \zeta(t)\}^2 \to 0, \tag{22}$$

uniformly for $\tau \geq 0$. Note that $E\{\zeta(t)\} \equiv 1$,

$$\Delta(t,\tau) = Var\zeta(t+\tau) + Var\zeta(t) - 2Cov\{\zeta(t), \zeta(t+\tau)\}, \tag{23}$$

$$Var\zeta(t) = W(t)/\ M^2(t) \text{ and } Cov\{\zeta(t), \zeta(t+\tau)\} = C(t,\tau)/M(t)M(t+\tau).$$

One can obtain from (2), (3), and (7) that under the conditions of the theorem as $t \to \infty$

$$W(t) \sim K_1 e^{\rho t}, \quad K_1 = r\frac{\gamma(\beta-\alpha) + (\gamma+\gamma_2)(\rho-\alpha)}{(\rho-\alpha)(\rho-2\alpha)}. \tag{24}$$

From (10) and (2) it is not difficult to show that

$$A(t,\tau) = e^{\alpha(t+\tau)}[\frac{\beta}{\alpha}(e^{\alpha t} - 1) + 1]. \tag{25}$$

Therefore, from (11), (25), and (2) one gets as $t \to \infty$

$$C(t,\tau) \sim K_2 e^{\rho t + \alpha\tau}, \quad K_2 = r\frac{\gamma\beta - \gamma_2\alpha}{(-\alpha)(\rho-2\alpha)}. \tag{26}$$

Now (22) follows from (23) applying (13), (24), and (26). Similarly to (24) and (26) one can calculate that

$$W(t+\tau)/\ M^2(t+\tau) \sim K_1^* e^{-\rho t - \rho\tau} \to 0, \quad \tau \to \infty,$$

$$C(t,\tau)/M(t)M(t+\tau) \sim K_2^* e^{-\rho t} e^{-(\rho-\alpha)\tau} \to 0, \quad \tau \to \infty,$$

where K_1^* and K_2^* are some positive constants.

Hence, from (23) one obtains that

$$\Delta(t) = \lim_{\tau\to\infty} \Delta(t,\tau) = E\{\zeta(t) - 1\}^2 = W(t)/M^2(t) \sim\ K_1 e^{-\rho t}.$$

Therefore $\int_0^\infty \Delta(t)dt < \infty$ and by Theorem 21.1 of Harris (1963) it follows that $\zeta(t)$ converges to 1, a.s. □

Remark 3. Theorem 2 can be interpreted as a LLN. Hence one can conjecture the CLT.

Theorem 3. *Let* $r(t) = re^{\rho t}$, $r > 0$, $\rho > 0$, *and* $\beta < \infty, \gamma_2 < \infty$. *Then*

$$X(t) = [Y(t) - M(t)]/\sqrt{W(t)} \to N(0, \sigma^2) \text{ in distribution as } t \to \infty,$$

where

$$\sigma^2 = \frac{\gamma\beta + \gamma_2(\rho - 2\alpha)}{\gamma(\beta - \alpha) + (\gamma + \gamma_2)(\rho - \alpha)}. \tag{27}$$

Proof. From (5) and (26) one can obtain the characteristic function

$$\begin{aligned} \varphi_t(z) &= E\{e^{izX(t)}\} = e^{-izM(t)/\sqrt{W(t)}} E\{e^{izY(t)/\sqrt{W(t)}}\} \\ &= e^{-izM(t)/\sqrt{W(t)}} \Psi(t; e^{iz/\sqrt{W(t)}}). \end{aligned}$$

Hence applying (5) one has

$$\log \varphi_t(z) = -izM(t)/\sqrt{W(t)} - \int_0^t r(t-u) \left[1 - g(F(u; e^{iz/\sqrt{W(t)}}))\right] du.$$

Now one can use the following asymptotic relations as $s \to 1$ (see e.g. Sevastyanov, 1971)

$$\begin{aligned} 1 - g(s) &\sim \gamma(1-s) - \gamma_2(1-s)^2/2, \\ 1 - F(u; s) &\sim A(u)(1-s) - B(u)(1-s)^2/2. \end{aligned}$$

Applying also that $1 - e^{cx} \sim -cx$ as $x \to 0$ one can obtain as $t \to \infty$

$$\begin{aligned} \log \varphi_t(z) &\sim -izM(t)/\sqrt{W(t)} \\ &- \int_0^t r(t-u) \left\{\gamma \left[1 - F(u; e^{iz/\sqrt{W(t)}})\right] - \gamma_2 \left[1 - F(u; e^{iz/\sqrt{W(t)}})\right]^2 /2\right\} du. \end{aligned} \tag{28}$$

Not that as $t \to \infty$

$$\begin{aligned} 1 - F(u; e^{iz/\sqrt{W(t)}}) &\sim A(u)(1 - e^{iz/\sqrt{W(t)}}) - B(u)(1 - e^{iz/\sqrt{W(t)}})^2/2 \\ &\sim -izA(u)/\sqrt{W(t)} + z^2 B(u)/\sqrt{W(t)}/2. \end{aligned}$$

Therefore as $t \to \infty$ one has

$$H(t) = \int_0^t r(t-u)\left\{\gamma\left[1-F(u;e^{iz/\sqrt{W(t)}})\right] - \gamma_2\left[1-F(u;e^{iz/\sqrt{W(t)}})\right]^2/2\right\}du$$

$$\sim -iz\gamma\int_0^t r(t-u)A(u)du/\sqrt{W(t)} + (z^2/2)\gamma\int_0^t r(t-u)B(u)du/W(t)$$

$$+ (z^2/2)\gamma_2\int_0^t r(t-u)A^2(u)du/W(t).$$

Then applying the relations (6) and (7) one obtains that

$$H(t) \sim -izM(t)/\sqrt{W(t)} + (z^2/2)[1-M(t)/W(t)], \quad t \to \infty.$$

Come back to (28) one gets

$$\log\varphi_t(z) \sim -(z^2/2)[1-M(t)/W(t)], \quad t\to\infty. \tag{29}$$

Now from (13) and (24) it is not difficult to see that

$$M(t)/W(t) \to D = \frac{\gamma(\rho-2\alpha)}{\gamma(\beta-\alpha)+(\gamma+\gamma_2)(\rho-\alpha)},$$

as $t \to \infty$.

One can also calculate that $\sigma^2 = 1 - D$ is just given by (27).

Therefore from (29) we finally obtain that

$$\lim_{t\to\infty}\varphi_t(\ z) = e^{-z^2/2\sigma^2}$$

which is just a characteristic function of a corresponding normal distribution. Then by the continuity theorem (see e.g. Feller, 1971) the assertion of the theorem follows. □

Remark 4. From Theorem 3 using (13) and (24) one can obtain the following relation which presents more convenient interpretation for the rate of convergence:

$$Y(t)/e^{\rho t} \sim N(\gamma r/(\rho-\alpha), C^2 e^{-\rho t})$$

where

$$C^2 = r\frac{\gamma\beta+\gamma_2(\rho-2\alpha)}{(\rho-\alpha)(\rho-2\alpha)}.$$

Note that this relation is also useful for constructing of asymptotic confident intervals.

Theorem 4. *Assume* $\gamma < \infty$, (16), *and* (14) *with* $\theta < 0$. *Then* (17) *is fulfilled where*

$$\Psi^*(s) = \sum_{k=1}^{\infty} P_k^* s^k = 1 - \exp\left\{\alpha \int_0^s dx/f(x)\right\}, \Psi^*(1) = 1. \tag{30}$$

Proof. We will consider conditional p.g.f. (18). From (5) under the conditions of the theorem one has

$$\Psi(t;s) = \exp\{-r(t+1)^\theta J(t;s)\}, \tag{31}$$

where

$$\begin{aligned} J(t;s) &= \int_0^t \left(1 - \frac{u}{t+1}\right)^\theta [1 - g(F(u;s))]du \\ &= (t+1)\int_0^{1-1/(t+1)} (1-x)^\theta [1 - g(F(x(t+1);s))]dx. \end{aligned} \tag{32}$$

Note that for $s = 0$ as $t \to \infty$

$$1 - g(F(x(t+1);0) \sim \gamma[1 - F(x(t+1);0))] \sim \gamma K e^{\alpha x(t+1)}.$$

Therefore from (32) one has

$$J(t;0) \sim \gamma K(t+1)\int_0^1 (1-x)^\theta e^{\alpha x(t+1)} dx, t \to \infty.$$

On the other hand as $t \to \infty$

$$(t+1)\int_0^1 (1-x)^\theta e^{\alpha x(t+1)} dx = \frac{1}{\alpha}(\theta \int_0^1 (1-x)^{\theta-1} e^{\alpha x(t+1)} dx - 1) \to \frac{1}{(-\alpha)}.$$

Hence

$$\lim_{t\to\infty} J(t;0) = \gamma K/(-\alpha)$$

and from (31) with $s = 0$ one obtains

$$1 - \Psi(t;0) \sim 1 - \exp\{(r\gamma K/\alpha)t^{\theta}\} \sim -(r\gamma K/\alpha)t^{\theta}, t \to \infty. \tag{33}$$

Now using (20) one has as $t \to \infty$

$$1 - g(F(xt;s)) \sim \gamma[1 - F(xt;s)] \sim \gamma K e^{\alpha x t} \exp\left\{\alpha \int_0^s dx/f(x)\right\}.$$

Then from (32) one gets

$$J(t;s) \sim \gamma K(t+1) \int_0^1 (1-x)^{\theta} e^{\alpha x(t+1)} dx \exp\left\{\alpha \int_0^s dx/f(x)\right\}, t \to \infty.$$

Therefore

$$\lim_{t\to\infty} J(t;s) = \frac{\gamma K}{(-\alpha)} \exp\left\{\alpha \int_0^s dx/f(x)\right\}$$

and from (31) one obtains as $t \to \infty$

$$\begin{aligned} 1 - \Psi(t;s) &\sim 1 - \exp\left\{(r\gamma K/\alpha)t^{\theta} \exp\left\{\alpha \int_0^s dx/f(x)\right\}\right\} \\ &\sim -(r\gamma K/\alpha)t^{\theta} \exp\left\{\alpha \int_0^s dx/f(x)\right\}. \end{aligned} \tag{34}$$

Hence from (18) applying (33) and (34) one proves that

$$\lim_{t\to\infty} \Psi^*(t;s) = \Psi^*(s),$$

where $\Psi^*(s)$ is just given in (30). □

Remark 5. It is interesting to point out that the limiting distribution (30) is just the same as in the classical Markov branching process without immigration. The difference is only in the rate of convergence of $\mathbf{P}\{Y(t) > 0\}$ obtained in (33).

Theorem 5. *Assume $\beta < \infty, \gamma_2 < \infty$ and (14) with $\theta > 0$. Then as $t \to \infty$,*

$$\zeta(t) = \frac{Y(t)}{M(t)} \to 1 \text{ in } L_2 \text{ and in probability.} \tag{35}$$

The convergence is almost surely if $\theta > 1$.

P r o o f. From (7) using (2) ,(3), and (14) it is not difficult to obtain that as $t \to \infty$,

$$W(t) \sim K_1^* t^\theta, \text{ where } K_1^* = r[\gamma(2 - \frac{\beta}{\alpha}) + \gamma_2]/(-2\alpha). \tag{36}$$

Similarly from (11) using (2) and (25) one can prove that uniformly for $\tau \geq 0$

$$C(t,\tau) \sim K_1^* e^{\alpha\tau} t^\theta, \ t \to \infty. \tag{37}$$

Now from (15), (36), and (37) it is easy to check that as $t \to \infty$

$$Var\zeta(t) = W(t)/\ M^2(t) \sim \overline{K_1} t^{-\theta}, \overline{K_1} = (-\alpha)\left[\gamma\left(2 - \frac{\beta}{\alpha}\right) + \gamma_2\right]/2r, \tag{38}$$

$$Cov\{\zeta(t), \zeta(t+\tau)\} = C(t,\tau)/M(t)M(t+\tau) \sim \overline{K_1} e^{\alpha\tau}(t+\tau)^{-\theta}. \tag{39}$$

Then (22) follows from (23) using (38) and (39) which proves (35) with the convergence in L_2.

Since $E\{\zeta(t)\} \equiv 1$ then for each $\varepsilon > 0$

$$\mathbf{P}\{|\zeta(t) - 1| \geq \varepsilon\} \leq \varepsilon^{-2} Var\zeta(t) \to 0, \quad t \to \infty,$$

which proves the convergence *in probability.*

Similarly to (38) and (39) one can calculate that

$$W(t+\tau)/\ M^2(t+\tau) \sim \overline{K_1}(t+\tau)^{-\theta} \to 0, \quad \tau \to \infty,$$

$$C(t,\tau)/M(t)M(t+\tau) \sim \overline{K_1} e^{\alpha\tau}(t+\tau)^{-\theta} \to 0, \quad \tau \to \infty.$$

Hence, from (23) one obtains that

$$\Delta(t) = \lim_{\tau\to\infty} \Delta(t,\tau) = E\{\zeta(t) - 1\}^2 = W(t)/M^2(t) \sim \ \overline{K_1} t^{-\theta}, \quad t \to \infty.$$

If $\theta > 1$ then $\int_0^\infty \Delta(t)dt < \infty$ and by Theorem 21.1 of Harris (1963) it follows that in this case $\zeta(t)$ converges to 1 a.s. □

Theorem 6. *Assume* $\beta < \infty, \gamma_2 < \infty$ *and* (14) *with* $\theta > 0$. *If additionally* $\dfrac{\beta}{-\alpha} = \dfrac{m_2}{1-m} > \dfrac{2-(\gamma_2+2\gamma)}{\gamma}$ *then*

$$X(t) = [Y(t) - M(t)]/\sqrt{W(t)} \to N(0,\sigma^2) \text{ in distribution as } t \to \infty, \tag{40}$$

where

$$\sigma^2 = \frac{\gamma(2-\beta/\alpha)+\gamma_2-2}{\gamma(2-\beta/\alpha)+\gamma_2}. \tag{41}$$

P r o o f. One has to investigate the characteristic function of $X(t)$ as it is shown in the proof of Theorem 3. The only difference is that in (29) one has to use now (15) and (36) to obtain that

$$\lim_{t\to\infty} \frac{M(t)}{W(t)} = \frac{r}{(-\alpha)K_1^*} = \frac{2}{\gamma(2-\frac{\beta}{\alpha})+\gamma_2}.$$

Then (40) with (41) follows where under the conditions of the theorem $\sigma^2 > 0$. □

Remark 6. Note that (35) can be interpreted as a LLN and (40) – as a CLT which admits also the following presentation:

$$Y(t)t^{-\theta} \sim N(\gamma r/(-\alpha), K_1^* t^{-\theta}),$$

where K_1^* is given in (24).

Theorem 7. *Let* $\gamma < \infty$ *and* $\lim_{t\to\infty} r(t) = r > 0$. *Then*

$$\lim_{t\to\infty} \mathbf{P}\{Y(t) = k\} = Q_k, \qquad \sum_{k=0}^{\infty} Q_k = 1$$

and

$$Q(s) = \sum_{k=0}^{\infty} Q_k s^k = \exp\left\{-r\int_s^1 \frac{1-g(x)}{f(x)}dx\right\}, \qquad Q'(1) = e^{-r\gamma/\alpha}.$$

P r o o f. Since $|1-g(s)| \le \gamma|1-s)|$ and $|1-F(u;s)| \le e^{\alpha u}|1-s|$ then

$$\left|\int_0^t r(t-u)[1-g(F(u;s))]du\right| \le \gamma|1-s| \int_0^t r(t-u)e^{\alpha u}du \to -\frac{\gamma r|1-s|}{\alpha}, \ t\to\infty.$$

Therefore $\lim_{t\to\infty}\Psi(t;s) = \exp\left\{r\int_0^\infty [1-g(F(u;s))]du\right\}$ uniformly for $|s|\le 1$. Now we obtain

$$\begin{aligned}\frac{d}{ds}\int_0^\infty [1-g(F(u;s))]du &= -\int_0^\infty \frac{dg(F(u;s))}{dF}\frac{\partial F(u;s)}{\partial s}du \\ &= -\frac{1}{f(s)}\int_0^\infty \frac{dg(F(u;s))}{dF}\frac{\partial F(u;s)}{\partial u}du = -\frac{1-g(s)}{f(s)},\end{aligned}$$

where we used the well known forward Kolmogorov equation

$$\frac{\partial}{\partial t}F(t;s) = f(s)\frac{\partial}{\partial s}F(t;s)$$

(see e.g. Harris, 1963) and the fact that $F(\infty;s)=1$ and $F(0;s)=s$. Hence

$$\int_0^\infty [1-g(F(u;s))]du = \int_s^1 \frac{1-g(x)}{f(x)}dx,$$

which completes the proof of the theorem. □

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