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EXISTENCE OF CLASSICAL SOLUTIONS OF QUASI-LINEAR NON-COOPERATIVE ELLIPTIC SYSTEMS

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Method of sub- and super-solutions is applied in investigation of solvability in classical $C^2(\Omega)\bigcap C(\overline{\Omega})$ sense of quasi-linear non-cooperative weakly coupled systems of elliptic second-order PDE.

1. Introduction

In this paper is considered a major application of the comparison principle, namely the method of sub- and super-solutions, in order to derive some sufficient conditions for solvability in C^2 of a quasi-linear non-cooperative elliptic system.

Let $\Omega \in R^n$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper are considered quasi-linear weakly-coupled elliptic systems of the type

$$Q^l(u) = -\operatorname{div} a^l(x, u^l, Du^l) + F^l(x, u^1, \dots, u^N, Du^l) = f^l(x) \quad \text{in} \quad \Omega \tag{1}$$

$$u^l(x) = g^l(x) \quad \text{on} \quad \partial\Omega \tag{2}$$

for $l = 1, \dots, N$.

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System (1) is strictly elliptic one, i.e. there are monotonously decreasing continuous function $\lambda(|u|) > 0$ and monotonously increasing one $\Lambda(|u|) > 0$, depending only on $|u| = \left(\left(u^1\right)^2 + \cdots + \left(u^N\right)^2\right)^{1/2}$, such that

$$\lambda(|u|)\left|\xi^l\right|^2 \leq \sum_{i,j=1}^{n} \frac{\partial a^{li}}{\partial p_j^l}(x, u^1, \ldots, u^N, p^l)\xi_i^l \xi_j^l \leq \Lambda(|u|)\left|\xi^l\right|^2 \tag{3}$$

holds for every u^l and $\xi^l = (\xi_1^l, \ldots, \xi_n^l) \in R^n$, $l = 1, 2, \ldots, N$.

Coefficients $a^l(x,u,p)$, $F^l(x,u,p)$, $f^l(x)$ and $g^l(x)$ are supposed at least measurable functions in Ω with respect to x variable, and locally Liepshitz continuous with respect to u^l, u and p, i.e.

$$\begin{aligned} &\left|F^l(x,u,p) - F^l(x,v,q)\right| \leq C(K)\left(|u-v| + |p-q|\right), \\ &\left|a^l(x,u^l,p) - a^l(x,v^l,q)\right| \leq C(K)\left(\left|u^l - v^l\right| + |p-q|\right) \end{aligned} \tag{4}$$

holds for every $x \in \Omega$, $|u| + |v| + |p| + |q| \leq K$, $l = 1, \ldots, N$.

Furthermore we suppose $a^l(x,u,p)$ and $F^l(x,u,p)$ to be differentiable on u^l and p^l, and

$$\frac{\partial a^{li}}{\partial p_j}, \frac{\partial a^{li}}{\partial u^k}, \frac{\partial F^l}{\partial p_l}, \frac{\partial F^l}{\partial u^k} \in L^1(\Omega).$$

Hereafter by $f^-(x) = \min(f(x), 0)$ and $f^+(x) = \max(f(x), 0)$ are denoted the non-negative and, respectively, the non-positive part of the function f. The same convention is valid for matrices as well. For instance, we denote by M^+ the non-negative part of M, i.e. $M^+ = \{m_{ij}^+(x)\}_{i,j=1}^N$.

2. Comparison principle for quasi-linear elliptic systems

Let $u(x) \in \left(C^2(\Omega) \cap C(\overline{\Omega})\right)^N$ be classical sub-solution of (1), (2). Then

$$\int_\Omega \left(a^{li}(x,u^l,Du^l)\eta_{x_i}^l + F^l(x,u^1,\ldots,u^N,Du^l)\eta^l - f^l(x)\eta^l\right) dx \leq 0$$

for $l = 1, \ldots, N$ and for every non-negative vector-function $\eta \in \left(W_c^1(\Omega) \cap C(\overline{\Omega})\right)^N$ (i.e. $\eta = (\eta^1, \ldots, \eta^N)$, $\eta^l \geq 0$, $\eta^l \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ and $\eta^l = 0$ on $\partial\Omega$).

Analogously, let $v(x) \in \left(C^2(\Omega) \cap C(\overline{\Omega})\right)^N$ be a classical super-solution of (1), (2). Then

$$\int_\Omega \left(a^{li}(x, v^l, Dv^l)\eta^l_{x_i} + F^l(x, v^1, \dots, v^N, Dv^l)\eta^l - f^l(x)\eta^l\right) dx \geq 0$$

for $l = 1, \dots, N$ and every non-negative vector-function $\eta \in \left(W^1_c(\Omega) \cap C(\overline{\Omega})\right)^N$.

Recall that the comparison principle holds for (1), (2), if $Q(u) \leq Q(v)$ in Ω and $u \leq v$ on $\partial\Omega$ yields $u \leq v$ in Ω.

Since $u(x)$ and $v(x)$ are sub- and super-solutions, then $\widetilde{w}(x) = u(x) - v(x)$ is weak sub-solution of the following problem

$$-\sum_{i,j=1}^n D_i\left(B^{li}_j D_j \widetilde{w}^l + B^{li}_0 \widetilde{w}^l\right) + \sum_{k=1}^N E^l_k \widetilde{w}^k + \sum_{i=1}^n H^l_i D_i \widetilde{w}^l = 0 \text{ in } \Omega$$

with non-positive boundary data on $\partial\Omega$, i.e.

$$\int_\Omega \left(\sum_{i,j=1}^n \left(B^{li}_j D_j \widetilde{w}^l + B^{li}_0 \widetilde{w}^l\right)\eta^l_{x_i} + \sum_{k=1}^N E^l_k \widetilde{w}^k \eta^l + \sum_{i=1}^n H^l_i D_i \widetilde{w}^l \eta^l\right) dx \leq 0 \text{ in } \Omega.$$

Here

$$\begin{aligned}
B^{li}_j &= \int_0^1 \frac{\partial a^{li}}{\partial p_j}(x, P^l)ds, \quad B^{li}_0 = \int_0^1 \frac{\partial a^{li}}{\partial u^l}(x, P^l)ds, \\
P^l &= \left(v^l + s(u^l - v^l), Dv^l + sD(u^l - v^l)\right) \\
E^l_k &= \int_0^1 \frac{\partial F^l}{\partial u^k}(x, S^l)ds, \quad H^l_i = \int_0^1 \frac{\partial F^l}{\partial p_i}(x, S^l)ds, \\
S^l &= \left(v + s(u - v), Dv^l + sD(u^l - v^l)\right).
\end{aligned}$$

Therefore $\widetilde{w}_+(x) = \max\left(\widetilde{w}(x), 0\right)$ is weak sub-solution of

$$-\sum_{i,j=1}^n D_i\left(B^{li}_j D_j \widetilde{w}^l_+ + B^{li}_0 \widetilde{w}^l_+\right) + \sum_{k=1}^N E^l_k \widetilde{w}^k_+ + \sum_{i=1}^n H^l_i D_i \widetilde{w}^l_+ = 0 \text{ in } \Omega \tag{5}$$

with null boundary data on $\partial\Omega$.

Equation (5) is equivalent to

$$B_E\widetilde{w}_+ = (B+E)\widetilde{w}_+ = 0 \ \text{ in } \ \Omega, \tag{6}$$

where $B = \operatorname{diag}(B_1, B_2, \ldots, B_N)$, $B_l = -\sum_{i,j=1}^{n} D_i\left(B_j^{li} D_j \widetilde{w}_+^l + B_0^{li}\widetilde{w}_+^l\right) + \sum_{i=1}^{n} H_i^l D_i \widetilde{w}_+^l$ and $E = \{E_k^l\}_{l,k+1}^N$.

Then the following theorem (Theorem (8) in [1]) holds:

Theorem 1. *Let* (1), (2) *be quasi-linear system and corresponding system* B_{E^-} *in* (6) *is elliptic one. Then comparison principle holds for system* (1), (2) *if* B_{E^-} *is irreducible one and for every* $j = 1, \ldots, n$ *hold*

$$(i) \quad \lambda + \left(\sum_{k=1}^{N} \frac{\partial F^k}{\partial p^j}(x,p,q^l) + \sum_{i=1}^{N} D_i \frac{\partial a^{ji}}{\partial p^j}(x,p^j,q^j)\right)^+ > 0 \ \textit{ for some } \ x_0 \in \Omega,$$

$$(ii) \qquad \lambda + \left(\sum_{i=1}^{n} D_i \frac{\partial a^{ji}}{\partial p^j}(x,p^j,q^j) + \frac{\partial F^j}{\partial p^j}(x,p,q^j)\right)^+ \geq 0 \ \textit{ for every } \ x \in \Omega,$$

where $p, q \in R^n$ *and* λ *is the first eigenvalue of operator* B_{E^-} *in* Ω*;*
or if B_{E^-} *is reducible one and for every* $j = 1, \ldots, n$ *hold*

$$(i') \quad \lambda_j + \left(\sum_{k=1}^{N} \frac{\partial F^k}{\partial p^j}(x,p,q^j) + \sum_{i=1}^{N} D_i \frac{\partial a^{ji}}{\partial p^j}(x,p^j,q^j)\right)^+ > 0 \ \textit{ for some } \ x_0 \in \Omega,$$

$$(ii') \qquad \lambda_j + \left(\sum_{i=1}^{n} D_i \frac{\partial a^{ji}}{\partial p^j}(x,p^j,q^j) + \frac{\partial F^j}{\partial p^j}(x,p,q^j)\right)^+ \geq 0 \ \textit{ for every } \ x \in \Omega,$$

where $p, q \in R^n$ *and* λ_l *is the first eigenvalue of operator* B_l *in* Ω.

Note: We remind the reader that B_{E^-} stands for the negative part of B_E. Irreducible matrix is one that can not be decomposed to matrices of lower rank, and respectively, the reducible matrix can be decomposed.

3. Existence of classical solutions

In order to use the method of sub- and super-solutions we need some constraints on the growth of the coefficients. Assume that for every $l = 1, \dots, N$

$$\left\{ \sum_{i=1}^{n} \left(\sum_{j=1}^{n} D_j B_j^{li} + \left(B_0^{li} + H_i^l \right) \right)^2, \left| \sum_{i=1}^{n} \left(D_i B_0^{li} \right) \right| \right\} \leq b \tag{7}$$

holds for $x \in \overline{\Omega}$, where b is a positive constant,

$$\left[\sum_{i=1}^{n} \left(B_0^{li} + H_i^l \right) .p_i.u^l + \sum_{i=1}^{n} \left(D_i B_0^{li} \right) u^l + \sum_{k=1}^{n} E_k^l.u_k(x) \right] u^l \geq c_1 |u|^2 - c_2 \tag{8}$$

for every $x \in \Omega$, $l = 1, \dots, N$ and arbitrary vectors u and p, where $c_1 = \text{const} > 0$ and $c_2 = \text{const} \geq 0$,

$$\left| \sum_{i=1}^{n} \left(B_0^{li} + H_i^l \right) .p_i.u^l + \sum_{i=1}^{n} \left(D_i B_0^{li} \right) u^l + \sum_{k=1}^{n} E_k^l.u_k(x) \right| \leq \tag{9}$$

$$\leq \varepsilon(C_M) + P(p, C_M)(1 + |p|^2),$$

where $P(p, C_M) \to 0$ for $|p| \to \infty$ and $\varepsilon(C_M)$ is sufficiently small and depends only on n, N, C_M, λ and Λ. λ and Λ are the constants from condition (3) and

$$C_M = \max \left\{ \max_{\partial\Omega} |u|, \frac{2 \max |f(x)|}{c_1 n}, \sqrt{\frac{2c_2}{c_1 n}} \right\}. \tag{10}$$

Then the following theorem holds

Theorem 2. *Suppose system* (1), (2) *satisfies conditions* (3) *to* (9), *and* (*i*), (*ii*) *or* (*i*′), (*ii*′), *according to the structure of matrix* $E = (E_k^l)$. *Assume that* $v(x)$ *is a classical super-solution and* $w(x)$ *is a a classical sub-solution of* (1), (2). *Then there exists a classical* $C^2(\Omega) \bigcap C(\overline{\Omega})$ *solution* $u(x)$ *of the problem* (1), (2) *with null boundary data.*

Theorem 2 is proved by the method of sub- and super-solutions. A key-point of the method is the validity of the comparison principle. Unlike the cooperative systems, for non-cooperative ones there is no complete theory for the validity of the comparison principle. In [1] are given some sufficient conditions such that the comparison principle holds, which are recalled in section "Comparison principle for non-cooperative linear elliptic systems" below.

Since the system (1) is a quasi-linear one, we assume in the following proof without loss of generality that $g(x) = 0$.

P r o o f o f T h e o r e m 2. Let us denote

$$\Phi_l^-(x, u^1, \dots, u^N) = \sum_{k=1}^{n} E_k^{l-} u^k + \sum_{i=1}^{n} \left(D_i B_0^{li} \right) u^l$$

and

$$\Phi_l^+(x, u^1, \dots, u^N) = \sum_{k=1}^{n} E_k^{l+} u^k.$$

1. Consider the sequence of vector-functions $u_0, u_1, \dots, u_k, \dots$, where $u_0 = v(x)$ and $u_k \in H_0^1(\Omega)$ defines u_{k+1} by induction as a solution of the problem

(11)
$$-\sum_{i,j=1}^{n} D_i \left(B_j^{li} D_j u_{k+1}^l + B_0^{li} u_{k+1}^l \right) + \sum_{i=1}^{n} H_i^l D_i u_{k+1}^l + \Phi_l^-(x, u_{k+1}^1, \dots, u_{k+1}^N) + \sigma u_{k+1}^l =$$

$$= f^l(x) - \Phi_l^+(x, u_k^1, \dots, u_k^N) + \sigma u_k^l \ \text{ in } \ \Omega$$

with null boundary conditions

(12) $$u_{k+1}^l(x) = 0 \ \text{ on } partial\Omega$$

for every $l = 1, \dots, N$, $\sigma < 0$ is a constant.

Let us denote the left-hand side of (11) by $A^k(x, u, \sigma)$, and the right-hand side – by $B^k(x, u, \sigma)$, $k = 1, \dots, N$.

The problem (11), (12) is cooperative system and by Theorem (1) in [2], page 161, it is solvable. Even more, for the solution $u_{k+1}^l(x) \in C^2(\overline{\Omega})$ there is constant $\beta \in (0, 1)$, β depends on $(l+1)$, such that

(13) $$\|u_{k+1}^l\|_{C^\beta(\overline{\Omega})} < c,$$

(14) $$\left\| \frac{\partial u_{k+1}^l}{\partial x_i} \right\|_{C^\beta(\overline{\Omega})} < c_1 \ \text{ for every } i = 1, \dots, n \ \text{ and } \ \gamma = 1, \dots, m.$$

(15) $$\text{For every compact set } K \subset \Omega \text{ holds } \left\| \frac{\partial^2 u_{k+1}^l}{\partial x_i \partial x_j} \right\|_{C^\beta(K)} < c_7(\rho)$$

for every $i, j = 1, \dots, n$, $\rho = \text{dist}(K, \partial\Omega)$, and the constants c_4–c_7 are independent on k. By Theorem 1 in [4] conditions (3)–(10) are necessary for solvability of the corresponding PDEs, while by Theorem 4 in [4, p. 120], conditions (13)–(15) are derived in every subset of the domain where the coefficients of the diffraction problem are smooth. In our case this is the whole domain Ω.

Furthermore $u_0^l \geq u_1^l \geq \cdots \geq u_{k+1}^l \geq \cdots$ by the comparison principle and the fact that

$$f^l(x) - \Phi_l^+(x, u_k^1, \ldots, u_k^N) + \sigma u_k^l - f^l(x) + \Phi_l^+(x, u_{k-1}^1, \ldots, u_{k-1}^N) - \sigma u_{k-1}^l =$$

$$= -\Phi_l^+(x, u_k^1 - u_{k-1}^1, \ldots, u_k^N - u_{k-1}^N) + \sigma(u_k^l - u_{k-1}^l) \geq 0$$

since $u_l^k \leq u_{l-1}^N$ and $-m_{ki}^+(x) \leq 0$

The proof of $u_0^l \geq u_1^l$ is trivial since u_0^l is a super-solution of (1), (2).

3. Obviously the inequality $u_{k+1}(x) \geq w(x)$ holds for every k, since $w(x)$ is a sub-solution of the same system (1), (2).

4. The sequence of vector-functions $\{u_k\}$ is monotonously decreasing and bounded from below in Ω. Therefore there is a function u such that $u_k(x) \to u(x)$ point-wise in Ω. Furthermore, (13) yields $\{u_k\}$ is uniformly equicontinuous in $\overline{\Omega}$ and $\{u_k\} < const$, since $u_k^l(x)$ is Holder continuous and therefore $|u_k^l(x) - u_k^l(x_0)| \leq c(|x - x_0|^\beta)$ for every $l = 1, \ldots, N$. By Arzela - Ascoli compactness criterion there is a sub-sequence $\{u_{k_j}\}$ that converges uniformly to $u \in C(\overline{\Omega})$. For convenience we denote $\{u_{k_j}\}$ by $\{u_k\}$.

Since $u \in C(\overline{\Omega})$ and all functions $\{u_{k_j}\}$ satisfy the null boundary conditions, then u satisfies the boundary conditions as well.

The functions u_k are Holder continuous with the same Holder constant, therefore u is Holder continuous as well with the same Holder constant, i.e. $u \in C^\beta(\overline{\Omega})$.

Since $u_{k+1}(x)$ is monotone and $u(x)$ is continuous, then $\{(u^k)^2\} \to u^2$ in Ω. Then the Dominated Convergence Theorem (Theorem 5 at p.648 in [3]) yields $u^k \to u(x)$ in $(L^2(\Omega))^N$.

5. Analogously to the previous step, (14) yields $\{D_i u_k\}$ is uniformly equicontinuous in $\overline{\Omega}$ and $\{D_i u_k\} < const$. According to Arzela–Ascoli compactness criterion there is sub-sequence $\{D_i u_{k_j}\}$ that converges uniformly to $D_i u \in C(\overline{\Omega})$. For convenience we denote again $\{u_{k_j}\}$ by $\{u_k\}$.

6. For every $0 < \eta(x) = (\eta^1(x), \ldots, \eta^N(x)) \in (H_0^1(\Omega))^N$

$$\int_\Omega \left(\sum_{i,j=1}^n \left(B_j^{li} D_j u_{k+1}^l + B_0^{li} u_{k+1}^l \right) D_i \eta^l(x) + \sum_{i=1}^n H_i^l D_i u_{k+1}^l \eta^l(x) \right) dx +$$

$$+ \int_\Omega \left(\Phi_l^-(x, u_{k+1}^1, \ldots, u_{k+1}^N) + \sigma u_{k+1}^l \right) \eta^l(x) dx =$$

$$= \int_\Omega (f^l(x) - \Phi_l^+(x, u_k^1, \ldots, u_k^N) + \sigma u_k^l)\eta^l(x)dx$$

holds and for $k \to \infty$ we obtain

$$\int_\Omega \left(\sum_{i,j=1}^n \left(B_j^{li} D_j u^l + B_0^{li} u^l \right) D_i \eta^l(x) + \sum_{i=1}^n H_i^l D_i u^l \eta^l(x) \right) dx +$$

$$+ \int_\Omega \left(\Phi_l^-(x, u^1, \ldots, u^N) + \sigma u^l \right) \eta^l(x) dx =$$

$$= \int_\Omega (f^l(x) - \Phi_l^+(x, u^1, \ldots, u^N) + \sigma u^l)\eta^l(x)dx,$$

that is $u(x)$ is solution of (1), (2).

7. Since the coefficients $a_{ij}^k(x)$ of the principal symbol in (1) are $C^{1+\alpha}(\Omega)$ smooth and $D_x^2 u_k(x)$ are locally bounded, then $D_x^2 u(x) \in C(\Omega)$.

In fact by the exhaustion of Ω by compact sets κ_r, $\kappa_r \subset \kappa_{r+1} \subset \Omega$ and $\bigcup \kappa_r = \Omega$, and by (15) we have $D_x^2 u_k \in C^\beta(K_r)$ are uniformly bounded and equicontinuous in κ_r. Applying Arzela–Ascoli theorem and Cantor diagonal process (for sub-sequence and compact) yields C^2 smoothness in Ω of the limit function $u(x)$.

Therefore $u(x) \in C^2(\Omega))^N$ is classical solution of (1), (2). $\square$

4. Model example

Consider the system

$$(16) \qquad \left| \begin{array}{l} \left(K^2 - \chi^2\right)^{1/4} \Delta_2 \ln|\chi - K| = 2(2K - \chi) \\ \\ \left(K^2 - \chi^2\right)^{1/4} \Delta_2 \ln|\chi + K| = 2(2K + \chi) \end{array} \right. ,$$

where $\Delta_2 = \partial_x^2 + \partial_y^2$, $K^2 > \chi^2$, $K < 0$, $K = K(x,y)$ and $\chi = \chi(x,y)$. Here K is the Gaussian curvature and χ is the curvature of the normal connection on minimal non-super-conformal surface M^2 in R^4.

Every couple of solutions (K, χ) define uniquely minimal non-super-conformal surface M^2 in R^4 with Gaussian curvature K and normal curvature χ.

Let $K > \chi$. Then we denote

$$(17) \qquad \left| \begin{array}{l} K - \chi = e^u \\ \\ K + \chi = e^v \end{array} \right.$$

and transform (16) to

$$(18) \qquad \left| \begin{array}{l} \Delta u = 3e^{(3u-v)/4} + e^{(3v-u)/4} \\ \\ \Delta v = e^{(3u-v)/4} + 3e^{(3v-u)/4} \end{array} \right. .$$

Equation (18) is quasi-linear non-cooperative elliptic system. In this case

$$\begin{aligned}
B_j^{li} &= \int_0^1 \frac{\partial a^{li}}{\partial p_j}(x,P^l)ds = \delta_{i,j}, \quad B_0^{li} = \int_0^1 \frac{\partial a^{li}}{\partial u^l}(x,P^l)ds = 0, \\
E_1^1 &= \int_0^1 \frac{\partial F^1}{\partial u^1}(x,S^l)ds = \int_0^1 \frac{9}{4}e^{(3u-v)/4} - \frac{1}{4}e^{(3v-u)/4}ds, \\
E_2^1 &= \int_0^1 \frac{\partial F^1}{\partial u^2}(x,S^l)ds = \int_0^1 -\frac{3}{4}3e^{(3u-v)/4} + \frac{3}{4}e^{(3v-u)/4}ds, \\
E_1^2 &= \int_0^1 \frac{\partial F^2}{\partial u^1}(x,S^l)ds = \int_0^1 \frac{3}{4}e^{(3u-v)/4} - \frac{3}{4}e^{(3v-u)/4}ds, \\
E_2^2 &= \int_0^1 \frac{\partial F^2}{\partial u^2}(x,S^l)ds = \int_0^1 -\frac{1}{4}e^{(3u-v)/4} + \frac{9}{4}e^{(3v-u)/4}ds, \\
H_i^l &= \int_0^1 \frac{\partial F^l}{\partial p_i}(x,S^l)ds = 0,
\end{aligned}$$

where $\delta_{i,j}$ is Kronecker delta (symbol), $P^l = \Big(v^l + s(u^l - v^l), Dv^l + sD(u^l - v^l)\Big)$ and $S^l = \Big(v + s(u-v), Dv^l + sD(u^l - v^l)\Big)$.

Since K is the Gaussian curvature and χ is the curvature of the normal connection on minimal non-super-conformal surface M^2 in R^4, by (17) we presume u, v do not blow up. In other words we suppose there is constant $C(\Omega)$ such that $e^u \leq C(\Omega)$ and $e^v \leq C(\Omega)$.

Assume that Ω is a map from $M^2 \to M^2$. The smaller is the map, the smaller is $C(\Omega)$ and the larger is the first eigenvalue of system (11). Therefore, if Ω is sufficiently small, conditions (i), (ii) (or (i'), (i'')) hold and by Theorem 1 comparison principle holds for system (18). Furthermore, conditions (7)–(9) hold as well. This way we have constructed (locally) a classical solution of system (16) having intreresting applications in differential geometry. Details of the proofs of the results in this short note will be published elsewhere.

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