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A COMPACT ALTERNATING DIRECTION IMPLICIT SCHEME FOR TWO-DIMENSIONAL FRACTIONAL OLDROYD-B FLUIDS*

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The two-dimensional Rayleigh-Stokes problem for a generalized Oldroyd-B fluid is considered in the present work. The fractional time derivatives are discretized using $L1$ and $L2$ approximations. A fourth order compact approximation is implemented for the space derivatives and two variants of an alternating direction implicit finite difference scheme are numerically investigated.

1. Introduction

In recent years, increasing attention has been devoted to the study of viscoelastic non-Newtonian fluids modeled by constitutive equations which involve fractional derivatives, see [1, 2, 3, 4, 5].

In the present work we consider the following two-dimensional initial-boundary value problem for the velocity distribution of a viscoelastic flow with generalized

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fractional Oldroyd-B constitutive model (see [2, 3, 4, 5] for details on the derivation):

$$
\begin{aligned}
&(1) \quad (1 + aD_t^{\alpha})u_t &&= \mu(1 + bD_t^{\beta})\Delta u + F(x, y, t), \quad (x, y) \in (0, 1)^2,\ t > 0,\\
&(2) \quad u(x, y, 0) &&= u_t(x, y, 0) = 0, \quad (x, y) \in [0, 1]^2,\\
&(3) \quad u(x, y, t) &&= v(x, y, t), \quad t > 0,\ x = 0 \text{ or } x = 1 \text{ or } y = 0 \text{ or } y = 1.
\end{aligned}
$$

Here $a, b \geq 0$, $\mu > 0$, D_t^{α} and D_t^{β} are Riemann-Liouville fractional time derivatives of orders $\alpha \in (0, 1)$ and $\beta \in (0, 1)$, $F(x, y, t)$, $v(x, y, t)$ are given functions.

Let us recall that the Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined as follows (see e.g. [6]):

$$
{}_{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t - s)^{n-\alpha-1} f(s) \mathrm{d}s,
$$

where n is a positive integer, such that $n - 1 < \alpha < n$. Due to the prescribed initial conditions (2), the Riemann-Liouville derivatives in Equation (1) can be replaced by Caputo derivatives, therefore we have not denoted the type of the fractional derivatives in Equation (1). Indeed, the Caputo derivative ${}_{C}D_t^{\alpha}$ is defined as (see e.g. [6]):

$$
{}_{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} f^{(n)}(s) \mathrm{d}s
$$

and is related to the Riemann-Liouville derivative ${}_{RL}D_t^{\alpha}$ via the identity (see [6])

$$
{}_{RL}D_t^{\alpha} f(t) = {}_{C}D_t^{\alpha} f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(0) t^{k-\alpha}}{\Gamma(k + 1 - \alpha)},
$$

which together with (2) implies

$$
(4) \quad {}_{RL}D_t^{\alpha} u = {}_{C}D_t^{\alpha} u, \quad {}_{RL}D_t^{\alpha} u_t = {}_{C}D_t^{\alpha} u_t = {}_{C}D_t^{\alpha+1} u = {}_{RL}D_t^{\alpha+1} u, \quad \alpha \in (0, 1).
$$

Theoretical studies on unidirectional flows of fractional Oldroyd-B fluids are performed in [2, 3, 4, 5, 7, 8], where eigenfunction expansions of the solutions in 1D and 2D are obtained. In [7] estimates for the time-dependent components in these eigenfunction expansions are established, which imply convergence of the series, i.e., problem (1)-(3) admits a unique solution under appropriate smoothness requirements on the data.

The present work is concerned with numerical computation of the solution of problem (1)-(3). Reviews of different approximations of the fractional derivatives

can be found in [9, 6, 10, 11]. The so-called $L1$ and $L2$ approximations of the fractional Caputo and Riemann-Liouville derivatives are introduced in [9]. The $L1$ scheme is suitable to approximate the fractional derivative D_t^α for $\alpha \in (0,1)$. The order of convergence of this approximation is $O(\tau^{2-\alpha})$, where τ is the time step. The $L2$ scheme approximates the fractional derivative of order $\alpha \in (1,2)$ with order of convergence $O(\tau^{3-\alpha})$. Based on the $L1$ or $L2$ approximation, implicit schemes for the time-fractional diffusion, diffusion-wave and diffusion-wave equation with damping are developed and investigated in [12, 13, 14, 15, 16, 17]. The schemes are unconditionally stable and convergent in the H_1, discrete H_1 or discrete maximum norm. In [18, 19, 16] the second order spatial derivatives are approximated by compact fourth order finite differences.

Alternating direction implicit (ADI) schemes are proposed and investigated for time-fractional diffusion or diffusion-wave equations in [15, 16, 19, 20]. They use an operator splitting technique to replace the solution of multidimensional problems by solution of independent one-dimensional problems.

Although numerical algorithms for particular cases of problem (1)-(3) when $a = 0$ or $b = 0$ are studied extensively, numerical studies concerning the general case $a \neq 0$ and $b \neq 0$ are still very limited. To the best of the authors' knowledge, the only such work is [5], where a numerical method is developed and analyzed for the 1D version of problem (1)-(3).

In this work we use the $L1$ and $L2$ approximation of the fractional time derivatives and the compact fourth order discretization in space. The construction of the corresponding ADI finite-difference scheme is described in the next section. Numerical experiments are presented in the third section in order to numerically investigate the stability and convergence. Finally, we present conclusions about the properties of the scheme.

2. Finite-Difference Scheme

2.1. $L1$ and $L2$ approximation of fractional derivatives

Let us introduce a uniform discretization in time and in space

$$t_k = k\tau,\ t_{k+1/2} = t_k + \tau/2,\ k = 0, 1, \ldots, N_t,\ T = N_t\tau,$$

$$x_i = ih_x,\ i = 0, \ldots, N_x,\ h_x = 1/N_x,\ y_j = jh_y, j = 0, \ldots, N_y,\ h_y = 1/N_y,$$

where $N_t + 1$, $N_x + 1$, $N_y + 1$ are the number of nodes in the corresponding direction. Let the function $f(t)$ satisfy $f(0) = f'(0) = 0$. Then

Finite L1 time derivative of f may be defined as (see [9, 12, 13, 14, 15, 10, 11, 5])

$$\delta^{\beta} f(t_{k+1}) = \frac{\tau^{-\beta}}{\Gamma(2-\beta)} \sum_{m=0}^{k} b_m [f(t_{k-m+1}) - f(t_{k-m})],$$

where $b_m = (m+1)^{1-\beta} - m^{1-\beta}$, $m \geq 0$. Then

$$\tilde{\delta}^{\beta} f(t_{k+1/2}) := \frac{\delta^{\beta} f(t_k) + \delta^{\beta} f(t_{k+1})}{2} = \frac{\tau^{-\beta}}{2\Gamma(2-\beta)} \left[f(t_{k+1}) + \sum_{m=0}^{k-1} \omega_m^{\beta} f(t_{k-m}) \right],$$

where $\omega_m^{\beta} = (b_{m+1} - b_{m-1})$, $m \geq 1$, $\omega_0^{\beta} = 1$, and the local truncation error is of order $O(\tau^{2-\beta})$, if the function f is twice continuously differentiable.

Finite L2 time derivative of f may be defined as (see [9, 13, 20, 16, 17, 10, 11, 5]

$$\begin{aligned} \delta^{\alpha+1} f(t_{k+1/2}) &= \frac{\tau^{-\alpha-1}}{\Gamma(2-\alpha)} \{ f(t_{k+1}) - f(t_k) \\ &+ \sum_{m=1}^{k} (a_m - a_{m-1}) [f(t_{k-m+1}) - f(t_{k-m})] \} \\ &= \frac{\tau^{-\alpha-1}}{\Gamma(2-\alpha)} \left[f(t_{k+1}) + \sum_{m=0}^{k-1} \omega_m^{\alpha+1} f(t_{k-m}) \right], \end{aligned}$$

where $\omega_m^{\alpha+1} = (a_{m+1} - 2a_m + a_{m-1})$, $m \geq 1$, $\omega_0^{\alpha+1} = a_1 - 2a_0$, $a_m = (m+1)^{1-\alpha} - m^{1-\alpha}$, $m \geq 0$, and the local truncation error is of order $O(\tau^{2-\alpha})$, if the function f is three times continuously differentiable.

2.2. Compact ADI discretization

As was mentioned in the Introduction, the authors found only one work [5], where numerical methods are developed and analyzed for the case $a \neq 0$, $b \neq 0$. In this paper Galerkin finite elements are used in space in combination with L_1 and L_2 approximations of the fractional time derivatives, and the unconditional stability and convergence in H_1-norm is discussed (without a complete proof). There is a vast number of recent numerical studies on various fractional evolution equations, modeling time-fractional subdiffusion, diffusion-wave, multi-term time-fractional diffusion, or generalized second grade fluids. Compact schemes and ADI methods for fractional diffusion and diffusion-wave equations are developed and investigated in [19, 16] and many ideas from these works are used here.

Let U_{ij}^k be the values of the approximate solution at the nodes $x_i,\ y_j,\ t_k,\ i = 0,\dots,N_x,\ j = 0,\dots,N_y,\ k = 0,\dots,N_t$. Due to the first initial condition in (2) the solution for $k = 0$ is the trivial one $U_{ij}^0 = 0,\ i = 0,\dots,N_x,\ j = 0,\dots,N_y$.

Let Λ be the usual second-order discretization of the Laplacian $\Lambda = \Lambda_{xx}+\Lambda_{yy}$,

$$\Lambda_{xx}U_{ij} = (U_{i+1,j} - 2U_{ij} + U_{i-1,j})/h_x^2,\quad \Lambda_{yy}U_{ij} = (U_{i,j+1} - 2U_{ij} + U_{i,j-1})/h_y^2.$$

Let us define (see also [18, 19, 16])

$$\Theta_x := I + \frac{h_x^2}{12}\Lambda_{xx},\quad \Theta_y := I + \frac{h_y^2}{12}\Lambda_{yy},\quad \Theta := \Theta_y\Lambda_{xx} + \Theta_x\Lambda_{yy}.$$

Then

$$\Theta_x U_{ij} = (U_{i+1,j} + 10U_{ij} + U_{i-1,j})/12,\quad \Theta_y U_{ij} = (U_{i,j+1} + 10U_{ij} + U_{i,j-1})/12,$$

and

$$\Theta_x\frac{\partial^2 u}{\partial x^2} = \Lambda_{xx}u + O(h_x^4),\quad \Theta_y\frac{\partial^2 u}{\partial y^2} = \Lambda_{yy}u + O(h_y^4).$$

Multiplying Equation (1) by $\Theta_x\Theta_y$, and using the $L1$ and $L2$ formulas in time, we can easily write a Crank-Nicolson type implicit discretization of the problem

$$\begin{aligned}(5)\quad &\Theta_x\Theta_y\left[\frac{U_{ij}^{k+1} - U_{ij}^k}{\tau} + \frac{a}{\Gamma(2-\alpha)\tau^{\alpha+1}}\left(U_{ij}^{k+1} + \sum_{m=0}^{k-1}\omega_m^{\alpha+1}U_{ij}^{k-m}\right)\right] =\\ =\ &0.5\mu\Theta\left[U_{ij}^{k+1} + U_{ij}^k + \frac{b}{\Gamma(2-\beta)\tau^{\beta}}\left(U_{ij}^{k+1} + \sum_{m=0}^{k-1}\omega_m^{\beta}U_{ij}^{k-m}\right)\right] + \Theta_x\Theta_y F_{ij}^{k+1/2},\end{aligned}$$

where $i = 1,\dots,N_x-1,\quad j = 1,\dots,N_y-1,\quad k = 0,\dots,N_t-1$.

The Dirichlet boundary conditions (3) are imposed in the usual way

$$U_{ij}^{k+1} = v(x_i, y_j, t_{k+1}),\ \ i = 0,\ \text{or } i = N_x,\ \text{or } j = 0,\ \text{or } j = N_y,\ k = 0,\dots,N_t-1.$$

Let

$$c := \frac{\mu\tau^{1+\alpha-\beta}\Gamma(2-\alpha)(\Gamma(2-\beta)\tau^{\beta} + b)}{2(\Gamma(2-\alpha)\tau^{\alpha} + a)\Gamma(2-\beta)}.$$

Multiplying (5) by $\Gamma(2-\alpha)\tau^{\alpha+1}/(\Gamma(2-\alpha)\tau^{\alpha} + a)$ and adding

$$A_{ij}^{k\,1} = c^2\Lambda_{xx}\Lambda_{yy}(U_{ij}^{k+1} - U_{ij}^k)\ \text{ or }\ A_{ij}^{k\,2} = c^2\Lambda_{xx}\Lambda_{yy}(U_{ij}^{k+1} - 2U_{ij}^k - U_{ij}^{k-1})$$

to the left hand side of (5) we obtain

$$(6)\quad \left[\left(I+\frac{h_x^2}{12}\Lambda_{xx}\right)\left(I+\frac{h_y^2}{12}\Lambda_{yy}\right)-c\left(I+\frac{h_y^2}{12}\Lambda_{yy}\right)\Lambda_{xx}-c\left(I+\frac{h_x^2}{12}\Lambda_{xx}\right)\Lambda_{yy}+c^2\Lambda_{xx}\Lambda_{yy}\right]U_{ij}^{k+1}=G_{ij}^k,$$

where

$$G_{ij}^k(U^k,U^{k-1},\dots,U^0,x_i,y_j,t_{k+1/2},\tau):=$$
$$\frac{\tau^\alpha\Gamma(2-\alpha)}{\Gamma(2-\alpha)\tau^\alpha+a}\left[\Theta_x\Theta_y\left(U_{ij}^k-\frac{a}{\Gamma(2-\alpha)\tau^\alpha}\sum_{m=0}^{k-1}\omega_m^{\alpha+1}U_{ij}^{k-m}\right)+\right.$$
$$\left.\frac{\mu\tau}{2}\Theta U_{ij}^k+\frac{\mu\tau b}{2\Gamma(2-\beta)\tau^\beta}\sum_{m=0}^{k-1}\omega_m^\beta\Theta U_{ij}^{k-m}+\tau\Theta_x\Theta_y F_{ij}^{k+1/2}\right]+R_{ij}^{k\,l},$$

$$R_{ij}^{k\,1}=c^2\Lambda_{xx}\Lambda_{yy}U_{ij}^k,\quad R_{ij}^{k\,2}=c^2\Lambda_{xx}\Lambda_{yy}(2U_{ij}^k-U_{ij}^{k-1}).$$

The left hand side of (6) may be rearranged as

$$\left[I-\left(c-\frac{h_x^2}{12}\right)\Lambda_{xx}-\left(c-\frac{h_y^2}{12}\right)\Lambda_{yy}+\left(\frac{h_y^2}{12}\frac{h_x^2}{12}-c\frac{h_y^2}{12}-c\frac{h_x^2}{12}+c^2\right)\Lambda_{xx}\Lambda_{yy}\right]U_{ij}^{k+1}$$
$$=(I-\tilde{c}_x\Lambda_{xx}-\tilde{c}_x\Lambda_{yy}+\tilde{c}_x\tilde{c}_y\Lambda_{xx}\Lambda_{yy})\,U_{ij}^{k+1}=(I-\tilde{c_x}\Lambda_{xx})(I-\tilde{c_y}\Lambda_{yy})U_{ij}^{k+1},$$

where $\tilde{c_x}=c-h_x^2/12$, $\tilde{c_y}=c-h_y^2/12$. Thus we obtain the following factorized scheme

$$(I-\tilde{c_x}\Lambda_{xx})(I-\tilde{c_y}\Lambda_{yy})U_{ij}^{k+1}=G_{ij}^k.$$

Note, we add one of the terms

$$\frac{\Gamma(2-\alpha)\tau^\alpha+a}{\Gamma(2-\alpha)\tau^{\alpha+1}}A_{ij}^{k\,l},\quad l=1 \text{ or } l=2$$

to (5). The use of $A_{ij}^{k\,2}$ is proposed in [19] in the discretization of the time-fractional diffusion equation with the Grünwald-Letnikov approximation of the fractional derivatives. The reason was to preserve the first order approximation of the equation. In our case, when $a,b\neq 0$, the order of the first term is $O(\tau^{2+2\alpha-2\beta-(\alpha+1)+1})=O(\tau^{2+\alpha-2\beta})$ and the order of the second term is $O(\tau^{3+\alpha-2\beta})$.

Let us introduce

$$U_{ij}^*:=(I-\tilde{c_y}\Lambda_{yy})U_{ij}^{k+1},\quad i=0,\dots,N_x,\ j=1,\dots,N_y-1.$$

Then, we have to solve for each $j = 1, \ldots, N_y - 1$ the following tridiagonal system of linear equations

$$\begin{aligned} (I - \tilde{c_x}\Lambda_{xx})U^*_{ij} &= G^k_{ij}, \quad i = 1, \ldots, N_{x-1} \\ U^*_{0,j} &= (1 - \tilde{c_y}\Lambda_{yy})v(0, y_j, t_{k+1}) \\ U^*_{N_x,j} &= (1 - \tilde{c_y}\Lambda_{yy})v(x_{N_x}, y_j, t_{k+1}) \end{aligned} \tag{7}$$

and then for each $i = 1, \ldots, N_x - 1$

$$\begin{aligned} (I - \tilde{c_y}\Lambda_{yy})U^{k+1}_{ij} &= U^*_{ij}, \quad j = 1, \ldots, N_{y-1} \\ U^{k+1}_{i,0} &= v(x_i, 0, t_{k+1}) \\ U^{k+1}_{i,N_y} &= v(x_i, y_{N_y}, t_{k+1}) \end{aligned} \tag{8}$$

As all linear systems have strictly diagonally dominant matrices, the Thomas algorithm can be successfully used in order to solve them. Note, each of the first $N_y - 1$ linear systems (7) is independent of the others, thus they can be solved in parallel. After that parallel computations can be performed for the solution of the remaining $N_x - 1$ independent linear systems (8). The most-time consuming part is the calculation of the fractional derivatives, i.e., of the right hand side G^k_{ij} in (7) for large k ($k \gg 1$).

As was already mentioned, the additional term in the discretization is of the order $O(\tau^{1+l+\alpha-2\beta})$, $a, b \neq 0$. Therefore, the order of approximation in time of the ADI scheme is

$$r = O(\tau^{\min\{2-\alpha, 2-\beta, 1+l+\alpha-2\beta\}}), \quad l = 1, 2.$$

Therefore we use $l = 1$ when

$$\min\{2 - \alpha, 2 - \beta\} \leq 2 + \alpha - 2\beta,$$

and $l = 2$ when

$$\min\{2 - \alpha, 2 - \beta\} > 2 + \alpha - 2\beta.$$

Let us note that in the last case $\min\{2-\alpha, 2-\beta\} \leq 2-\beta \leq 3-2\beta \leq 3+\alpha-2\beta$, i.e., the order of approximation of the equation is not destroyed by the order of the additional term. Due to the initial conditions the solution on the first time step U^1_{ij} is of the order $O(\tau^2)$. Thus for $k = 0$ we may always use the first additional term, as then its order is $O(\tau^{3+\alpha-2\beta})$.

3. Numerical experiments

Extensive numerical experiments are performed in order to investigate the stability and the accuracy of the solutions for different values of the parameters α and β. The initial data and the right-hand side are chosen to correspond to an exact solution

$$u(x,y,t) = \mathrm{e}^{x+y} t^{\gamma+1}, \quad \text{where } \gamma = 3.5.$$

The order of convergence p is computed using Runge's rule

$$p = \log_2[\delta(U_{s-1})/\delta(U_s)],$$

where s is the number of the corresponding grid and

$$\delta(U) := \max\{|u(x_i,y_j,t_k) - U(x_i,y_j,t_k)|, \quad 0 \le i \le N_x,\ 0 \le j \le N_y,\ 0 \le k \le N_t\}$$

is the maximum of the absolute value of the difference between the exact and the numerical solution. In all numerical experiments we will take $a = b = \mu = 1$, $N_x = N_y$, and $T = N_t\tau = 1$.

Numerical experiments for various values of α and β are presented in Table 1 for $l = 1$ and in Table 2 for $l = 2$. In the computations we use two different grids in space with $N_x = N_y = 25$ points and $N_x = N_y = 50$ points respectively and various values for the number of time steps N_t. In the all cases the computed order of convergence p is the same or slightly better than the order of approximation r of the equation. As it can be seen, the results for $N_x = N_y = 25$ and $N_x = N_y = 50$ are almost the same. The only exception is in the case $\alpha = \beta = 0.1$, where the order of convergence on the corresponding last two rows of Table 1 is not well established for $N_x = N_y = 25$. For $N_x = N_y = 50$ the computations show almost perfect second order convergence ($p \approx 1.99$), although the order of approximation of the equation is $r = 1.9$.

In general, the small and constant number of 25 space steps does not destroy the convergence, which indicates that the truncation error in the space discretization is much smaller than the corresponding error in the time-fractional approximation.

4. Summary

The presented here ADI method leads to the successive solution of two cycles of tridiagonal linear systems (7) and (8). These cycles can be easily parallelized.

The compact fourth order scheme allows us to do fast computations and to save a lot of memory. Let us note, that in order to compute the fractional derivatives we need to store the solution on the all time levels. The order of

Table 1: The error $\delta(U)$ and the order of convergence p for $l = 1$

N_t	$N_x = N_y = 25$		$N_x = N_y = 50$	
	$\alpha = 0.1,\ \beta = 0.1,\ r = 1.9$			
100	4.8757e-5		4.8831e-5	
200	1.2198e-5	1.9989	1.2219e-5	1.9987
400	3.0568e-6	1.9965	3.0640e-6	1.9956
800	7.6651e-7	1.9956	7.7027e-7	1.9920
1600	1.9140e-7	2.0017	1.9409e-7	1.9886
3200	4.6894e-8	2.0291	4.9039e-8	1.9847
6400	1.0812e-8	2.1168	1.2338e-8	1.9908
	$\alpha = 0.5,\ \beta = 0.5,\ r = 1.5$			
100	2.3470e-4		2.3502e-4	
200	7.9009e-5	1.5707	7.9123e-5	1.5706
400	2.6959e-5	1.5513	2.6999e-5	1.5512
800	9.2904e-6	1.5370	9.3052e-6	1.5368
1600	3.2243e-6	1.5268	3.2304e-6	1.5263
3200	1.1240e-6	1.5203	1.1271e-6	1.5191
6400	3.9154e-7	1.5214	3.9355e-7	1.5180
	$\alpha = 0.9,\ \beta = 0.9,\ r = 1.1$			
100	2.5529e-3		2.5570e-3	
200	1.1847e-3	1.1076	1.1866e-3	1.1076
400	5.5094e-4	1.1046	5.5184e-4	1.1045
800	2.5655e-4	1.1027	2.5697e-4	1.1027
1600	1.1956e-4	1.1015	1.1976e-4	1.1015
3200	5.5740e-5	1.1009	5.5835e-5	1.1009
6400	2.5993e-5	1.1006	2.6039e-5	1.1005
	$\alpha = 0.9,\ \beta = 0.1,\ r = 1.1$			
100	1.0620e-2		1.0646e-2	
200	4.9568e-3	1.0993	4.9688e-3	1.0993
400	2.3130e-3	1.0996	2.3186e-3	1.0996
800	1.0792e-3	1.0998	1.0819e-3	1.0997
1600	5.0352e-4	1.0998	5.0475e-4	1.0999
3200	2.3491e-4	1.0999	2.3549e-4	1.0999
6400	1.0959e-4	1.1000	1.0986e-4	1.1000

Table 2: The error $\delta(U)$ and the order of convergence p for $l = 2$

N_t	$N_x = N_y = 25$		$N_x = N_y = 50$	
	$\alpha = 0.1,\ \beta = 0.5,\ r = 1.5$			
100	4.1501e-4		4.1576e-4	
200	1.5576e-4	1.4138	1.5604e-4	1.4138
400	5.7438e-5	1.4392	5.7542e-5	1.4392
800	2.0929e-5	1.4565	2.0965e-5	1.4566
1600	7.5639e-6	1.4684	7.5746e-6	1.4687
3200	2.7190e-6	1.4761	2.7205e-6	1.4773
6400	9.7497e-7	1.4796	9.7315e-7	1.4831
	$\alpha = 0.5,\ \beta = 0.9,\ r = 1.1$			
100	3.6896e-3		3.6961e-3	
200	1.7901e-3	1.0434	1.7933e-3	1.0434
400	8.5851e-4	1.0601	8.6005e-4	1.0601
800	4.0861e-4	1.0711	4.0934e-4	1.0711
1600	1.9345e-4	1.0788	1.9380e-4	1.0787
3200	9.1244e-5	1.0842	9.1407e-5	1.0842
6400	4.2922e-5	1.0880	4.2996e-5	1.0881
	$\alpha = 0.1,\ \beta = 0.9,\ r = 1.1$			
100	4.6066e-3		4.6181e-3	
200	2.1573e-3	1.0945	2.1626e-3	1.0945
400	1.0056e-3	1.1012	1.0080e-3	1.1013
800	4.6776e-4	1.1042	4.6889e-4	1.1042
1600	2.1739e-4	1.1055	2.1791e-4	1.1055
3200	1.0101e-4	1.1058	1.0125e-4	1.1058
6400	4.6936e-5	1.1057	4.7046e-5	1.1058
	$\alpha = 0.0,\ \beta = 1.0,\ r = 1.0$			
100	8.0290e-3		8.0496e-3	
200	4.0493e-3	0.9875	4.0601e-3	0.9874
400	2.0333e-3	0.9938	2.0388e-3	0.9938
800	1.0188e-3	0.9970	1.0216e-3	0.9969
1600	5.0997e-4	0.9984	5.1135e-4	0.9984
3200	2.5513e-4	0.9992	2.5581e-4	0.9992
6400	1.2760e-4	0.9996	1.2794e-4	0.9996

convergence in time is limited by $\min\{2-\alpha, 2-\beta\}$ and relatively large numbers of time steps are needed to obtain high accuracy in time. Therefore the possibility to use relatively small numbers of space steps N_x and N_y is important. Of course, the compact scheme has fourth order accuracy for sufficiently smooth solutions.

Our future work includes theoretical analysis of the stability and convergence, numerical study of the acceleration for the parallel computations, numerical and theoretical investigation for other approximations of the time-fractional derivatives, numerical experiments for practical problems.

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