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BIFURCATIONS IN KURAMOTO–SYVASHINSKY EQUATION*

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Kuramoto–Sivashinsky equation with periodic boundary-value conditions is considered. The stability of the homogeneous equilibrium is investigated as well as the local bifurcation of the spatially nonhomogeneous t-periodic solutions. It is shown that the two-dimensional invariant manifolds are composed of these solutions. These manifolds can be stable or unstable, but all solutions belonging to these manifolds are always unstable.

The bifurcation problem can be reduced to investigate certain system of ordinary differential equations (normal form). This normal form was constructed by a modified Krylov–Bogolubov algorithm. These normal forms can be used to explain a ripple topography induced by ion bombardment.

1. Introduction

Consider the equation

$$h_t = -\nu_0 + \gamma h_x + \nu_1 h_{xx} + \nu_2 h_{yy} + \sigma_1 h_{xxx} + \sigma_2 h_{xyy} - D_1 h_{xxxx} - \\ -D_2 h_{yyyy} - D_3 h_{xxyy} + \gamma_1 h_x^2 + \gamma_2 h_y^2 + \xi_1 h_x h_{xx} + \xi_2 h_x h_{yy}, \tag{1.1}$$

where $h = h(t, x, y)$ defines the shape of the ion bombardment. It is known as the Bradley–Harper (BH) equation [1] or the generalized Kuramoto–Sivashinsky equation [2]. The coefficients of this equation depend on parameters of the phenomenon. A detailed discussion of physical aspects can be found in [1–4] (see also

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the references in these articles). The positive value ν_0 is known as the erosion coefficient. D_1, D_2, D_3 are also positive. They depend on the surface diffusion of the material. At last, $\gamma, \nu_1, \nu_2, \sigma_1, \sigma_2, \gamma_1, \gamma_2, \xi_1, \xi_2 \in R$, but $\gamma_1 \neq 0$. If the deformations $h(t,x,y)$ are independent of y (the deformations are "cylindrical"), then equation (1.1) may be rewritten in the following form

$$h_t = -\nu_0 + \gamma h_x + \nu_1 h_{xx} + \sigma_1 h_{xxx} - D_1 h_{xxxx} + \gamma_1 h_x^2 + \xi_1 h_x h_{xx}. \tag{1.2}$$

This equation (1.2) (also (1.1)) has the solution $h(t,x) = -\nu_0 t + \nu_3$, where $\nu_3 \in R$. For example, we can set $\nu_3 = 0$, since its value depends on the chosen system of coordinates.

Define $h(t,x) = u(t,x) - \nu_0 t$. Then the corresponding equation is

$$u_t = \gamma u_x + \nu_1 u_{xx} + \sigma_1 u_{xxx} - D_1 u_{xxxx} + \gamma_1 u_x^2 + \xi_1 u_x u_{xx}. \tag{1.3}$$

It is convenient and reasonable to change the variable as $x = h_1 x_1,\ t_1 = h_2 t, h_1, h_2 > 0$. In new variables, the equation (1.3) is rewritten as

$$u_{t_1} = a_1 u_{x_1} - b u_{x_1 x_1} - u_{x_1 x_1 x_1 x_1} + a_2 u_{x_1 x_1 x_1} + c_1 u_{x_1}^2 + 2c_2 u_{x_1} u_{x_1 x_1}, \tag{1.4}$$

where

$$a_1 = \frac{\gamma h_2}{h_1},\ \ a_2 = \sigma_1 \frac{h_2}{h_1},\ \ b = -\nu_1 \frac{h_2}{h_1^2},\ \ c_1 = \gamma_1 \frac{h_2}{h_1^2},\ \ c_2 = \frac{\xi_1 h_2}{2h_1^3},\ \ \frac{h_2}{h_1^4} D_1 = 1.$$

Below the indices "1" of the new variables x_1, t_1 are omitted for simplicity.

In this paper, we shall consider the resulting equation (1.4)

$$v_t = a_1 v_x - b v_{xx} - v_{xxxx} + a_2 v_{xxx} + c_1 (v^2)_x + c_2 (v^2)_{xx},\ v = u_x. \tag{1.5}$$

The equation (1.5) is called the Kuramoto–Sivashinsky equation [2,3]. Note that the function $v(t,x)$ has a basic physical sense (see, for example, [1,3]). As usual, we consider the equation (1.5) with periodic boundary-value condition

$$v(t, x + 2\pi) = v(t,x). \tag{1.6}$$

It means that the equation (1.4) is supplemented by boundary-value conditions

$$u_x(t, x + 2\pi) = u_x(t,x). \tag{1.7}$$

Boundary-value problem (1.5), (1.6) has solutions $v(t,x) = h_0$, where h_0 is a real constant. As usual, these solutions are called homogeneous equilibriums. The nonhomogeneous topography leads to x-dependent solutions of the boundary-value problem (1.5), (1.6). Let

$$v(0,x) = f(x) \in H_2^4, \tag{1.8}$$

where H_2^4 is the Sobolev space of 2π-periodic function of x with square integrable distributional partial derivatives up to the fourth order on the closed interval

$[-\pi,\pi]$ $(x \in [-\pi,\pi])$. The condition $f(x) \in H_2^4$ ensures the local solvability of problems (1.5), (1.6), (1.8) (see [5–6]).

In this paper, we analyse the stability of all equilibriums $v = h_0$ of problem (1.5), (1.6), (1.8) and we consider their local bifurcations under the change of their stability. The stability is treated in the sense of the norm of the phase space of solution (the space of the initial conditions). Here,

$$\|f\| = \|f\|_{H_2^4} = \|f\|_{L_2} + \|f'\|_{L_2} + \|f''\|_{L_2} + \|f'''\|_{L_2} + \|f^{(IV)}\|_{L_2}.$$

2. Stability of equilibrium

Let function $v(t,x)$ be a solution of problem (1.5), (1.6), (1.8), where $f(x) \in H_2^4$. In this case, the functions $f(x), v(t,x)$ admit the representation in the form of the Fourier series

$$f(x)=f_0 + g(x), \qquad g(x) = \sum_{n=-\infty, n\neq 0}^{\infty} g_n \exp(inx),$$

$$v(t,x)=w_0(t) + w(t,x), \qquad w(t,x) = \sum_{n=-\infty, n\neq 0}^{\infty} w_n(t) \exp(inx),$$

$$f_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = M_0(f), \qquad g_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\exp(inx)dx,$$

$$w_0(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi} w(t,x)dx = M_0(w), \quad w_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} w(t,x)\exp(inx)dx,$$

where $n = \pm 1, \pm 2, \ldots$

Therefore, $M_0(g) = 0$, $M_0(w) = 0$. The right part of equation (1.5) has the trivial average. Consequently, $\dot{w}_0(t) = 0$ or $w_0(t) = \alpha$, where α is a real constant. It implies that we can rewrite the equation (1.5) in the following form

$$\dot{w}_0 = 0 \ (w_0(t) = \alpha), \tag{2.1}$$

$$w_t = Aw + c_1[(\alpha + w)^2]_x + c_2[(\alpha + w)^2]_{xx}, \tag{2.2}$$

$$w(t, x + 2\pi) = w(t,x), M_0(w) = 0, \tag{2.3}$$

where the linear differential operator (LDO)

$$Ay = -y^{(IV)} - by'' + a_1 y' + a_2 y'''$$

is defined at smooth 2π-periodic function $y(x)$. Moreover, we can rewrite the problem (2.2), (2.3) in a convenient form for the investigation

$$w_t = A(\alpha)w + c_1(w^2)_x + c_2(w^2)_{xx}, \tag{2.4}$$

$$w(t, x + 2\pi) = w(t, x), M_0(w) = 0, \tag{2.5}$$

where $b(\alpha) = b - 2c_2\alpha, a(\alpha) = a_1 + 2c_1\alpha$,

$$A(\alpha)w = w_{xxxx} - b(\alpha)w_{xx} + a(\alpha)w_x + a_2 w_{xxx},$$

Let $v(t, x) = \alpha + w(t, x)$ be a solution of problem (2.1), (2.2), (2.3). Then $w(t, x) = w(t, x, \alpha)$ is the solution of auxiliaries problem (2.4), (2.5) for selected $\alpha \in R$. For the investigation of its stability, we must analyse the stability of the trivial solution ($w = 0$) of the problem (2.4), (2.5). To analyse the stability of this solution in the first approximation, we consider the linearised boundary-value problem

$$w_t = A(\alpha)w, \tag{2.6}$$

$$w(t, x + 2\pi) = w(t, x), M(w) = 0. \tag{2.7}$$

Lemma 1. *This operator $A(\alpha)$ has the eigenvalues*

$$\lambda = \lambda(n, \alpha) = b(\alpha)n^2 - n^4 + i(a(\alpha)n - a_2 n^3),$$

corresponding to the eigenfunctions $e_n(x) = \exp(inx), n \in Z \setminus \{0\}$.

The proof is by direct calculation and it based on the fact that the orthogonal system of functions $e_n(x)(n \neq 0)$ is complete in $L_2(-\pi, \pi), M_0(g) = 0$.

Clearly, if $Re\lambda(n, \alpha) < 0$ for all n in question, then all solutions of problem (2.6), (2.7) are exponentially stable. In particular, for all solutions of this problem the inequality $\|w\|_{H_2^4} \leq M \exp(-\gamma t)\|g\|_{H_2^4}$ is hold.

Here, $g(x) = w(0, x), M = M(\gamma) > 0, \ -\gamma = \max\limits_n(Re\lambda(n, \alpha))$ $(i.e. \gamma > 0)$.

Lemma 2. *Let $\|g\|_{H_2^4} \leq \delta$ and $w(t, x)$ is a solution of problem* (2.4), (2.5) *with the condition $w(0, x) = g(x)$. These solutions admit the following estimation*

$$\|w\|_{H_2^4} \leq M_1 \exp(-\gamma_1 t)\|g\|_{H_2^4}.$$

In particular, the solution $w = 0$ is exponentially (asymptotically) stable. Here, $M_1 = M + \delta_1, \gamma_1 = \gamma - \delta_1, \delta_1 = \delta_1(\delta), \lim\limits_{\delta \to 0} \delta_1(\delta) = 0$.

The proof of this lemma is the corollary from the Lemma 1 (see [7–9]).

Assume that there exists n_0 such that $Re\lambda(n_0, \alpha_0) > 0$. In this case, the equilibrium $w(t, x) = 0$ is unstable for the problems (2.6),(2.7) and (2.4),(2.5). The critical cases are singled out the following conditions:

i) For all $n \in Z \setminus \{0\}, Re\lambda(n, \alpha_0) \leq 0$.

ii) There exist $n = n_0 \in Z \setminus \{0\}$, such that $Re\lambda(n_0, \alpha_0) = 0$.

It should be stressed that in our case, $Re\lambda(n,\alpha) = -n^4 + b(\alpha)n^2$. Therefore, the stability condition becomes

$$b(\alpha) < 1 \ (b - 2c_2\alpha < 1).$$

Consequently, we have a critical case if $b(\alpha) = 1$ $(b - 2c_2\alpha = 1)$. In this critical case, LDO $A(\alpha)$ has two eigenvalues $\lambda_{\pm 1} = \lambda(\pm 1, \alpha) = \pm i\omega, \omega = a(\alpha) - a_2$, corresponding to the eigenfunctions $\exp(\pm ix)$.

Consider a perturbed case for the problem (2.4), (2.5) and let

$$a(\varepsilon) = a(\alpha)(1 + \beta_2\varepsilon), b(\varepsilon) = 1 + \beta_1\varepsilon, \beta_1, \beta_2 \in R, 0 < \varepsilon \ll 1.$$

Define the LDO by the equality

$$A(\varepsilon)y = -y^{(IV)} - b(\varepsilon)y'' + a(\varepsilon)y' + a_2y''',$$

where $y = y(x)$ is a smooth 2π-periodic function. Finally, $A_0y = A(0)y$. The LDO $A(\varepsilon)$ has two eigenvalues

$$\lambda_{\pm 1}(\varepsilon) = \tau(\varepsilon) \pm i\omega(\varepsilon), \tau(\varepsilon) = \beta_1\varepsilon, \omega(\varepsilon) = a(\alpha)(1 + \beta_2\varepsilon),\ \tau(0) = 0,\ \omega(0) = \omega.$$

The other eigenvalues satisfy the inequality $Re\lambda_n(\varepsilon) \le -\gamma < 0, n \ne 0, n \ne \pm 1$.

3. Nonlinear boundary-value problem

In this section, we consider the boundary-value problem (2.4), (2.5) for the perturbed case, i.e.

$$w_t = A(\varepsilon)w + c_1(w^2)_x + c_2(w^2)_{xx}, \tag{3.1}$$

$$w(t, x + 2\pi) = w(t, x). \tag{3.2}$$

It has the two-dimensional invariant manifold (the center manifold) $V_2(\varepsilon)$ (see, for example, [8-10]). Moreover, this manifold $V_2(\varepsilon)$ possesses the following properties:

i) If some solutions of problem (3.1), (3.2) do not belong to $V_2(\varepsilon)$, then these solutions approach it exponentially with time.

ii) If the solutions belong to $V_2(\varepsilon)$ then the behaviour of these solutions are determined by the solutions of the system of two ordinary differential equations which are called the normal form [10–11].

Consider the set of solutions belonging to $V_2(\varepsilon)$. Recall [12] (see also the references in [12]) that these solutions are sought in the form of the sum

$$w(t, x, \varepsilon) = \varepsilon^{1/2}w_0(t, s, x) + \varepsilon w_1(t, s, x) + \varepsilon^{3/2}w_2(t, s, x) + o(\varepsilon^{3/2}), \tag{3.3}$$

where $s = \varepsilon t$, all functions are t-periodic of period $2\pi/\omega$ and satisfy the boundary-value condition (3.2). We put

$$w_0(t, s, x) = z(s)\exp(i\omega t + ix) + \overline{z}(s)\exp(-i\omega t - ix) \tag{3.4}$$

and suppose that the functions w_1, w_2 have the trivial averages, i.e.

$$M_0(w_j)=0, M_{\pm 1}(w_j)=\frac{\omega}{(2\pi)^2}\int\limits_0^{2\pi/\omega}\int\limits_{-\pi}^{\pi} w_j(t,s,x)\times$$
$$\times\exp(\pm i\omega t\pm ix)dxdt=0, j=1,2.$$

Here, $z(s)$ in (3.4) is a complex-valued function which will be defined later.

To determine w_1, w_2, we substitute sum (3.3) in to (3.1) and (3.2) and successively equate coefficients at respective powers of ε, i.e. we use a version of Krylov-Bogolubov algorithm for the formation of the normal form [11]. As a result, we obtain two nonhomogeneous boundary-value problems for w_1, w_2.

For w_1 we obtain the following problem

$$w_{1t}-A_0w_1=c_1(w_0^2)_x+c_2(w_0^2)_{xx}, \tag{3.5}$$

$$w_1(t,s,x+2\pi)=w_1(t,s,x). \tag{3.6}$$

Finally, for w_2 this problem has the following form

$$\begin{gathered} w_{2t}-A_0w_2=2c_1(w_0w_1)_x+2c_2(w_0w_1)_{xx}- \\ -[z'\exp(i\omega t+ix)+\overline{z'}\exp(-i\omega t-ix)], \end{gathered} \tag{3.7}$$

$$w_2(t,s,x+2\pi)=w_2(t,s,x). \tag{3.8}$$

Remark. Let a continuous function $F(t,x)$ satisfies the periodic condition in question and $F(t,x)$ is $2\pi/\omega$-periodic of t, belongs to $H_2^p(p=0,1,2,\dots)$ for fixed t. Then the nonhomogeneous boundary-value problem

$$v_t-A_0v=F(t,x), v(t,x+2\pi)=v(t,x)$$

has a periodic solution of t which the period is equal to $2\pi/\omega$ if the following equalities $M_0(F)=M_{\pm1}(F)=0$ are valid. Here, in this remark

$$M_0(F)=\frac{\omega}{(2\pi)^2}\int\limits_0^{2\pi/\omega}\int\limits_{-\pi}^{\pi}F(t,x)dxdt,$$
$$M_{\pm1}(F)=\frac{\omega}{(2\pi)^2}\int\limits_0^{2\pi/\omega}\int\limits_{-\pi}^{\pi}F(t,x)\exp(\pm i\omega t\pm ix)dxdt.$$

The equalities $M_0(v)=M_{\pm1}(v)=0$ single out one such a solution. The above conditions ($M_0(F)=M_{\pm1}(F)=0$) are known as the solvability conditions for the boundary-value problem in question [13].

Consider the boundary-value problem (3.5), (3.6). It can be shown that the solvability conditions are always hold and the solution of this problem has the following form

$$w_2(t,s,x) = \eta \exp(2i\omega t + 2ix) + \overline{\eta} \exp(-2i\omega t - 2ix),$$

where $\eta = (ic_1 - 2c_2)/(3(2+ia_2))$. Applying the solvability condition to problem (3.7), (3.8) gives that

$$z' = (q_1 + iq_2)z - (l_1 + il_2)z^2\overline{z}, \tag{3.9}$$

where

$$l_1 = \frac{6c_1c_2a_2 - 4(2c_2^2 - c_1^2)}{3(4+a_2^2)},\ l_2 = \frac{12c_1c_2 + 2a_2(2c_2^2 - c_1^2)}{3(4+a_2^2)}, q_1 = \beta_1, q_2 = a(\alpha)\beta_2.$$

We have obtained the ordinary differential equation (3.9). It is the principal part of the normal form in the study of the Andronov-Hopf bifurcation [8]. In fluid dynamic, this equation is known as the Landau equation (see, for example, [14]). Evidently, the following assertion is just (see, for example, [7]).

Lemma 3. *If $q_1 l_1 > 0$, then the equation* (3.9) *has the family of periodic solutions*

$$z(s) = \sqrt{\frac{q_1}{l_1}} \exp(\sigma_1 s + \varphi_1), \sigma_1 = q_2 - l_2\frac{q_1}{l_1}, \varphi_1 \in R.$$

All solutions of this family are stable if $l_1 > 0$ and unstable if $l_1 < 0$. This family of solutions gives us the limit cycle $C_l(\alpha)$.

Using the results of [7–11,13,15], we can proof the following assertion.

Theorem 1. *There exist $\varepsilon_0 > 0$ such that the boundary-value problem* (3.1), (3.2) *has the solutions corresponding to $C_l(\alpha)$ for all $\varepsilon \in (0, \varepsilon_0)$*

$$\begin{aligned} w_\alpha(t,x,\varepsilon) = \sqrt{\frac{q_1}{l_1}}\varepsilon^{1/2}[\exp(i(\omega+\varepsilon\sigma_1)t + ix + i\varphi_1) + \\ + \exp(-i(\omega+\varepsilon\sigma_1)t - ix - i\varphi_1)] + \frac{q_1}{l_1}\varepsilon[\eta\exp(2i(\omega+\varepsilon\sigma_1)t + \\ + 2ix + 2i\varphi_1) + \overline{\eta}\exp(-2i(\omega+\varepsilon\sigma_1)t - 2ix - 2i\varphi_1)] + o(\varepsilon). \end{aligned} \tag{3.10}$$

The cycle $C_p(\alpha)$ of boundary-value problem (3.1), (3.2) *inherits the stability properties of cycle $C_l(\alpha)$ of equation* (3.9)*. Therefore, the cycle $C_p(\alpha)$ exists if $l_1 q_1 > 0$. It is stable if $l_1 > 0$ and unstable if $l_1 < 0$.*

Remark. If $a_2 = c_2 = 0$, then we have the classic case for the Kuramoto–Sivashinsky equation. Then $l_1 = c_1^2/6, l_2 = 0$.

Therefore, the problem (3.1), (3.2) (with α in question) have the stable periodic solution if the trivial equilibrium ($w = 0$) has lost the stability.

Let α be fixed ($\alpha = \alpha_0$). Suppose that in this case the theorem 1 remains valid. Let now $\alpha = \alpha_0 + \alpha_1$, where α_1 is a sufficiently small real constant. Hence, we obtain the cycle $C_p(\alpha)$. This cycle $C_p(\alpha)$ lies in the neighbourhood of $C_p(\alpha_0)$ and the formula (3.10) remains a valid with $q_1(\alpha), q_2(\alpha)$ instead of $q_1(\alpha_0), q_2(\alpha_0)$.

4. Original boundary problem

We now return to the problem (1.5), (1.6). Let $\varepsilon = |(b - 2c_2\alpha) - 1|, \beta_1 = \mathrm{sign}((b_2 - 2c_2\alpha) - 1)$ and suppose that b, α are chosen such that $\varepsilon \ll 1$. The results of the section 2,3 allows us to affirm that the following assertions are valid.

Theorem 2. *There exist $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the boundary-value problem* (1.5), (1.6) *has the two-dimensional invariant manifold $V_2(\alpha)$ is formed by the set of the solutions* (3.10)

$$v(t, x, \varepsilon, \alpha) = \alpha + w_\alpha(t, x, \varepsilon), \tag{4.1}$$

where the function $w_\alpha(t, x, \varepsilon)$ was indicated in Section 3 *by the formula* (3.10)*.*

This manifold $V_2(\alpha)$ exists if $q_1 l_1 > 0$, where the constants q_1, l_1 was determined in Section 3*. If $l_1 > 0$, we can affirm that our manifold $V_2(\alpha)$ is exponential stable. This manifold is unstable if $l_1 < 0$.*

Note that all solutions from the set (3.10) depend on two parameters α and φ_1 (see formula (3.10)) and the periodic solutions (4.1) have the different period $T(\alpha) = 2\pi/\omega(\alpha)$. It implies that $T'(\alpha) \neq 0$.

The reader will easily prove that all solutions belonging to $V_2(\alpha)$ are unstable in the sense of the definition of Lyapunov.

The proof follows from the assertion.

Lemma 4. *Let $\omega, \omega_n \in R, n = 1, 2, 3, \ldots$. Consider the sequence of functions $\psi_n(t) = \cos\omega_n t - \cos\omega t, n = 1, 2, 3, \ldots$. There exists a sequence $\{\omega_n\}$ such that*

$$(i)\ \lim_{n\to\infty} \omega_n = \omega;\ (ii)\ \max_{0\le t\le\infty} |\psi_n(t)| = 2.$$

The proof of this lemma is trivial.

Finally, we have proved that boundary-value problem (1.5),(1.6) can have certain exponentially stable manifold $V_2(\alpha)$, but the solutions belonging to this manifold are unstable.

Corollary. *The boundary-value problem* (1.4), (1.7) *has the manifolds* $V_3(\varepsilon)$ ($\dim V_3(\varepsilon) = 3$) *containing the solutions*

$$u(t, x, \varepsilon, \alpha) = \beta t + \alpha x + u_\alpha(t, x, \varepsilon) + \gamma, \tag{4.2}$$

where $\alpha, \gamma \in R, \beta = a\alpha + a_2\alpha^2$,

$$u_\alpha(t, x, \varepsilon) = \int_0^x w_\alpha(t, y, \varepsilon) dy.$$

All solutions (4.2) *are also unstable as the solutions* (4.1) *of boundary-value problem* (1.5), (1.6).

Note that the periodic boundary-value problem for the equation (1.1) was studied in [12], i.e. in this article, the equation (1.1) was considered with the conditions $h(t, x + 2\pi, y) = h(t, x, y + 2\pi) = h(t, x, y)$.

In the paper [15], the equation (14) with $a_1 = a_2 = 0$ was considered. Such equation was investigated with the boundary-value conditions

$$w_{xx}|_{x=0,x=1} = w_{xxx}|_{x=0,x=1} = 0, \ x \in [0, 1].$$

The similar results was obtained.

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