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EXISTENCE AND UNIQUENESS OF THE SOLUTION FOR THE STOCHASTIC EQUATION OF MOTION OF A VISCOUS GAS IN A DISCRETIZED ONE-DIMENSIONAL DOMAIN

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A stochastic equation system corresponding to the description of the motion of a barotropic viscous gas in a discretized one-dimensional domain with a weight regularizing the density is considered. In [3], [4], the existence and uniqueness of the solution of this discretized problem in the stationary case was established. In this paper, by applying the technics used in [3], we generalize this result in the periodic case.

1. Introduction

The motion of a barotropic viscous gas in one dimension can be expressed in massic Lagrangian coordinates $\xi \in \mathbb{R}$ by the following stochastic system

$$dv = [\eta\partial_\xi(\varrho\partial_\xi v) - h\partial_\xi(\varrho^\gamma)]dt + dW, \tag{1.1}$$

$$\partial_t \frac{1}{\varrho} = \partial_\xi v, \tag{1.2}$$

with boundary conditions

$$v|_{\xi=0,1} = 0; \tag{1.3}$$

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where v and ϱ are the speed and density of the gas, while η and γ are the viscosity coefficient and adiabatic exponent (for example, for the air, $\gamma \approx 1.4$) and W is the stochastic perturbation represented by the Brownian motion in the Hilbert space $L^2(0,1)$.

We consider the system of equations (1.1)–(1.2) in the domain $0 < \xi < 1$; for the pressure, we use the relation

$$p = h\varrho^\gamma \qquad (h : \text{positive constant}).$$

The deterministic problem has been studied in [1]. To prove the existence and uniqueness of the solution of the system (1.1)–(1.3) with initial conditions, we refer to [9]; an analog problem has been studied in [5].

In [3],[4], the application of the Ito's formula in an infinite dimension space to a specific functional allows us to obtain an estimate useful to analyse the behavior of the solution. But, therefore, for technical reasons, it is difficult to exploit this estimate. Therefore, an approximate problem is considered which concerns the equation of the barotropic viscous gas in lagrangian coordinates in a discritized one dimensional domain with mesh size $\delta = \dfrac{1}{N}$, $(N \geq 2)$ and a "weight" $\varepsilon > 0$ regularizing the density. More precisely, they prove the existence and uniqueness of the solution for the problem

$$dv_i = \left[\frac{\eta}{\delta^2}(\varrho_{i+1}v_{i+1} - (\varrho_{i+1} + \varrho_i)v_i + \varrho_i v_{i-1}) - \frac{h}{\delta}((\varrho_{i+1})^\gamma - (\varrho_i)^\gamma)\right] dt \tag{1.4}$$

$$+\lambda_i dW_i, \qquad i = 1, \ldots, N-1,$$

$$\frac{d}{dt}\frac{1}{\varrho_i} = \frac{1}{\delta}[v_i - v_{i-1}] + \varepsilon - \frac{\varepsilon}{\varrho_i} \qquad i = 1, \ldots, N, \tag{1.5}$$

with

$$v_0 = v_N = 0, \tag{1.6}$$

where the λ_i, $i = 1, \ldots, N-1$, are positive constants and the W_i, $i = 1, \ldots, N-1$, are Brownian independent canonical motions with real values defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$. This is a real system of equations for $2N-1$ unknowns $v_1, \ldots, v_{N-1}, \varrho_1, \ldots, \varrho_N$.

In this paper, the result presented in [3] is generalized in the periodic case. The result will be presented as a theorem of existence and uniqueness of a periodic solution for the system of equations of a barotropic viscous gas in a discretised domain in one dimension.

2. Position of problem

Assume that the pressure p is given by

$$p = h(t)\varrho^{\gamma}. \tag{2.1}$$

Consider the system of equations

$$dv_i = \left[\frac{\eta}{\delta^2}(\varrho_{i+1}v_{i+1} - (\varrho_{i+1} + \varrho_i)v_i + \varrho_i v_{i-1}) - \frac{h(t)}{\delta}((\varrho_{i+1})^{\gamma} - (\varrho_i)^{\gamma})\right] dt + \lambda_i(t)dW_i, \qquad i = 1, \dots, N-1, \tag{2.2}$$

$$\frac{d}{dt}\frac{1}{\varrho_i} = \frac{1}{\delta}[v_i - v_{i-1}] + \varepsilon - \frac{\varepsilon}{\varrho_i}, \qquad i = 1, \dots, N, \tag{2.3}$$

with

$$v_0 = v_N = 0. \tag{2.4}$$

When we consider the system (2.2)–(2.4) with an initial condition, in addition to the natural condition for the density

$$\varrho_i(0) > 0, \qquad i = 1, \dots, N,$$

we require by normalization that

$$\sum_{i=1}^{N} \frac{\delta}{\varrho_i(0)} = 1. \tag{2.5}$$

In this case, from the equation (2.3), we prove that

$$\sum_{i=1}^{N} \frac{\delta}{\varrho_i(t)} = 1, \qquad \forall\, t \geq 0. \tag{2.6}$$

Now, if $\varrho_i > 0$, then, by letting

$$\sigma_i = \log \varrho_i, \tag{2.7}$$

the system of equations (2.2)–(2.3) can be transformed into

(2.8)
$$dv_i = \left[\frac{\eta}{\delta^2}(e^{\sigma_{i+1}}v_{i+1} - (e^{\sigma_{i+1}} + e^{\sigma_i})v_i + e^{\sigma_i}v_{i-1}) - \frac{h(t)}{\delta}((e^{\sigma_{i+1}})^{\gamma} - (e^{\sigma_i})^{\gamma})\right] dt + \lambda_i(t)dW_i, \qquad i = 1, \dots, N-1,$$

$$\frac{d}{dt}\sigma_i = -\frac{e^{\sigma_i}}{\delta}[v_i - v_{i-1}] - \varepsilon(e^{\sigma_i} - 1), \qquad i = 1, \dots, N, \tag{2.9}$$

with

$$v_0 = v_N = 0. \tag{2.10}$$

The change of variables (2.7) is useful from a technical point of view as the system (2.8)–(2.9) to be considered is of values in $\mathbb{R}^{2N-1}$ instead of $\mathbb{R}^{N-1} \times (\mathbb{R}_+)^N$.

Here, we assume that $h(t)$ and $\lambda_i(t)$, $i = 1, \ldots, N-1$, are T-periodic and satisfy

$$\sup_{t\in[0,T]} \sum_{i=1}^{N-1} \lambda_i^2(t) \le M, \qquad M \text{ is a positive constant,} \tag{2.11}$$

$$\min_{t\in[0,T]} h(t) = m > 0 \tag{2.12}$$

and

$$0 \le h'(t) < C, \quad \forall\, t \in [0,T]\,, \qquad C \quad \text{is a positive constant.} \tag{2.13}$$

3. Existence and uniqueness of a T-periodic solution

Consider the system of equations (2.8)–(2.9) with the condition (2.10) and the initial conditions

$$v_i(0) = v_{0i}, \quad i = 1, \ldots, N-1, \qquad \sigma_i(0) = \sigma_{0i}, \quad i = 1, \ldots, N, \tag{3.1}$$

where $X_0 = (v_{01}, \ldots, v_{0N-1}, \sigma_{01}, \ldots,\ \sigma_{0N})$ is a random variable with values in $\mathbb{R}^{2N-1}$ satisfying the condition

$$\delta \sum_{i=1}^{N} e^{-\sigma_{0i}} = 1 \qquad \text{a.s..} \tag{3.2}$$

For this problem, we are going to show the existence and uniqueness of the $T-$periodic solution.

Proposition 3.1. *Assume that the random variable $X_0 = (v_{01}\ , \ldots, v_{0N-1},$ $\sigma_{01}, \ldots, \sigma_{0N})$ satisfies, in addition to* (3.2), *the condition*

$$\mathbb{E} \sum_{i=1}^{N-1} {v_{0i}}^2 < \infty, \qquad \mathbb{E} \sum_{i=1}^{N} e^{\sigma_{0i}} < \infty. \tag{3.3}$$

Furthermore, assume that the functions $h(t)$ and $\lambda_i(t)$ are T-periodic and satisfy the conditions (2.11), (2.12) *and* (2.13). *Then, for all $t_1 \in [0,T]$, the system of equations* (2.8)–(2.9) *with the conditions* (2.10) *and* (3.1) *has, on the* *interval* $[0,t_1]$*, a unique* T*-periodic solution* $(v_1, \ldots, v_{N-1}, \sigma_1, \ldots, \sigma_N)$*. In addition, it satisfies the relation*

$$\delta \sum_{i=1}^{N} e^{-\sigma_i(t)} = 1 \qquad a.s., \qquad \forall\, t \in [0,T]. \tag{3.4}$$

P r o o f. The proof is similar to one given in [3]. More precisely, a sequence of solution which converges to the desired solutions is constructed. Let $\nu \in \mathbb{N}$, $\nu \geq 1$. We introduce the function θ_ν defined by

$$\theta_\nu = \theta_\nu(\{\sigma_i^{(\nu)}\}_{i=1}^N) = \min(1, \frac{\nu}{\sum_{i=1}^N e^{\sigma_i^{(\nu)}}}) \tag{3.5}$$

and we consider the system of equations in $v_i^{(\nu)}, \sigma_i^{(\nu)}$

$$dv_i^{(\nu)} = [\frac{\eta}{\delta^2}\theta_\nu(e^{\sigma_{i+1}^{(\nu)}}v_{i+1}^{(\nu)} - (e^{\sigma_{i+1}^{(\nu)}} + e^{\sigma_i^{(\nu)}})v_i^{(\nu)} + e^{\sigma_i^{(\nu)}}v_{i-1}^{(\nu)}) \tag{3.6}$$

$$-\frac{h(t)}{\delta}\theta_\nu^\gamma((e^{\sigma_{i+1}^{(\nu)}})^\gamma - (e^{\sigma_i^{(\nu)}})^\gamma)]dt + \lambda_i(t)dW_i, \qquad i = 1, \ldots, N-1,$$

$$\frac{d}{dt}\sigma_i^{(\nu)} = -\frac{\theta_\nu^\gamma e^{\sigma_i^{(\nu)}}}{\delta}[v_i^{(\nu)} - v_{i-1}^{(\nu)}] - \varepsilon\theta_\nu(e^{\sigma_i^{(\nu)}} - 1), \qquad i = 1, \ldots, N. \tag{3.7}$$

with

$$v_0^{(\nu)} = v_N^{(\nu)} = 0, \tag{3.8}$$

$$v_i^{(\nu)}(0) = v_{0i}, \quad i = 1, \ldots, N-1, \qquad \sigma_i^{(\nu)}(0) = \sigma_{0i}, \quad i = 1, \ldots, N. \tag{3.9}$$

Lemma 3.1. *For all* $t_1 \in [0,T]$*, the system of equations* (3.6)–(3.7) *with the conditions* (3.8)–(3.9) *has, on the interval* $[0,t_1]$*, a* T*-periodic solution* $(v_1^{(\nu)}, \ldots, v_{N-1}^{(\nu)}, \sigma_1^{(\nu)}, \ldots, \sigma_N^{(\nu)})$ *which is unique and it satisfies the inequality*

$$\frac{\delta}{2}\mathbb{E}\sum_{i=1}^{N-1}(v_i^{(\nu)}(t))^2 + \frac{m\delta}{\gamma-1}\mathbb{E}\sum_{i=1}^{N}(e^{\sigma_i^{(\nu)}(t)})^{\gamma-1} \leq C_0 + C_1 t, \quad \forall t \in [0,t_1], \tag{3.10}$$

where C_0 *and* C_1 *are two non negative constants independent of* ν*.*

P r o o f o f L e m m a 3.1. From the definition (3.5) of θ_ν, il follows immediately that

$$0 < \theta_\nu \leq 1, \qquad 0 < \theta_\nu e^{\sigma_i^{(\nu)}} \leq \frac{\nu}{\delta}.$$

From these inequalities, it follows that the coefficients of (3.6) and (3.7) satisfy the Lipschitz's condition for $(v_1^{(\nu)}, \ldots, v_{N-1}^{(\nu)}, \sigma_1^{(\nu)}, \ldots, \sigma_N^{(\nu)}) \in \mathbb{R}^{2N-1}$. Furthermore, as the coefficients of (3.6)–(3.7) are T-periodic (by hypothesis, $h(t)$ and $\lambda_i(t)$ are T-periodic), by the classical theorems of existence and uniqueness of solution (see for example [6], Theorem 3.2), for all $t_1 > 0$, the problem (3.6)–(3.7) admits, in the interval $[0, t_1]$, a unique T-periodic solution $(v_1^{(\nu)}, \ldots, v_{N-1}^{(\nu)}, \sigma_1^{(\nu)}, \ldots, \sigma_N^{(\nu)})$.

To prove the inequality (3.10), we consider the function

$$\psi(t) = \frac{\delta}{2}\sum_{i=1}^{N-1}(v_i^{(\nu)}(t))^2.$$

As

$$\frac{\partial \psi}{\partial v_i^{(\nu)}} = \delta v_i^{(\nu)},$$

by applying Ito's formula to $\psi(t)$, it follows that

$$\psi(t) - \psi(0) = \int_0^t \delta \sum_{i=1}^{N-1} v_i^{(\nu)} G_i^{(\nu)} ds + \int_0^t \delta \sum_{i=1}^{N-1} v_i^{(\nu)} \lambda_i(s) dW_i + \frac{1}{2}\int_0^t \delta \sum_{i=1}^{N-1} \lambda_i^2(s) ds, \tag{3.11}$$

where

$$G_i^{(\nu)} = \frac{\eta}{\delta^2}\theta_\nu(e^{\sigma_{i+1}^{(\nu)}} v_{i+1}^{(\nu)} - (e^{\sigma_{i+1}^{(\nu)}} + e^{\sigma_i^{(\nu)}})v_i^{(\nu)} + e^{\sigma_i^{(\nu)}} v_{i-1}^{(\nu)}) - \frac{h(s)}{\delta}\theta_\nu^\gamma((e^{\sigma_{i+1}^{(\nu)}})^\gamma - (e^{\sigma_i^{(\nu)}})^\gamma).$$

From (3.8), the equality (3.11) is written by

$$\begin{aligned}\psi(t) - \psi(0) = &-\eta \int_0^t \sum_{i=1}^{N} \frac{\theta_\nu e^{\sigma_i^{(\nu)}}}{\delta}(v_i^{(\nu)} - v_{i-1}^{(\nu)})^2 ds \\ &- \frac{\delta}{\gamma - 1}\int_0^t \sum_{i=1}^{N} h(s)\frac{d(e^{\sigma_i^{(\nu)}(s)})^{\gamma-1}}{ds} ds - \int_0^t h(s)\varepsilon\theta_\nu \sum_{i=1}^{N} \delta((e^{\sigma_i^{(\nu)}})^\gamma - (e^{\sigma_i^{(\nu)}})^{\gamma-1}) ds \\ &+ \frac{1}{2}\int_0^t \delta \sum_{i=1}^{N} \lambda_i^2(s) ds + \int_0^t \delta \sum_{i=1}^{N-1} v_i^{(\nu)} \lambda_i(s) dW_i.\end{aligned} \tag{3.12}$$

As we have

$$\int_0^t h(s)\frac{d(e^{\sigma_i^{(\nu)}(s)})^{\gamma-1}}{ds} dt' = h(t)(e^{\sigma_i^{(\nu)}(t)})^{\gamma-1} - h(0)(e^{\sigma_{0i}})^{\gamma-1} \tag{3.13}$$

$$-\int_0^t \frac{dh(s)}{ds}(e^{\sigma_i^{(\nu)}(s)})^{\gamma-1}ds,$$

the equality (3.12) becomes

$$\frac{\delta}{2}\sum_{i=1}^{N-1}(v_i^{(\nu)}(t))^2 - \frac{\delta}{2}\sum_{i=1}^{N-1}(v_{0i})^2 = -\eta\int_0^t \sum_{i=1}^{N}\frac{\theta_\nu e^{\sigma_i^{(\nu)}}}{\delta}(v_i^{(\nu)} - v_{i-1}^{(\nu)})^2 ds \tag{3.14}$$

$$-\frac{h(t)\delta}{\gamma-1}\sum_{i=1}^{N}(e^{\sigma_i^{(\nu)}(t)})^{\gamma-1} + \frac{h(0)\delta}{\gamma-1}\sum_{i=1}^{N}(e^{\sigma_{0i}})^{\gamma-1} + \frac{\delta}{\gamma-1}\int_0^t\sum_{i=1}^{N}\frac{dh(s)}{ds}(e^{\sigma_i^{(\nu)}(s)})^{\gamma-1}ds$$

$$-\int_0^t h(s)\varepsilon\theta_\nu\sum_{i=1}^{N}\delta((e^{\sigma_i^{(\nu)}})^{\gamma} - (e^{\sigma_i^{(\nu)}})^{\gamma-1})ds + \frac{1}{2}\int_0^t \delta\sum_{i=1}^{N}\lambda_i^2(s)ds$$

$$+\int_0^t \delta\sum_{i=1}^{N-1}v_i^{(\nu)}\lambda_i(s)dW_i.$$

From the condition (2.12), we have

$$\frac{\delta}{2}\sum_{i=1}^{N-1}(v_i^{(\nu)}(t))^2 + \frac{m\delta}{\gamma-1}\sum_{i=1}^{N}(e^{\sigma_i^{(\nu)}(t)})^{\gamma-1} \tag{3.15}$$

$$\leq \frac{\delta}{2}\sum_{i=1}^{N-1}(v_{0i})^2 + \frac{h(0)\delta}{\gamma-1}\sum_{i=1}^{N}(e^{\sigma_{0i}})^{\gamma-1} + \frac{\delta}{\gamma-1}\int_0^t\sum_{i=1}^{N}\frac{dh(s)}{ds}(e^{\sigma_i^{(\nu)}(s)})^{\gamma-1}ds$$

$$-\eta\int_0^t\sum_{i=1}^{N}\frac{\theta_\nu e^{\sigma_i^{(\nu)}}}{\delta}(v_i^{(\nu)} - v_{i-1}^{(\nu)})^2 ds - \int_0^t h(s)\varepsilon\theta_\nu\sum_{i=1}^{N}\delta((e^{\sigma_i^{(\nu)}})^{\gamma} - (e^{\sigma_i^{(\nu)}})^{\gamma-1})ds$$

$$+\frac{1}{2}\int_0^t\delta\sum_{i=1}^{N-1}\lambda_i^2(s)ds + \int_0^t\delta\sum_{i=1}^{N-1}v_i^{(\nu)}\lambda_i(s)dW_i.$$

By taking the expectation of this inequality, we get

$$\frac{\delta}{2}\mathbb{E}\sum_{i=1}^{N-1}(v_i^{(\nu)}(t))^2 + \frac{m\delta}{\gamma-1}\mathbb{E}\sum_{i=1}^{N}(e^{\sigma_i^{(\nu)}(t)})^{\gamma-1} \leq \tag{3.16}$$

$$\leq \frac{\delta}{2}\mathbb{E}\sum_{i=1}^{N-1}(v_{0i})^2 + \frac{\delta}{\gamma-1}\mathbb{E}\sum_{i=1}^{N}h(0)(e^{\sigma_{0i}})^{\gamma-1} + \frac{\delta}{\gamma-1}\mathbb{E}\int_0^t\sum_{i=1}^{N}\frac{dh(s)}{ds}(e^{\sigma_i^{(\nu)}(s)})^{\gamma-1}ds$$

$$-\eta\mathbb{E}\int_0^t\sum_{i=1}^N\frac{\theta_\nu e^{\sigma_i^{(\nu)}}}{\delta}(v_i^{(\nu)}-v_{i-1}^{(\nu)})^2ds-\mathbb{E}\int_0^t h(s)\varepsilon\delta\theta_\nu\sum_{i=1}^N((e^{\sigma_i^{(\nu)}})^\gamma-(e^{\sigma_i^{(\nu)}})^{\gamma-1})ds$$

$$+\frac{1}{2}\mathbb{E}\int_0^t\delta\sum_{i=1}^{N-1}\lambda_i^2(s)ds.$$

As we have

$$-((e^{\sigma_i^{(\nu)}})^\gamma-(e^{\sigma_i^{(\nu)}})^{\gamma-1})\leq\sup_{s>0}(s^{\gamma-1}-s^\gamma)=(\frac{\gamma-1}{\gamma})^{\gamma-1}-(\frac{\gamma-1}{\gamma})^\gamma, \tag{3.17}$$

then, taking into account the conditions (2.12), (2.13) et (2.11), the inequality (3.16) becomes

$$\frac{\delta}{2}\mathbb{E}\sum_{i=1}^{N-1}(v_i^{(\nu)}(t))^2+\frac{m\delta}{\gamma-1}\mathbb{E}\sum_{i=1}^N(e^{\sigma_i^{(\nu)}(t)})^{\gamma-1}\leq$$

$$\frac{\delta}{2}\mathbb{E}\sum_{i=1}^{N-1}(v_{0i})^2+\frac{\delta}{\gamma-1}\mathbb{E}\sum_{i=1}^N h(0)(e^{\sigma_{0i}})^{\gamma-1}$$

$$+\left(\varepsilon\left((\frac{\gamma-1}{\gamma})^{\gamma-1}-(\frac{\gamma-1}{\gamma})^\gamma\right)\sup_{0\leq t\leq T}h(t)+\frac{\delta}{2}\sup_{0\leq t\leq T}\sum_{i=1}^{N-1}\lambda_i^2(t)\right)t,$$

so, the inequality (3.10) is proved by letting

$$C_0=\frac{\delta}{2}\mathbb{E}\sum_{i=1}^{N-1}(v_{0i})^2+\frac{\delta}{\gamma-1}\mathbb{E}\sum_{i=1}^N h(0)(e^{\sigma_{0i}})^{\gamma-1},$$

$$C_1=\varepsilon\left((\frac{\gamma-1}{\gamma})^{\gamma-1}-(\frac{\gamma-1}{\gamma})^\gamma\right)\sup_{0\leq t\leq T}h(t)+\frac{\delta}{2}\sup_{0\leq t\leq T}\sum_{i=1}^{N-1}\lambda_i^2(t).$$

The lemma is proved. □

Lemma 3.2. *We have*

$$\mathbb{P}\{\sup_{0\leq t\leq T}\sum_{i=1}^N\delta e^{\sigma_i^{(\nu)}(t)}\geq\nu-1\}\to 0\qquad \textit{for}\quad \nu\to\infty. \tag{3.18}$$

P r o o f o f L e m m a 3.2. As the inequality (3.17) is true independently of $\omega \in \Omega$, from (3.15), one can also deduce that

$$\frac{m\delta}{\gamma-1}\sum_{i=1}^{N}(e^{\sigma_i^{(\nu)}(t)})^{\gamma-1} \tag{3.19}$$

$$\leq \frac{\delta}{2}\sum_{i=1}^{N-1}(v_{0i})^2+\frac{h(0)\delta}{\gamma-1}\sum_{i=1}^{N}(e^{\sigma_{0i}})^{\gamma-1}+C_1t+\int_0^t\sum_{i=1}^{N-1}\lambda_i(s)\delta v_i^{(\nu)}dW_i,$$

where C_1 is the same constant introduced for (3.10). It follows that

$$\mathbb{E}\frac{m\delta}{\gamma-1}\sup_{0\leq t\leq T}\sum_{i=1}^{N}(e^{\sigma_i^{(\nu)}(t)})^{\gamma-1}\leq C_0+C_1T+\mathbb{E}\sup_{0\leq t\leq T}\int_0^t\sum_{i=1}^{N-1}\delta v_i^{(\nu)}\lambda_i(s)dW_i, \tag{3.20}$$

with the same constant C_0 used for (3.10). Now, by using the Cauchy Schwarz's and Doob's inequalities, we have

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}\int_0^t\delta v_i^{(\nu)}\lambda_i(s)dW_i &\leq (\mathbb{E}(\sup_{0\leq t\leq T}|\int_0^t\delta v_i^{(\nu)}\lambda_i(s)dW_i|^2))^{\frac{1}{2}}\\ &\leq 2(\mathbb{E}(\int_0^T\delta v_i^{(\nu)}\lambda_i(s)dW_i)^2)^{\frac{1}{2}}=2(\mathbb{E}\int_0^T(\delta v_i^{(\nu)}\lambda_i(s))^2ds)^{\frac{1}{2}}\\ &=2\delta(\mathbb{E}\int_0^T(\lambda_i(s)v_i^{(\nu)})^2ds)^{\frac{1}{2}}.\end{aligned} \tag{3.21}$$

It follows from the condition (2.11) and the inequality (3.10) that

$$\mathbb{E}\sup_{0\leq t\leq T}\int_0^t\sum_{i=1}^{N-1}\delta v_i^{(\nu)}\lambda_i(s)dW_i$$

$$\leq 2M^{\frac{1}{2}}\delta^{\frac{1}{2}}(\mathbb{E}\int_0^T\sum_{i=1}^{N-1}(v_i^{(\nu)})^2ds)^{\frac{1}{2}}\leq 2M^{\frac{1}{2}}\delta(2C_0T+C_1T^2)^{\frac{1}{2}}.$$

Substituting this inequality in (3.20), we get

$$\mathbb{E}\sup_{0\leq t\leq T}\frac{m\delta}{\gamma-1}\sum_{i=1}^{N}(e^{\sigma_i^{(\nu)}(t)})^{\gamma-1}\leq C_0+C_1T+2M^{\frac{1}{2}}\delta^{\frac{1}{2}}(2C_0T+C_1T^2)^{\frac{1}{2}}. \tag{3.22}$$

For $\gamma>2$ and using Cauchy-Schwarz's inequality, we have

$$\sum_{i=1}^{N}e^{\sigma_i^{(\nu)}(t)}\leq N^{\frac{\gamma-2}{\gamma-1}}\left(\sum_{i=1}^{N}(e^{\sigma_i^{(\nu)}(t)})^{\gamma-1}\right)^{\frac{1}{\gamma-1}}$$

and it is deduced from (3.22) that

$$\delta^{2-\gamma}\mathbb{E}\sup_{0\le t\le T}\left(\sum_{i=1}^{N}\delta e^{\sigma_i^{(\nu)}(t)}\right)^{\gamma-1} \le \mathbb{E}\sup_{0\le t\le T}\delta\sum_{i=1}^{N}(e^{\sigma_i^{(\nu)}(t)})^{\gamma-1} \le K, \tag{3.23}$$

where

$$K = \frac{\gamma-1}{m}N^{\gamma-2}(C_0 + C_1T + 2M^{\frac{1}{2}}\delta^{\frac{1}{2}}(2C_0T + C_1T^2)^{\frac{1}{2}}).$$

By applying the Markov's inequality, the inequality (3.23) gives

$$\mathbb{P}\{\sup_{0\le t\le T}\left(\sum_{i=1}^{N}\delta e^{\sigma_i^{(\nu)}(t)}\right)^{\gamma-1} \ge (\nu-1)^{\gamma-1}\} \le \frac{1}{(\nu-1)^{\gamma-1}}\cdot\frac{K}{\delta^{2-\gamma}}, \tag{3.24}$$

the lemma is proved. □

Follow up of the proof of Lemma 3.1. Having proved the existence and uniqueness of the approximate solution $(v_1^{(\nu)}, \ldots, v_{N-1}^{(\nu)}, \sigma_1^{(\nu)}, \ldots, \sigma_N^{(\nu)})$ for every ν, we now prove the proposition 3.1. The definition (3.5) of θ_ν implies that, if $0 < \nu < \nu'$, we have

$$v_i^{(\nu)} = v_i^{(\nu')}, \qquad \sigma_i^{(\nu)} = \sigma_i^{(\nu')} \qquad \text{a.s.}$$

on $\left\{\sup_{0\le t\le T}\sum_{i=1}^{N}\delta e^{\sigma_i^{(\nu)}(t)} \le \nu - 1\right\}$.

Let

$$A_\nu = \{\sup_{0\le t\le T}\sum_{i=1}^{N}\delta e^{\sigma_i^{(\nu)}} > \nu - 1\}.$$

From the inequality (3.24), one has

$$\sum_{\nu=1}^{\infty}\mathbb{P}\{A_\nu\} < \infty$$

and using Borel-Cantelli's lemma, one gets

$$\mathbb{P}\{\overline{\lim}A_\nu\} = 0.$$

It deduces that the sequence $\{(v_1^{(\nu)}, \ldots, v_{N-1}^{(\nu)}, \sigma_1^{(\nu)}, \ldots, \sigma_N^{(\nu)})\}_{\nu=1}^{\infty}$ converges almost certainly to a limit which is denoted by $(v_1, \ldots, v_{N-1}, \sigma_1, \ldots, \sigma_N)$. It is easy to note that $(v_1, \ldots, v_{N-1}, \sigma_1, \ldots, \sigma_N)$ satisfies the system of equations (2.8)–(2.9).

The uniqueness of the solution $(v_1, \ldots, v_{N-1}, \sigma_1, \ldots, \sigma_N)$ is shown by using the standard arguments. In fact, we simply have to note that, on $\left\{ \sup_{0\le t\le T} \sum_{i=1}^{N} \delta e^{\sigma_i(t)} \le \nu - 1 \right\}$ the two possible solutions must coincide and we have

$$\mathbb{P}\{ \sup_{0\le t\le T} \sum_{i=1}^{N} \delta e^{\sigma_i(t)} \le \nu - 1\} \to 1 \qquad \text{for} \quad \nu \to \infty.$$

To prove (3.4), we recall the reasoning for getting (2.6); we easily find the relation (3.4). The result is proved.

The proposition is proved. □

Remark 3.1. *If we let* $\varrho_i = e^{\sigma_i}$, $i = 1, \ldots, N$, *then it is clear that* $(v_1, \ldots, v_{N-1}, \varrho_1, \ldots, \varrho_N)$ *is the solution of the system of equations* (2.2)–(2.3) *with the condition* (2.4) *and the initial conditions*

$$v_i(0) = v_{0i}, \quad i = 1, \ldots, N-1, \qquad \varrho_i(0) = \varrho_{0i} \equiv e^{\sigma_{0i}}, \quad i = 1, \ldots, N$$

and we get

$$\sum_{i=1}^{N} \frac{\delta}{\varrho_i(t)} = 1 \qquad a.s., \qquad \forall t \ge 0.$$

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