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A STUDY ON THE STABILITY OF SOME FREE FILMS*

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In this work the special case of free films of liquid (bounded by two interfaces between liquid and gas or liquid and two other liquids) is considered. The films are assumed to be viscous (Newtonian or non-Newtonian), with fully mobile interfaces (with unknown velocity, but with given shear stresses on the interfaces), laterally bounded (with fixed film thickness or fixed thickness gradient on the lateral boundaries), with planar located midsurface (symmetric interfaces with respect to the middle plane). Additional action of different forces on the interfaces is applied, such as capillary forces (through a constant or temperature/surfactant dependent surface tension), van der Waals forces, etc.

Typical solutions of the nonlinear evolution equations are discussed. In the cases when the film static shapes exist (which depends on the combination of the different parameters), a linear and nonlinear stability analysis is also presented for them, when squeezing perturbations are applied on the shape itself, on the velocity or on the temperature, etc.

1. Introduction

The study of the dynamics of a thin liquid film has been substantially investigated in the last few decades. Usually it incorporates some aspects of modeling, mathematical analysis and simulation connected with the respective physical experiments. By use of the fluid mechanics fundamentals the obtained results for the thin liquid film behavior received a wide variety of industrial, biological and

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medical applications [1], [2]. The films of non-Newtonian liquids have received much less attention, although their application as ceramic or polymer melts, liquid metals, suspensions, biological solutions, etc. [3], [4], [5].

The thin fluid film (layer) problems can be referred to a class of fluid mechanics problems for which the thickness disparity of the length scale (small film thickness with respect to its length) is an advantage to reduce with one dimension the correspondent system of partial differential equations describing the fluid flow (and eventually the temperature or concentration distribution). Finally, this leads to one highly non-linear evolution equation or to a system of such equations. In general these equations have no analytical solution and appropriate numerical schemes are applied to solve them. If there exists a static shape of the film, the main question is to examine this shape for stability, when subjected to different types of perturbations.

There exist two generic types of thin films: supported films, spread on solid substrates and free films – bounded by two free surfaces. The free thin films are considered as films of symmetric systems. They could be infinitely long [6], or of finite length, as obtained experimentally in engineering devices or in foams [3]. The latter films may have fixed boundary conditions on both sides perpendicular to the film middle plane [1], [7], [8].

Linear and non-linear stability analysis of different factors is usually accounted. The destabilizing attractive intermolecular forces of van-der Waals are expressed by many authors as an extra pressure (disjoining pressure) for periodic or semi-infinite film [6], [9], [10] and for laterally bounded [7], [8], [11], [12]. The Marangoni soluto-capillary convection due to insoluble surfactants presence on the film interfaces (an additional PDE for the surfactant concentration to the PDE dynamic system) for a periodic film [9] and for a laterally bounded film [13] could be regarded as a stabilizing effect. The heat transfer with the surrounding gas (an additional PDE for the temperature coupled with the dynamic system) and its effect on the film stability for the periodic film has been studied in [14] and for the laterally bounded film in [12].

The dynamics of non-Newtonian films with nonlinear stress-strain relation has been treated in [3], [4] for periodic films and in [5] for laterally bounded films.

In the present paper we shall present in a systematic way the problems connected with the statics, dynamics, linear and non-linear stability of a finite free film under the action of intermolecular forces. The problems of heat and surfactant presence will be discussed. Finally, some non-Newtonian models will be applied when dealing with polymer liquid or blood.

2. The model

2.1. Governing equations

In this paper we shall concentrate only on free films whose middle surface is planar and one of the lateral dimensions is much larger than the other one. Then the film could be regarded as 2D, i.e., in (x, z) space, where x is the lateral coordinate and z is the transverse coordinate. The fluid is assumed as incompressible with density ρ, but with variable dynamic viscosity, named apparent viscosity, μ_{app}, which is constant for the Newtonian films and is a function of the shear rate for the non-Newtonian films. The 2D equations of motion and continuity in vector form are the following:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \nabla\cdot\mathbf{T}, \tag{1}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2}$$

where $\mathbf{v} = (u, w)$ is the velocity vector, p is the pressure and the viscous stress tensor is $\mathbf{T} = 2\mu_{app}(\dot{\gamma})\mathbf{E}$ with $\mathbf{E}$ as the strain rate tensor and $\dot{\gamma}$ is the shear rate given by the relation:

$$\dot{\gamma}^2 = 2Tr\mathbf{E}^2 = 2(u_x^2 + w_z^2) + (u_z + w_x)^2$$

For the non-isothermal film, the heat transfer equation is added:

$$\rho c\left(\frac{\partial \theta}{\partial t} + \mathbf{v}\cdot\nabla\theta\right) = -\nabla\cdot\mathbf{q}, \tag{3}$$

with c standing for the heat capacity, θ – for the temperature, $\mathbf{q} = -\kappa\nabla\theta$ – for the heat flux and κ is the thermal conductivity.

2.2. Boundary conditions

The free film is considered to be symmetrical of thickness h, mean thickness $\bar{h}$ and length a, where $a \gg \bar{h}$. The aspect ratio of both dimensions is then taken as a small parameter $\varepsilon = \bar{h}/a \ll 1$. The free film symmetrical surfaces are defined as $z = \pm h/2$, where $h(x, t)$ is regarded as the unknown function of the film shape.

The symmetry assumed along the middle plane $z = 0$ leads to the following relations for the unknown functions at $z = 0$: $w = 0$, $u_z = p_z = \theta_z = 0$. At the lateral boundaries, $x = \pm a$, the film could have fixed wetting angles $h_x = \pm\varepsilon\tan\alpha$, where α is the complimentary wetting angle, and unknown width or could have a right wetting angle and known width $h(\pm 1, t) = \varepsilon a$. In the first case the non-slip condition is applied for the velocities $\mathbf{v} = \mathbf{0}$ and a fixed

temperature $\theta = \theta_1$, where θ_1 is the boundary temperature. In the second case the velocity is fixed, e.g. $u(\pm 1, t) = \pm U$ for symmetrical drainage, but could be more complicated (in the present paper we do not consider this case, for further reading cf. [11]).

On the free film surfaces $z = \pm h/2$ the kinematic boundary condition $h_t + uh_x = \pm 2w$ is applied. The dynamic normal boundary condition is $\mathbf{T.n.n}|_{outer}^{film} - \Pi = 2\sigma H$, where $\mathbf{t}$, $\mathbf{n}$ are the tangent and normal unit vectors, H is the surface mean curvature, σ is the surface tension and Π is the disjoining pressure. The surface tension could be constant $\sigma = \sigma_0$, but could depend on the temperature $\sigma = \sigma_{t0} - \varphi(\theta - \theta_0)$, where σ_{t0} is the surface tension at temperature θ_0 and φ is the rate of the surface tension dependence on the temperature. The surface tension could also depend on the surfactant, if present: $\sigma = \sigma_{s0} + \Gamma\sigma_\Gamma|_{\Gamma=0}$, where σ_{s0} is the surface tension of a clean interface at $\Gamma = 0$. The dynamic tangential boundary condition on the mobile interfaces reads $\mathbf{T.n.t} = (\nabla_s \sigma)\mathbf{.t}$. The heat transfer with the surrounding gas is given by the relation $\mathbf{q.n} = \beta(\theta - \theta_0)$. For the insoluble surfactants, the surfactant concentration $\Gamma(x, t)$ transport along the interface leads to the following equation:

$$\frac{\partial \Gamma}{\partial t} + \nabla_s \mathbf{.}(\Gamma \mathbf{v}) = D_s \nabla_s^2 \Gamma, \tag{4}$$

where ∇_s is the surface gradient and D_s is the surface diffusion coefficient of the surfactant.

For the non-Newtonian liquid films usually several generalized models are applied [15], [16], [3]: the power law viscosity model

$$\mu_{app} = \mu_p = K(\dot{\gamma}^2)^{(n_p - 1)/2} \tag{5}$$

where K and n_p are given parameters; the Carreau viscosity model

$$\mu_{app} = \mu_c = \mu_\infty + (\mu_0 - \mu_\infty)[1 + \lambda^2 \dot{\gamma}^2]^{(n_c - 1)/2} \tag{6}$$

where λ and n_c are parameters empirically determined and μ_0, μ_∞ are the upper and lower limits of the viscosity correspondent to low and high shear rates.

The disjoining pressure is represented as a sum [11]:

$$\Pi = \Pi_1 + \Pi_2 + \Pi_3,$$

where $\Pi_1 = \dfrac{A_{tot}}{6\pi h^3}$ is the van der Waals and hydrophobic attraction disjoining pressure, $A_{tot} = A_H + K_{323}$ is the sum of the Hamaker const. and the hydrophobic const, Π_2 and Π_3 are the electrostatic and steric repulsion disjoining pressure. The attraction disjoining pressure Π_1 is the most dangerous one for the film rupture, further included in the analysis.

2.3. Thin film equations

The following scaling is applied to dimensionalize all the variables and unknowns: a for the length, U for rhe velocity, a/U for the time, εa for the film thickness, μ for the viscosity, $\dfrac{\sigma\varepsilon}{2a}$ for the disjoining pressure, $\Delta\theta = \theta_1 - \theta_0$ for the temperature and Γ_0 (initial concentration) for the surfactant concentration. Since $z = O(\varepsilon)$ then u, θ, p and w are developed in power series of z: even and odd, respectively. From (1) - (6) and the boundary conditions the following dimensionless 1D system of $O(\varepsilon)$ is obtained:

$$h_\tau + (uh)x = 0, \tag{7}$$

$$u_\tau + uu_x = \frac{\varepsilon}{We}h_{xxx} + \frac{4}{Re\,h}(h\bar{\mu}_{app}u_x)_x + \frac{A}{h^4}h_x - \frac{2M_t}{h}T_x - \frac{2M_s}{h}\gamma_x, \tag{8}$$

$$T_\tau + uT_x = \frac{1}{Peh}(hT_x)_x - \frac{Bi}{Peh}T, \tag{9}$$

$$\gamma_\tau + (\gamma u)_x = \frac{1}{Pe_s}\gamma_{xx}, \tag{10}$$

$$\bar{\mu}_{app} = \bar{\mu}_p = \tilde{K}(2u_x)^{n_p-1}, \tag{11}$$

$$\bar{\mu}_{app} = \bar{\mu}_c = 1 + \left(\frac{\mu_0}{\mu_\infty} - 1\right)[1 + 4\tilde{\lambda}^2 u_x^2]^{(n_c-1)/2}, \tag{12}$$

where τ is the dimensionless time, γ is the dimensionless surfactant concentration, T is the dimensionless temperature, etc. The correspondent dimesionless numbers in the upper system are: $Re = \rho aU/\mu$ – the Reynolds number, $We = ReCa$ – the Weber number, $Ca = 2\mu U/\sigma_0$ – the capillary number, $A = A_H/2\pi\varrho U^2a^3\varepsilon^3$ – the dimensionless Hamaker constant, $M_t = \varphi(\theta_1 - \theta_0)/\sigma_0\varepsilon We$ – the thermal Marangoni number, $M_s = -\dfrac{1}{\varepsilon We}\dfrac{\Gamma_0}{\sigma_0}\sigma_\Gamma|_{\Gamma=0}$ – the solutal Marangoni number, $Pe = PrRe$ – the Peclet number, $Pr = \mu/\rho\kappa$ – the Prandtl number, $Bi = 2\beta a/\varepsilon\kappa$ – the Biot number, $Pe_s = ScRe$ – the solutal Peclet number and $Sc = \dfrac{\mu}{\rho D_s}$ – the Schmidt number, $\tilde{\lambda} = \dfrac{\lambda U}{a}$, $\tilde{K} = \dfrac{K}{\mu_\infty}\left(\dfrac{U}{a}\right)^{n_p-1}$.

The boundary conditions for the films with fixed wetting angles are transformed into the relations: $h_x(\pm1, \tau) = \pm\tan\alpha$, $u(\pm1, \tau) = 0$, $T(\pm1, t) = 1$, $\dfrac{\partial\gamma}{\partial x}(\pm1, \tau) = 0$, $\displaystyle\int_{-1}^{1} h dx = W_0$, where W_0 is the initial film volume conserved in time.

The initial conditions are given in a general form, since they are different for the relaxation and for the stability problem: $h(x, 0) = h_0(x), u(x, 0) =$

$u_0(x), T(x,0) = T_0(x), \gamma(x,0) = \gamma_0(x)$. The functions $h_0(x)$, $u_0(x)$, $T_0(x)$ and $\gamma_0(x)$ will be specified in the next sections.

3. Stability analysis

3.1. Static solution and relaxation

The static solution is obtained as a solution of the relaxation problem with the initial conditions: $h_0(x) = 1, \quad u_0(x) = 0, \quad T_0(x) = 0, \quad \gamma_0(x) = 1$ at $W_0 = 2$. For $\alpha > 0$ there exist only numerical static solutions at different values of B, Ma_s or Ma_t and Bi ($B = WeA/\varepsilon$, $Ma_s = WeM_s/\varepsilon$, $Ma_t = WeM_t/\varepsilon$). This problem is solved numerically: by a finite-difference conservative implicit scheme on staggered grids of accuracy $O((\Delta x)^2 + \Delta t)$ (full description in [17]) and steps $\Delta x = 0.01$, $\Delta t = 10^{-5}$ or 10^{-6}; by the Crank-Nicolson method in combination with the Newton linearization of accuracy $O((\Delta x)^2 + (\Delta t)^2)$ [11] and $\Delta x = \Delta t = 10^{-4}$. The results by both methods are similar. However, the first method is faster, while the second one is more accurate near the rupture points.

If no disjoining pressure is present, i.e., $A = 0$, and the film is isothermal and clean, the static solution is analytical:

$$h_s(x) = 0.5(x^2 - 1/3)\tan\alpha + 0.5W_0, \qquad u_s(x) \equiv 0.$$

It occurs that the static shape (hereafter denoted as $h_s(x)$) existence is limited only for some values of B and α, which is shown in Figure 1 At $W_0 = 2$ and $B \geq 9$ the static film shapes exist only for $\alpha = 0$.

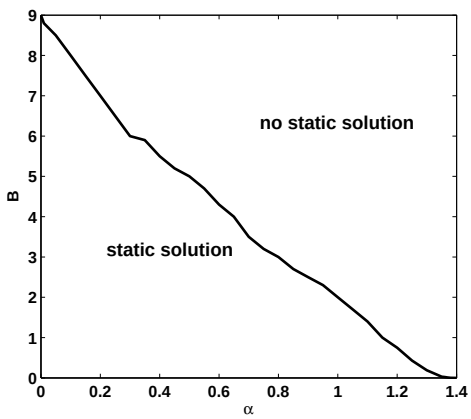


Figure 1: Static solution limit $\alpha(B)$ for $h_s(x)$ at $W_0 = 2$

If the film shape is static for clean interfaces, it also remains static for surfactant coated interfaces at different values of the solutal Marangoni number M_s. For the non-isothermal film, if the film shape is static for $Ma_t = Bi = 0$, it remains also static at $Ma_t > 0$ and $Bi > 0$. If at $Ma_t = Bi = 0$ the film ruptures, at $Ma_t > 0$ and $Bi > 0$ the rupture time increases and even a static shape can be reached.

3.2. Linear stability theory

The following small disturbances $\tilde{h}(x,t)$, $\tilde{u}(x,t)$ are imposed on the static solution of the clean isothermal film problem:

$$h = h_s + \tilde{h}, \qquad u = \tilde{u} \tag{13}$$

After inserting the upper form of the film shape and velocity in (7), (8) at $M_s = 0$, $M_t = 0$ and neglecting the 2^{-nd} order disturbances, the following linearized system is obtained:

$$\frac{\partial \tilde{h}}{\partial t} + \frac{\partial (h_s \tilde{u})}{\partial x} = 0, \tag{14}$$

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\varepsilon}{We}\frac{\partial^3 \tilde{h}}{\partial x^3} + \frac{4}{Re\, h_s}\frac{\partial}{\partial x}(h_s \frac{\partial \tilde{u}}{\partial x}) + \frac{A}{h_s^4}\frac{\partial \tilde{h}}{\partial x} - \frac{4A\tilde{h}}{h_s^5}\frac{\partial h_s}{\partial x}, \tag{15}$$

$$\tilde{u}(\pm 1, t) = 0, \qquad \frac{\partial \tilde{h}}{\partial x}(\pm 1, t) = 0 \tag{16}$$

The disturbances are sought in the following form for $\tilde{h}$ and $\tilde{u}$:

$$\tilde{h} = H(x)\exp(\omega t), \qquad \tilde{u} = V(x)\exp(\omega t), \tag{17}$$

where $\omega = \omega_r + i\omega_i$.

Finally the system (14)–(16) is reduced to the eigenvalue equation for $\Omega = \dfrac{We}{\varepsilon Re}\omega$:

$$h_s(h_s V)^{IV} - 4\Omega(h_s V')' + P\Omega^2 h_s V + \frac{B}{h_s^3}(V h_s)'' - \frac{4Bh_s'}{h_s^4}(V h_s)' = 0, \tag{18}$$

where $P = \dfrac{\varepsilon Re^2}{We}$ and the boundary conditions are as follows:

$$V(\pm 1) = 0, \qquad (h_s V)''(\pm 1) = 0 \tag{19}$$

At $\alpha = 0$ the eigenvalue problem (18)–(19) has the analytical solution:

$$\Omega_{1,2,3,4} = \frac{-m_{1,2}^2\left(2 \mp \sqrt{4 - P + \frac{BP}{m_{1,2}^2}}\right)}{P}, \tag{20}$$

where $m_1 = (2k_1 + 1)\pi/2$, $(k_1 = 0, 1, \ldots)$ corresponds to asymmetrical disturbances, while $m_2 = k_2\pi$, $(k_2 = 1, 2, \ldots)$ – to symmetrical disturbances. The "cutoff" wave number m_c at which $\Omega = 0$ is $m_c = \sqrt{B}$. This is exactly the result obtained in [6] for infinite films. At given P and B, the eigenvalue with the greatest real part corresponds to Ω_1 at $k_1 = 0$, $(m_1 = \pi/2)$.

For $\alpha > 0$ there are no analytical solution. The eigenvalues are sought numerically by the differential Gauss elimination method [12], [18], [19].

If a surfactant is present or the film is non-isothermal, the linear stability analysis is performed in the same way. Small disturbances are imposed on the static surfactant ($\gamma_{static} = 1$) or temperature ($T_{static} = T_s$)

$$\gamma = 1 + \tilde{\gamma}, \qquad T = T_s + \tilde{T} \tag{21}$$

$$\tilde{\gamma} = G(x)\exp(\omega\tau), \qquad \tilde{T} = F(x)\exp(\omega\tau) \tag{22}$$

An extra equation and some extra terms are added to the eqs.(18)–(19):

- in the surfactant case:

$$V^{IV} - \frac{4\Omega}{h_s^2}(h_s V')' + \frac{P\Omega^2}{h_s}V + \frac{B}{h_s^5}(Vh_s)'' - \frac{4Bh_s'}{h_s^6}(Vh_s)' + \frac{2Ma\Omega}{h_s}G' = 0,$$

$$G'' - \Omega PScG - Pe_s V' = 0, \tag{23}$$

$$V(\pm 1) = 0, \tag{24}$$

$$(h_s V)''(\pm 1) = 0, \tag{25}$$

$$G'(\pm 1) = 0, \tag{26}$$

At $\alpha = 0$ the eigenvalue problem (23)–(26) leads to the characteristic equation for Ω:

$$\Omega^3 + \frac{m^2}{P}(\frac{1}{Sc} + 4)\Omega^2 + \frac{m^2}{P}(m^2 - B + \frac{4m^2}{PSc} + 2Ma)\Omega + \frac{m^4}{ScP^2}(m^2 - B) = 0,$$

where $m = (2k_1 + 1)\pi/2$, $(k_1 = 0, 1, \ldots)$ corresponds to asymmetrical disturbances of the film shape and $m = k_2\pi$, $(k_2 = 1, 2, \ldots)$ – to symmetrical disturbances. Here again the cutoff wave number is $m_c = \sqrt{B}$. Thus, the surfactant presence does not influence the critical wave number, but it influences the values of the real root, Ω_r.

- in the non-isothermal case

$$\omega H = -(h_s V)' \tag{27}$$

$$\omega V = \frac{\varepsilon}{We} H''' + \frac{4}{Reh_s}(h_s V')' - \frac{2M_t}{h_s} F' + \frac{A}{h_s^4} H' + (\frac{2M_t T_s'}{h_s^2} - \frac{4Ah_s'}{h_s^5}) H$$

$$\omega G = -T_s' V + \frac{1}{Peh_s}(h_s F')' + \frac{1}{Peh_s}(T_s' H)' - \frac{Bi}{Peh_s} F - \frac{[(h_s T_s')' - BiT_s]}{Peh_s^2} H$$

with boundary conditions:

$$H'(\pm 1) = 0, \quad V(\pm 1) = 0, \quad F(\pm 1) = 0. \tag{28}$$

At $\alpha = 0$ and $Bi = 0$ (similar as in [14] for infinite sheet) the eigenvalue equation is independent of the Marangoni number M_t:

$$\omega^3 + (\frac{4}{Re} + \frac{1}{Pe}) m^2 \omega^2 + (\frac{4m^2}{RePe} + \frac{m^2 \varepsilon}{We} - A) m^2 \omega + \frac{m^4}{Pe}(\frac{m^2 \varepsilon}{We} - A) = 0 \tag{29}$$

with $\omega_1 = -m^2/Pe$ being the thermal root and the two other roots depending on the van der Waals force are given by eq. (20). The cutoff wave number is the same as for isothermal film. Thus, for plane static films, the thermal instability always decays, while the film shape instability may rupture the film.

3.3. Non-linear stability theory

With the linear stability analysis the static shape, the velocity, the surfactant or the temperature could be analyzed for stability, but the final shape, velocity, surfactant or temperature can be obtained only by the non-linear stability analysis. The non-linear stability analysis is based on the numerical solution of the dynamic problem (7), (8),(10) at the surfactant presence with the initial conditions:

$$h_0(x) = h_s(x) + 0.1 \sin(\pi/2x), \quad u_0(x) = 0, \quad \gamma_0(x) = 1. \tag{30}$$

On Table 1, the final time of the film rupture is given as a function of the solutal Marangoni number Ma_s. The surfactant concentration augments the final time for both states: static film shape and film rupture. The addition of surfactants in the film does not change the stability of the film, that is the film rests stable or unstable. The surfactants only affect its relaxation or rupture time.

Table 1

Ma_s	0	0.1	1
$\tau_{rupture}/\alpha = 0$	6.48	6.81	11.44
$\tau_{rupture}/\alpha = 0.5$	60.59	68.92	155.26
$\tau_{rupture}/\alpha = 0.8$	52.86	59.15	123.2

Table 2
$Re = Pe = 1$, $We = 0.01$, $\varepsilon = 0.01$, $A = 3$ $(B = 3)$

Ma_t, Bi	$Ma_t.Bi = 0$	$Ma_t = Bi = 0.1$	$Ma_t = 1$, $Bi = 0.1$	$Ma_t = 0.1$, $Bi = 1$	$Ma_t = Bi = 1$
$\alpha = 0/type$	rupture	rupture	rupture	rupture	rupture
τ_{end}/x_{min}	6.5/-1	6.5/-1	6/-1	6.2/-1	4/-1
$\alpha = 0.6/type$	static	static	rupture	static	rupture
τ_{end}/x_{min}	272.6/0	286/0	118.2/-0.425	1.1/0	25.2/-0.525
$\alpha = 0.8/type$	rupture	rupture	rupture	rupture	rupture
τ_{end}/x_{min}	52.9/-0.185	53.1/-0.195	53.9/-0.205	53.8/-0.205	37.9/-0.315

Similarly to the surfactant problem the numerical solution of the thermodynamic problem (7) – (9) with different disturbances on the static solution:

$$h_0(x) = h_s(x) + 0.1\sin(\pi/2x), u_0(x) = 0, \qquad T_0(x) = T_s(x). \tag{31}$$

are shown on Table 2.

If the thermal Marangoni number Ma_t is bigger, i.e., at the higher temperature difference $\triangle\theta = \theta_1 - \theta_0$, it will promote instability and lead to rupture process. Here again, the wave number $m = \pi/2$ is "the most dangerous" one. If other initial disturbances are applied, e.g. $h_0(x) = h_s(x)$, $u_0(x) = 0.1\cos(\pi/2x)$, $T_0 = T_s + 0.1\cos(\pi/2x - \delta\pi/2)$, where $\delta = 0, \pm 1$, a delay of rupture or a rapid return to stability is observed, which is shown on Table 3.

Table 3

α	Bi	$\delta = -1$	$\delta = 1$	$\delta = 0$
		τ_{rup}/x_{rup}	τ_{rup}/x_{rup}	τ_{rup}/x_{rup}
0	0.1	20.1/-1	16.8/-1	18.4/-1
0	1	12.1/-1	10.4/-1	11.2/-1
0.2	0.1	43.7/-0.845	35.9/-0.845	39.5/-0.845
0.2	1	22.3/-0.915	19.0/-0.915	20.6/-0.915
0.8	1	static	182.7/-0.315	204.4/-0.315

It is evident that for all wetting angles the time for film rupture is shorter if there is a positive phase shift of the temperature disturbance with respect to

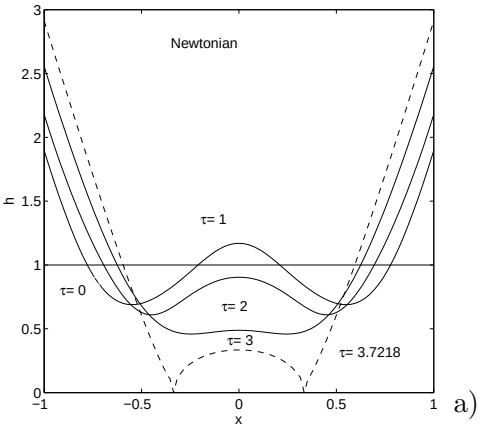
the velocity one. However, for the negative phase shift the rupture time is longer or even the rupture could be avoided, i.e., the film reaches its static shape (at $\alpha = 0.8$ and $Bi = 1$). The respective rupture points remain unchanged for the different phase shifts. This eventual delay or enhancement of the rupture onset by the initial thermal disturbances, which are in a phase shift with respect to the velocity disturbances, has been recently observed in [14] for an infinite periodic film and in [20] for laterally bound film.

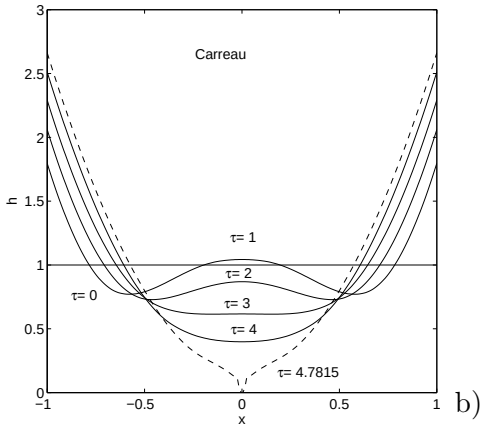
4. Non-Newtonian films

In the case of non-Newtonian films, the viscosity being a function of the shear rate, increases the nonlinearity of the dynamic problem. However, the static film shapes are the same as for the Newtonian film and the linear stability analysis gives also the same results. The nonlinear stability analysis shows a slight delay of the rupture process. In Figure 2a), 2b) and 2c) the shape evolution in time of a blood film is presented in the following cases: the Newtonian, the Carreau and the Power law viscosity models. The other parameters are kept the same: $A = 0.01$ ($B = 2.47$), $Re = 247$ and $\alpha = 1.37$, with initial conditions: $h_0(x) = 1$ and $u_0(x) = 0$. It is obvious that for all models the film ruptures, but with different rupture times and different types of rupture. The Newtonian film ruptures faster forming a "dimple" rupture at the points $x = \pm 0.335$. The Carreau model and the Power law model films show a "pimple" rupture at the center of the film $x = 0$.

5. Conclusions

A model describing the drainage and relaxation dynamics of symmetric thin liquid films with tangentially mobile surfaces is developed. The model describes the film profile, surface velocity and pressure for arbitrary values of the Reynolds and the capillary number, accounting for different components of the disjoining pressure. Films with fixed wetting angles are studied. The dynamics and heat transfer of a non-isothermal free thin viscous film, laterally bounded is studied numerically. The van der Waals attractive disjoining pressure and the Marangoni thermocapillary convection are taken into account. The action of insoluble surfactans, spread along the interfaces of a free thin film, is incorporated into the model. Two final states exist: static film shape and film rupture; the increase of the Marangoni number could change the rupture state into a static one when the other parameters are left the same. The linear stability of a free thin film with respect to an exponentially growing perturbation is studied at different wetting angles and different values of the dimensionless van der Waals attraction coefficient.





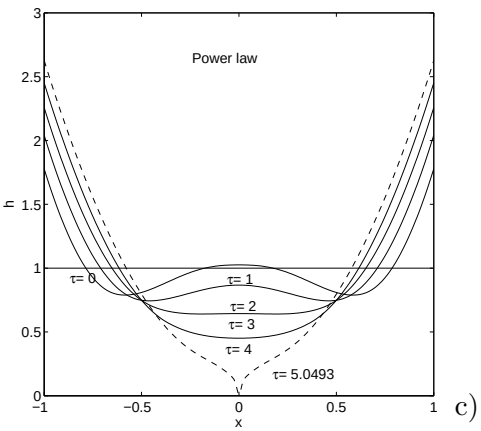


Figure 2: Shape evolution in time of a blood film starting from a planar shape at $\tau = 0$ and $A = 0.01$ ($B = 2.47$), $Re = 247$, $\alpha = 1.37$: a) Newtonian model; b) Carreau model (The final shapes at rupture are given as dashed lines.); c) Power law model

The non-linear stability analysis confirms approximately the results of the linear stability analysis. The importance of the wetting angle is confirmed: the film can be stable or unstable depending on the competition between the wetting with the lateral boundary, the thermo-capillary convection or soluto-capillary convection and the van der Waals attraction. The governing equations for the non-Newtonian thin liquid film at the van der Waals forces action are derived. Two non-Newtonian models are discussed: Power law and Carreau model. Power law model leads to viscosity singularity when $\dot{\gamma} \to 0$; Carreau model does not exhibit any viscosity singularity. At the calculated thinning rates, the two types of rupture: "pimple" and "dimple" can occur depending on the model used. It is expected the non-Newtonian viscosity to have a more stabilizing effect on the film drainage. The blood film results can be applied to some biomechanical problems

connected with humans. An idea to control the rupture process by appropriate initial thermal disturbances is proposed, which will be further developed in our future studies.

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