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STABILITY AND ACCURACY OF RBF DIRECT METHOD FOR SOLVING A DYNAMIC INVESTMENT MODEL

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In this paper we consider a Dynamic investment model. In the model, firm's objective is maximizaing discounted sum of profits over an interval of time. The model assumes that firm's capital in time t increases with investment and decreases with depreciation rate that can be expressed by means of differential equation.

We propose a direct method for solving the problem based on Radial Basis Functions(RBFs). The authors describe operational matrices of RBFs and use them to reduce the variational problem to a static optimization problem which can be solved via some optimization techniques. Next, we describe some economic interpretation of the solution. Finally, the accuracy and stability of the Multiquadric (MQ), and Gaussian (GA) RBFs are illustrated by conducting some numerical experiments.

1. Introduction

A direct method is a tool to convert a variational problem into a mathematical programming problem. The idea of direct methods for solving variational problems consists in replacing the problem of searching for the extremum (usually, for the point of stationary) of a functional in the function space by a problem of searching for a solution in a finite set of parameters.

Maleki and others [12] using nonclassical parametrization presented a direct method for solving a variational problem. Special attention has been given to

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applications of Walsh functions [3], Block-pulse functions [10], Laguerre polynomials [11], Legendre polynomials [2], Chebyshev polynomials [8], Legendre Wavelets [13], Triangular orthogonal functions [1], Sine-Cosine Wavelets [14], Walsh-Wavelets [6], Rationalized Haar functions [15].

In the last decades, Radial Basis Functios (RBFs) have been widely applied in different fields such as multivariate function interpolation and approximation, neural networks, and solution of differential and integral equations. The essence of RBF interpolation is to use linear combination of Radial Basis Function $\phi(\|x-x_j\|)$, which is radially symmetric about its center x_j to approximate the unknown function $y(x)$. Common choices for RBFs are listed in Table 1 where $r=\|x-x_j\|$ and $\|.\|$ denotes the Euclidean norm, and x_js are the centers of RBFs and c is a positive shape parameter which controls the flatness (width) of the basis function.

Franke and Schaback [5] used RBFs for solving PDEs, Golbabai and Seifollahi [7] used RBFs for solving the second kind Integral Equations (IEs). Moreover, it has been observed that certain class of RBFs such as multiqudric (MQ) and Gaussian(GA) RBFs exhibit superior error convergence properties. Some advantages of the RBF-Based methods are their ease of implementation, accuracy and efficiency, which are the reasons that this technique is getting popular.

In this paper we use RBFs for solving variational problems . For this purpose, we introduce the RBFs opertaional matrices of differentiation and the product of two RBF vectors. The RBFs operational matrices can be used to solve problems in fields analysis, calculus of variations and optimal control. Gauss-Legendre quadrature is used to approximate integration. Finally we evaluate the coefficients of RBF in such a way that the necessary conditions for extermination of a multivariate function are imposed.

Table 1: Some popular RBFs

Name	$\phi(r,c)$
Gaussian (GA)	$\phi(r)=\exp(-cr^2)$
Multiquadric (MQ)	$\phi(r)=\sqrt{c^2+r^2}$
Inverse multiquadric (IMQ)	$\phi(r)=(c^2+r^2)^{-1/2}$

2. Properties of RBFs

Approximation of a function $y(x)$ may be expanded as

$$y(x) \cong y^N(x) = \sum_{i=0}^{N} a_i \varphi(\|x - x_i\|), \quad (1)$$

where N is the number of RBFs and $a_i s$ are the unknown coefficients and x_i are the RBF centers.

Theorem 1. *Let* $y^N(x) = a^T \phi(x)$, $\varphi_i(x) = \sqrt{c^2 + \|x - x_i\|^2}$ *for* $i = 1, \ldots, N$. *Then there exist symmetric matrices* H, L *and vector* $\overline{\phi}(x)$, *square matrix* M *such that:*

$$a)\ (y^N(x))^2 = a^T H a, \quad (2)$$

where $H_{ij} = \varphi_i(x)\varphi_j(x)$.

$$b)\ \frac{d}{dx} y^N(x) = a^T \overline{\phi}(x), \quad (3)$$

where $\overline{\phi}(x) = [\overline{\varphi_1}(x), \ldots, \overline{\varphi_N}(x)]^T$ *and* $\overline{\varphi_i}(x) = (x - x_i)(c^2 + \|x - x_i\|^2)^{\frac{-1}{2}}$.

$$c)\ (\frac{d}{dx} y^N(x))^2 = a^T L a, \quad (4)$$

where $L_{ij} = (x - x_i)(x - x_j)\varphi_i^{-1}(x)\varphi_j^{-1}(x)$.

3. Dynamic investment model

Suppose that a firm's output depends on the amount of capital it employs. Let $Q = q(K)$ where Q is the firm's output level, q is the production function and K is the amount of capital employed.

We suppose that the firm must purchase its own capital . Once purchased, the capital lasts for a long time. Let $I(t)$ be the amount of capital purchased (investment) at time t and $c[I(t)]$ be a function that gives the cost of purchasing (investing) the amount $I(t)$ of capital at time t; then profit at time t is

$$\pi[K(t),\ I(t)] = pq[K(t)] - c(I(t)). \quad (5)$$

If the firm's objective is to choose K and I to maximize discounted sum of profits over an interval of time running from the present time ($t = 0$) to a given time horizone, T, this is given by the functional $J[I(t)]$

$$\max J[I] = \int_0^T e^{-\rho t}\ \pi[K(t)\ , I(t)]\ dt \quad (6)$$

where ρ is the firm's discount rate and $e^{-\rho t}$ is the continuous-time discounting factor.

Assume that capital depreciates at the rate δ. The amount (stock) of capital owned by the firm at time t is $K(t)$ and changes according to the differential equatian

$$K'(t) = I(t) - \delta K(t) \tag{7}$$

which says that, at each point in time, the firm's capital stock increases by the amount of investment and decreases by the amount of depreciation.

The problem facing the firm at each point in time is to decide how much capital to purchase . This is a truley dynamic problem because current investment affects current profit, because it is a current expense, and also affects future profits, because it affects the amount of capital available for future production.

The problem can be expressed as follows:

$$\max J[I] = \int_0^T e^{-\rho t}\, \pi[K(t),\ I(t)]\, dt \tag{8}$$

subject to

$$K' = I(t) - \delta K(t) \tag{9}$$

$$K(0) = K_0. \tag{10}$$

4. RBF direct method

Consider the problem of dynamic investment (8)-(9). Tonelli's theorem [4] says under assumptions

1. Coercivity of rank $r > 1$ for certain constants $\alpha > 0$ and β we have
$$L(t, x, v) \geq \alpha \|v\|^r + \beta. \qquad \text{for every } (t, x, v) \epsilon [a\ , b] \times R^N \times R^N$$
2. Convexity: L_{vv} is everywhere positive semidefinite ($L_{vv} \geq 0$),

there exists a solution of the basic problem (8)–(9) relative to the class of absolutly continuous (A.C.) functions.

Euler-Lagrange equation leading to a necessary condition for y(x) on extremizing $J(y)$ in the form of the well known Euler-Lagrange equation

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0, \tag{11}$$

which can be easily solved only in a few classes. Thus it is more practical to use numerical and direct methods such as the Ritz method.

Applying theorem (1) combined with Gauss-Legendre quadrature with m nodes yields

$$(12)\quad \int_0^T e^{-\rho t}\ \pi[K(t),\ I(t)]\ dt$$
$$\cong \frac{T}{2}\sum_{k=1}^{m} w_k e^{-\rho t_k}\ \pi[a^T\phi(\|t_k-t_i\|), a^T\frac{d\phi}{dt}(\|t_k-t_i\|)+\delta a^T\phi(\|t_k-t_i\|)]$$

where $t_k = \frac{T}{2}+\frac{T}{2}z_k$, $z_k s$ and $w_k s$ are nodes and coefficients of Gauss-Legendre quadrature and RBF centers $t_i,\ i=1,\dots,N$ are equi-spaced nodes in the interval $[0\ ,T]$.

Finally the problem (8)–(9) will be reduced to a constrained optimization problem of finding

(13)

$$\max J(a) = \frac{T}{2}\sum_{k=1}^{m} w_k e^{-\rho t_k}\ \pi[a^T\phi(\|t_k-t_i\|), a^T\frac{d\phi}{dt}(\|t_k-t_i\|)+\delta a^T\phi(\|t_k-t_i\|)],$$

subject to

$$(14)\qquad a^T\phi(\|t_i\|)=K_0,\quad a^T\phi(\|T-t_i\|)=K_T.$$

To solve the optimization problem (13)-(14) our choice is to use Lagrange multipliers method, so we establish Lagrangian

$$(15)\qquad J^*(a)=J(a)+\lambda_1(a^T\phi(\|x_i\|)-K_0),$$

then we should solve the following algebraic system in order to calculate coefficients $a^T=[a_1,\dots,a_N]$

$$(16)\qquad \frac{\partial J^*}{\partial a}=0,$$

$$(17)\qquad \frac{\partial J^*}{\partial \lambda_i}=0\qquad i=1,2.$$

It is easy to see that if Lagrangian in problem (17) is convex then objective function (13) will be convex function of unknown parameter $a=[a_1,\dots,a_N]^T$.

For example ([9]) assume in the investment model (8)–(9) the firm's production function is given by $\pi[K(t),I(t)]=K-aK^2-I^2\quad a>0$ and the price of the firm's output is a constant 1\$, and the cost of investment is equal to I^2\$ then the firm's profit at a point in time are $\pi=K(t)-aK(t)^2-I(t)^2$ So we are faced with following variational problem

$$(18)\qquad \max J[I]=\int_0^{+\infty} e^{-\rho t}\ (K(t)-aK(t)^2-I(t)^2)\ dt$$

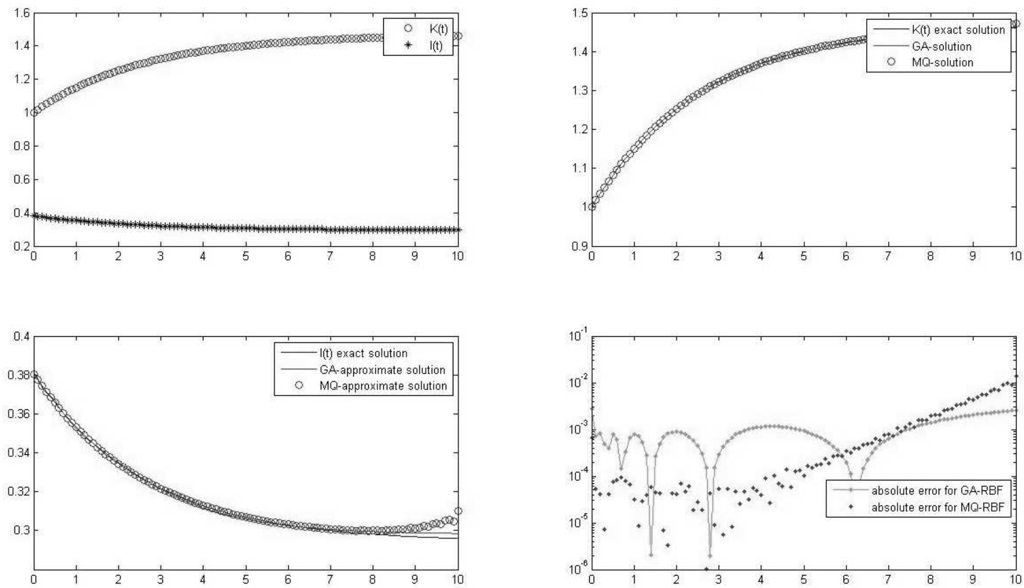

Figure 1: a) Capital Stock over time. b) Investment over time. c) Comparing accuracy of MQ and IMQ RBFs

subject to

$$K' = I(t) \; - \delta K(t) \tag{19}$$

$$K(0) = \; K_0 \tag{20}$$

If we assume $K(0) = 1$, using RBF direct method with $c = 2.5$ and $N = 100$, the solution will be obtained as in figure(1). The authors used the change of variable $t = f(t)$ where f is a differnetiable, strictly increasing function of τ which maps the finite interval $[-1, \; 1)$ onto $[0, \; +\infty)$.

We used RBF-direct method to solve the problem (18)-(19). Using the described procedure in sec. 4 the problem (18)-(19) will be reduced to an $(N+2) \times (N+2)$ algebraic linear system which have been solved with MATLAB 7.7.0.

Figure 1 – displays the optimal path of investment for the case in which $K_0 < K$. Along the optimal path, investment declines, with increasing in time, investment converges to a constant amount so that in the long run the firm's investment is just replacement of depreciation.

$$f(\tau) = \frac{1+\tau}{1-\tau} \tag{21}$$

The results are presented in the Figure 1.

5. Conclusion

In this paper a general formula for the MQ-RBFs and GA-RBFs operational matrix has been derived. This matrix can be used to solve problems in identification, analysis, optimal control and variational problems. The RBF operational matrix combined with Gauss- Legendre quadrature reduces a variational problem into a set of algebraic equations. The product of two MQ-RBF vectors have quadratic form, hence making MQ-RBF method computationally attractive. In this paper MQ-RBF and GA-RBF method was developed and employed for solving a Dynamc Investment problem over the specified domain. The high accuracy of the estimation and validity and applicability of the method was demonstrated by some examples.

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