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THE SIS-MODEL ON TIME SCALES

Martin Bohner, Sabrina H. Streipert

In this paper, we introduce the epidemic model following the hypothesis of the disease flow *Susceptible* $\rightarrow$ *Infected* $\rightarrow$ *Susceptible*, short *SIS*, on time scales. After a brief introduction of time scales, we present dynamic systems representing the SIS-model on time scales and derive its solution sets. We are discussing the stability of the steady states before investigating a modified SIS-model including a birth and death rate. Throughout, examples are used to illustrate the results.

1. Classical Epidemic Models

The earliest evidence of using mathematical models to describe the spread of disease was carried out in 1766 by Daniel Bernoulli [7], where he showed that universal inoculation against smallpox would increase the life expectancy. In 1927, Kermack and McKendrick published the first *susceptible-infectious-recovered* model in epidemics, which served as the foundation for many existing models. The authors continued their studies in epidemiology and contributed mainly to the development of mathematical modeling of epidemics. Since then, the progress is continuing steadily; in particular since 2002 with the outbreak of SARS (Severe Acute Respiratory Syndrome). The interest in modeling epidemics mathematically to predict the course of the disease and provide an optimal strategy to support political decisions [4] has been increasing. Most epidemic models are based on dividing the population into groups that are identical in terms of their

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status with respect to the modeled disease. The *SIR-model* distinguishes between three classes:

- *Susceptible:* individuals not yet infected with the disease,
- *Infectious:* individuals who have been infected with the disease and are capable of transmitting the disease to members in the susceptible category,
- *Recovered:* individuals with immunity, who can not transmit the disease.

Traditionally, a constant population is assumed, so that $S+I+R=N$, with N the total population.

In 1927, Kermack and McKendrick introduced the basic *SIR-model* for a constant population as

$$S' = -\frac{\beta SI}{N}, \quad I' = \frac{\beta SI}{N} - \gamma I, \quad R' = \gamma I,$$

where S represents the population in the class of susceptible, I the individuals that are infected, and R the population density in the class of individuals that are recovered/removed. The model is following the assumptions that an individual must be considered as having an equal probability as every other individual of contracting the disease with a rate of β, which is considered as the contact or infection rate of the disease. An infected individual is able to transmit the disease to individuals in the other groups. The population leaving the susceptible class is equal to the number entering the infected class [10]. However, infected members recovered from the disease enter the removed class. Birth and death rates are ignored in this model, by the assumption that the rate of infection and recovery is much faster than the rate of births and deaths. Using the assumption of a constant population, we can eliminate the recovered class in the model and analyze a system of dimension two.

Based on this model, a variety of models have been designed for infectious diseases, see [5, 6, 7, 9]. The complexity of the resulting models varies, and its analysis is therefore approached in different ways using bifurcation theory and Lyapunov functions to analyze stability [8, 11, 13].

In some epidemics, especially regarding sexually transmitted diseases such as gonorrhea [3], the recovered individuals are again susceptible to infection. These assumptions are establishing a special class of *SIR-model*, the *SIS-model*, which is given by

$$S' = -\frac{\beta SI}{N} + \gamma I, \quad I' = \frac{\beta SI}{N} - \gamma I, \tag{1.1}$$

where S and I represent the population of the class of susceptible and infected individuals respectively, with the transmission rate β and recovery rate γ.

In [12], the authors studied (1.1) without the normalizing factor N. Using the assumption of a constant population, $S' + I' = 0$, which enables the elimination of one variable. Applying further the transformation $z = 1/I$ yields a nonhomogeneous first-order differential equation, whose solution is

$$ze^{(\beta N-\gamma)t} = \int \beta e^{(\beta N-\gamma)t}\mathrm{d}t + C,$$

where C is an integrating constant and $N = S_0 + I_0$. Resubstituting yields the solution to the model described in (1.1).

2. Time Scales Preliminaries

To simulate real-life situations, a general time scale is used to make the demonstrations more realistic. That causes the interest in epidemic models on time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ [1, p. 1]. In $\mathbb{T}$, the forward jump operator σ is critical to define the graininess function μ.

Definition 2.1. (See [1, Definition 1.1]) *For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is*

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}.$$

Definition 2.2. (See [1, Definition 2.25]) *A function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided*

$$1 + \mu(t)p(t) \neq 0 \quad \textit{for all} \quad t \in \mathbb{T}, \quad \textit{where} \quad \mu(t) = \sigma(t) - t.$$

The set of all regressive functions is denoted by $\mathcal{R}$. Moreover, $p \in \mathcal{R}$ is called positively regressive, denoted by $\mathcal{R}^+$, if

$$1 + \mu(t)p(t) > 0 \quad \textit{for all} \quad t \in \mathbb{T}.$$

Definition 2.3. *If $t \in \mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.*

Definition 2.4. (See [2, Definition 1.24]) *A function $p : \mathbb{T} \to \mathbb{R}$ is rd-continuous provided p is continuous at all right-dense points and the left-sided limit exists at all left-dense points. The set of rd-continuous functions is denoted by* $\mathrm{C_{rd}} = \mathrm{C_{rd}}(\mathbb{T}) = \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$ *and the set of* regressive *and* rd-continuous functions *is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.*

Definition 2.5. (See [1, Definition 1.10]) *Assume $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. Then we define the derivative of f, denoted by f^Δ, to be the number, such that for all $\varepsilon > 0$ there exists $\delta > 0$, such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s| \quad \textit{for all} \quad s \in [t - \delta, t + \delta] \cap \mathbb{T}.$$

Theorem 2.6. *If f is any differentiable function, then*

$$f^\sigma(t) := f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

Theorem 2.7. (See [1, Theorem 2.33]) *If $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$, then the initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0) = 1 \tag{2.1}$$

has a unique solution.

Definition 2.8. *If $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$, then the unique solution of* (2.1) *is called the exponential function on the time scale and denoted by $e_p(\cdot, t_0)$.*

Some useful properties of the dynamic exponential function are the following.

Theorem 2.9. (See [1, Theorem 2.36]) *If $p \in \mathcal{R}$, then*

1. *$e_0(t, s) = 1$, and $e_p(t, t) = 1$,*
2. *the semigroup property holds: $e_p(t, r)e_p(r, s) = e_p(t, s)$.*

On $\mathcal{R}$, we define a "circle plus" and a "circle minus" operation as follows.

Definition 2.10. (See [2, p. 13]) *Define the "circle plus" addition on $\mathcal{R}$ as*

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t),$$

and the "circle minus" subtraction as

$$(p \ominus q)(t) = \frac{p(t) - q(t)}{1 + \mu(t)q(t)}.$$

Theorem 2.11. (Variation of Constants, see [2, Theorem 2.1]) *Let* $p \in \mathcal{R}$ *and* $f \in \mathrm{C}_{\mathrm{rd}}$. *Let* $t_0 \in \mathbb{T}$ *and* $y_0 \in \mathbb{R}$. *The unique solution of the IVP*

$$y^\Delta = p(t)y + f(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(s))f(s)\Delta s.$$

The unique solution of the IVP

$$x^\Delta = -p(t)x^\sigma + f(t), \quad x(t_0) = x_0$$

is given by

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^{t} e_{\ominus p}(t, s)f(s)\Delta s.$$

It is not hard to show the following identities.

Corollary 2.12. *Assume* p *and* $q \in \mathcal{R}$. *Then*

a) $e_{p\oplus q}(t, s) = e_p(t, s)e_q(t, s)$,

b) $e_{\ominus p}(t, s) = e_p(s, t) = \dfrac{1}{e_p(t, s)}$.

Lemma 2.13. (See [1, Theorem 2.39]) *If* $p \in \mathcal{R}$ *and* $a, b, c \in \mathbb{T}$, *then*

$$\int_a^b p(t)e_p(t, c)\Delta t = e_p(b, c) - e_p(a, c) \tag{2.2}$$

and

$$\int_a^b p(t)e_p(c, \sigma(t))\Delta t = e_p(c, a) - e_p(c, b). \tag{2.3}$$

The integration by parts formulas read as follows.

Theorem 2.14. (See [2, Theorem 1.28]) *If* $a, b \in \mathbb{T}$ *and* $f, g \in \mathrm{C}_{\mathrm{rd}}$, *then*

$$\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g(t)\Delta t \tag{2.4}$$

and

$$\int_a^b f(t)g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t. \tag{2.5}$$

This basic introduction is critical to understand the definition of the expressions of the dynamic susceptible-infected-susceptible epidemic model.

3. The Dynamic SIS-Model

We define the dynamic susceptible-infected-susceptible-model under the assumption of a constant population by

$$S^\Delta = -\beta S^\sigma I + \gamma I, \quad I^\Delta = \beta S^\sigma I - \gamma I, \tag{3.1}$$

where $S : \mathbb{T} \to \mathbb{R}^+$ represents the amount of infected individuals and $I : \mathbb{T} \to \mathbb{R}_0^+$ the number of susceptible individuals in the population. The transmission rate of the disease is given by the constant $\beta > 0$ and the recovery rate by $\gamma > 0$.

Theorem 3.1. *The unique solution to the IVP* (3.1) *with* $S_0 > 0$ *and* $I_0 \geq 0$ *is given by*

$$I(t) = \frac{\alpha}{e_{\ominus\alpha}(t, t_0)\left(\frac{\alpha}{I_0} - \beta\right) + \beta} \tag{3.2}$$

and

$$S(t) = \kappa - \frac{\alpha}{e_{\ominus\alpha}(t, t_0)\left(\frac{\alpha}{I_0} - \beta\right) + \beta}, \tag{3.3}$$

where $\alpha = \beta\kappa - \gamma \in \mathcal{R}^+$ *and* $\kappa = I_0 + S_0$.

Proof. If we add the two equations from (3.1), then we have

$$(S + I)^\Delta = 0, \quad \text{so} \quad S + I = \kappa,$$

where κ represents the constant population. We can now eliminate $S = \kappa - I$ in the second equation of (3.1) to get

$$I^\Delta(t) = \beta\left(\kappa - I^\sigma(t)\right) I(t) - \gamma I(t).$$

Applying the transformation $y = 1/I$ yields

$$y^{\Delta}(t) = -\beta\left(\kappa y^{\sigma}(t) - 1\right) + \gamma y^{\sigma}(t) = -y^{\sigma}(t)(\beta\kappa - \gamma) + \beta,$$

which by Theorem 2.11 has the solution

$$\begin{aligned} y(t) &= e_{\ominus\alpha}(t,t_0)y(t_0) + \int_{t_0}^{t} e_{\ominus\alpha}(t,s)\beta\Delta s \\ &= e_{\ominus\alpha}(t,t_0)y(t_0) + \frac{\beta}{\alpha}\int_{t_0}^{t} e_{\alpha}(s,t)\alpha\Delta s \\ &\overset{(2.2)}{=} e_{\ominus\alpha}(t,t_0)y(t_0) + \frac{\beta}{\alpha}(1 - e_{\alpha}(t_0,t)) \\ &= e_{\ominus\alpha}(t,t_0)\left(y(t_0) - \frac{\beta}{\alpha}\right) + \frac{\beta}{\alpha}, \end{aligned}$$

where $\alpha = \beta\kappa - \gamma$ is constant. To obtain the desired solution, we resubstitute $I = 1/y$ to get

$$I(t) = \frac{\alpha}{e_{\ominus\alpha}(t,t_0)\left(\frac{\alpha}{I_0} - \beta\right) + \beta},$$

where $I_0 = I(t_0)$. The amount of susceptible individuals in the population is then

$$S(t) = \kappa - \frac{\alpha}{e_{\ominus\alpha}(t,t_0)\left(\frac{\alpha}{\kappa - S_0} - \beta\right) + \beta},$$

which completes the proof. □

Theorem 3.2. (Stability Theorem) *The steady states of* (3.1) *are*

$$(S_1^*, I_1^*) = (\kappa, 0) \quad \textit{and} \quad (S_2^*, I_2^*) = \left(\frac{\gamma}{\beta}, \kappa - \frac{\gamma}{\beta}\right).$$

Let $R_0 = \dfrac{\kappa\beta}{\gamma}$ *be the reproduction rate. Then the following holds:*

a) If $R_0 < 1$*, then the steady state* (S_1^*, I_1^*) *is asymptotically stable for solutions with initial conditions* $(S_0, I_0) \in \mathbb{R}^+ \times \mathbb{R}_0^+$*.*

b) If $R_0 > 1$*, then the steady state* (S_2^*, I_2^*) *is asymptotically stable for solutions with initial conditions* $(S_0, I_0) \in \mathbb{R}^+ \times \mathbb{R}^+$*.*

Proof. It is easy to verify that (S_1^*, I_1^*) and (S_2^*, I_2^*) yield $S^\Delta = 0$ and $I^\Delta = 0$. By Theorem 3.1, the solution for (3.1) is given by (3.2) and (3.3). Assume $R_0 < 1$. Then $\alpha = \beta\kappa - \gamma < 0$ and $e_\alpha(t, t_0) \to 0$, i.e., $e_{\ominus\alpha}(t, t_0) \to \infty$ as $t \to \infty$. We then have

$$I(t) \to 0, \quad S(t) \to \kappa \quad \text{as} \quad t \to \infty.$$

Assume now $R_0 > 1$. Then $\alpha = \beta\kappa - \gamma > 0$ and $e_{\ominus\alpha}(t, t_0) \to 0$ as $t \to \infty$. We then have

$$I(t) \to \frac{\alpha}{\beta} = \frac{\beta\kappa - \gamma}{\beta} = \kappa - \frac{\gamma}{\beta}, \quad S(t) \to \frac{\gamma}{\beta} \quad \text{as} \quad t \to \infty,$$

which completes the proof. □

Remark 3.3. Note that Theorem 3.2 is consistent with the analysis for $\mathbb{R}$, where the classical SIS-model is given by (1.1). The steady state (S_1^*, I_1^*) is asymptotically stable for positive solutions if the reproduction rate $R_0 < 1$. If $R_0 > 1$, then the steady state (S_2^*, I_2^*) is asymptotically stable. Also, if $\mathbb{T} = \mathbb{R}$, then (3.2) and (3.3) are

$$I(t) = \frac{\alpha}{e^{-\alpha(t-t_0)}\left(\frac{\alpha}{I_0} - \beta\right) + \beta}$$

and

$$S(t) = \kappa - \frac{\alpha}{e^{-\alpha(t-t_0)}\left(\frac{\alpha}{\kappa - S_0} - \beta\right) + \beta},$$

where $\alpha = \beta\kappa - \gamma$ and $\kappa = I_0 + S_0$. This is consistent with the well-known solutions for (1.1).

Remark 3.4. If we normalize the population, i.e., $\kappa = S + I = 1$, then the solutions of (3.1) can be expressed as

$$I(t) = I_0 \frac{\beta - \gamma}{e_\alpha(t_0, t)\,(\beta S_0 - \gamma) + \beta I_0},$$

and the number of susceptible is

$$S(t) = \frac{e_\alpha(t_0, t)\,(\beta S_0 - \gamma) + \gamma I_0}{e_\alpha(t_0, t)\,(\beta S_0 - \gamma) + \beta I_0}.$$

Example 3.5. If $\mathbb{T} = \mathbb{Z}$, then Theorem 3.1 states that the unique solution to

$$\Delta S_n = -\beta S_{n+1} I_n + \gamma I_n, \quad \Delta I_n = \beta S_{n+1} I_n - \gamma I_n$$

with initial conditions $S_0 + I_0 = \kappa$ is given by

$$I_n = I_0 \frac{\alpha(1+\alpha)^n}{(\alpha - \beta I_0) + \beta I_0 (1+\alpha)^n}.$$

The amount of susceptible individuals in the population is then

$$S_n = S_0 + I_0 \frac{(\alpha - \beta I_0)(1 - (1+\alpha)^n)}{(\alpha - \beta I_0) + \beta I_0 (1+\alpha)^n},$$

where $\alpha = \beta\kappa - \gamma$.

In time scales, the SIS-model can appear in a slightly different form. Another expression of the dynamic SIS epidemic model is given by

$$S^\Delta = -\beta S I^\sigma + \gamma I^\sigma, \quad I^\Delta = \beta S I^\sigma - \gamma I^\sigma, \tag{3.4}$$

where I represents the amount of infected individuals in the population and S the susceptible individuals in the population, β is the transmission rate and γ the recovery rate.

Theorem 3.6. *The unique solution to the IVP* (3.4) *with* $I_0 \geq 0$ *and* $S_0 > 0$ *is given by*

$$I(t) = \frac{\tilde{\alpha}}{e_{\tilde{\alpha}}(t, t_0)\left(\frac{\tilde{\alpha}}{I_0} + \beta\right) - \beta} \tag{3.5}$$

and

$$S(t) = \kappa - \frac{\tilde{\alpha}}{e_{\tilde{\alpha}}(t, t_0)\left(\frac{\tilde{\alpha}}{\kappa - S_0} + \beta\right) - \beta}, \tag{3.6}$$

where $\tilde{\alpha} = \gamma - \beta\kappa \in \mathcal{R}^+$ *and* $\kappa = I_0 + S_0$.

P r o o f. Adding the two equations from (3.4) yields

$$(S + I)^\Delta = 0, \quad \text{so} \quad S + I = \kappa,$$

which corresponds to the assumption of a constant population κ. The relation aids to eliminate S, and the second equation of (3.4) becomes

$$I^{\Delta}(t) = \beta(\kappa - I(t))I^{\sigma}(t) - \gamma I^{\sigma}(t). \tag{3.7}$$

Applying the transformation $y = 1/I$ to (3.7) yields

$$y^{\Delta}(t) = -\beta(\kappa y(t) - 1) + \gamma y(t) = y(t)(\gamma - \beta\kappa) + \beta.$$

The unique solution is given by Theorem 2.11 as

$$\begin{aligned} y(t) &= e_{\tilde{\alpha}}(t, t_0)y(t_0) + \int_{t_0}^{t} e_{\tilde{\alpha}}(t, \sigma(s))\beta\Delta s \\ &= e_{\tilde{\alpha}}(t, t_0)y(t_0) + \frac{\beta}{\tilde{\alpha}}\int_{t_0}^{t} e_{\tilde{\alpha}}(t, \sigma(s))\tilde{\alpha}\Delta s \\ &\overset{(2.3)}{=} e_{\tilde{\alpha}}(t, t_0)y(t_0) + \frac{\beta}{\tilde{\alpha}}\left(e_{\tilde{\alpha}}(t, t_0) - 1\right) \\ &= e_{\tilde{\alpha}}(t, t_0)\left(y(t_0) + \frac{\beta}{\tilde{\alpha}}\right) - \frac{\beta}{\tilde{\alpha}}, \end{aligned}$$

where $\tilde{\alpha} = \gamma - \beta\kappa$ is constant. Resubstituting $I = 1/y$ results in

$$I(t) = \frac{\tilde{\alpha}}{e_{\tilde{\alpha}}(t, t_0)\left(\tilde{\alpha}y(t_0) + \beta\right) - \beta},$$

i.e.,

$$I(t) = \frac{\tilde{\alpha}}{e_{\tilde{\alpha}}(t, t_0)\left(\frac{\tilde{\alpha}}{I_0} + \beta\right) - \beta},$$

where $I_0 = I(t_0)$. The amount of susceptible individuals in the population is then

$$S(t) = \kappa - \frac{\tilde{\alpha}}{e_{\tilde{\alpha}}(t, t_0)\left(\frac{\tilde{\alpha}}{\kappa - S_0} + \beta\right) - \beta}.$$

This completes the proof. □

Theorem 3.7. (Stability Theorem) *The steady states of* (3.4) *are*

$$(S_1^*, I_1^*) = (\kappa, 0) \quad \text{and} \quad (S_2^*, I_2^*) = \left(\frac{\gamma}{\beta}, \kappa - \frac{\gamma}{\beta}\right).$$

Let $R_0 = \dfrac{\kappa\beta}{\gamma}$ *be the reproduction rate. Then the following holds:*

a) If $R_0 < 1$*, then the steady state* (S_1^*, I_1^*) *is asymptotically stable for solutions with initial conditions* $(S_0, I_0) \in \mathbb{R}^+ \times \mathbb{R}_0^+$*.*

b) If $R_0 > 1$*, then the steady state* (S_2^*, I_2^*) *is asymptotically stable for solutions with initial conditions* $(S_0, I_0) \in \mathbb{R}^+ \times \mathbb{R}^+$*.*

Proof. It is easy to verify that (S_1^*, I_1^*) and (S_2^*, I_2^*) yield $S^\Delta = 0$ and $I^\Delta = 0$. By Theorem 3.6, the solution for (3.4) is given by (3.5) and (3.6). Assume $R_0 < 1$. Then $\tilde{\alpha} = \gamma - \beta\kappa > 0$ and $e_{\tilde{\alpha}}(t, t_0) \to \infty$ as $t \to \infty$. We then have

$$I(t) \to 0, \quad S(t) \to \kappa \quad \text{as} \quad t \to \infty.$$

Assume now $R_0 > 1$. Then $\tilde{\alpha} = \gamma - \beta\kappa < 0$ and $e_{\tilde{\alpha}}(t, t_0) \to 0$ as $t \to \infty$. We then have

$$I(t) \to \frac{\tilde{\alpha}}{-\beta} = \frac{\gamma - \beta\kappa}{-\beta} = \kappa - \frac{\gamma}{\beta}, \quad S(t) \to \frac{\gamma}{\beta} \quad \text{as} \quad t \to \infty,$$

which completes the proof. □

Note that this is again consistent with the analysis for $\mathbb{R}$.

Example 3.8. If $\mathbb{T} = \mathbb{Z}$, then Theorem 3.1 states that the unique solution to

$$\Delta S_n = -\beta S_n I_{n+1} + \gamma I_{n+1}, \quad \Delta I_n = \beta S_n I_{n+1} - \gamma I_{n+1}$$

with initial conditions $S_0 + I_0 = S(t_0) + I(t_0) = \kappa$ is

$$I_n = \frac{\tilde{\alpha}}{e_{\tilde{\alpha}}(n, 0)\left(\frac{\tilde{\alpha}}{I_0} + \beta\right) - \beta} = I_0 \frac{\tilde{\alpha}}{(1 + \tilde{\alpha})^n (\tilde{\alpha} + \beta I_0) - \beta I_0}.$$

The group of susceptible is

$$S_n = \kappa - I_n = \kappa - \frac{\tilde{\alpha} I_0}{(1 + \tilde{\alpha})^n (\tilde{\alpha} + \beta I_0) - \beta I_0},$$

where $\tilde{\alpha} = \gamma - \beta\kappa$.

4. Equal Birth and Death Rate

Let us now consider the dynamic SIS-model with equal birth and death rate ν. In the continuous case, the model reads as

$$S' = -\beta SI + \gamma I + \nu\kappa - \nu S, \quad I' = \beta SI - \gamma I - \nu I.$$

On time scales, the model appears in different forms. One version is

$$S^{\Delta} = -\beta S^{\sigma} I + \gamma I + \nu\kappa - \nu S, \quad I^{\Delta}(t) = \beta S^{\sigma} I - \gamma I - \nu I, \tag{4.1}$$

where $S : \mathbb{T} \to \mathbb{R}^{+}$ represents the amount of infected individuals, $I : \mathbb{T} \to \mathbb{R}_0^{+}$ the number of susceptible individuals in the population. As before, the transmission rate of the disease is given by the constant $\beta > 0$ and the recovery rate by $\gamma > 0$. The birth rate is chosen to be equal to the death rate $\nu > 0$.

Theorem 4.1. *The unique solution to the IVP* (4.1) *with* $S_0 > 0$ *and* $I_0 \geq 0$ *is given by*

$$I(t) = \frac{I_0}{e_{\ominus\alpha}(t,t_0)\left(1 - \frac{I_0\beta}{\alpha}\right) + \frac{I_0\beta}{\alpha}} \tag{4.2}$$

and

$$S(t) = \kappa - \frac{\kappa - S_0}{e_{\ominus\alpha}(t,t_0)\left(\frac{S_0\beta - (\gamma+\nu)}{\alpha}\right) + \frac{(\kappa - S_0)\beta}{\alpha}}, \tag{4.3}$$

where $\alpha = \beta\kappa - \gamma - \nu \in \mathcal{R}^{+}$ *and* $\kappa = I_0 + S_0$ *with* $S_0 = S(t_0)$ *and* $I_0 = I(t_0)$.

Proof. System (4.1) can be rewritten as (3.1) and be treated in a similar fashion. Realize that $S^{\Delta} + I^{\Delta} = 0$ and therefore

$$S(t) + I(t) = S(t_0) + I(t_0) = \kappa.$$

Using $S = \kappa - I$ in the second equation of (4.1), we obtain

$$I^{\Delta} = \beta(\kappa - I^{\sigma})I - (\gamma + \nu)I.$$

Applying the substitution $y = 1/I$, we obtain the linear first-order dynamic equation

$$y^{\Delta} = -(\beta\kappa - \gamma - \nu)y^{\sigma} + \beta.$$

The solution is

$$y(t) = e_{\ominus\alpha}(t, t_0)y(t_0) + \int_{t_0}^{t} e_{\ominus\alpha}(t, s)\beta\Delta s,$$

where $\alpha = \beta\kappa - \gamma - \nu$. Integrating yields

$$y(t) = e_{\ominus\alpha}(t, t_0)y(t_0) + \frac{\beta}{\alpha}\left(1 - e_{\ominus\alpha}(t, t_0)\right).$$

Resubstituting $y = 1/I$ gives

$$I(t) = \frac{I_0}{e_{\ominus\alpha}(t, t_0)\left(1 - \frac{I_0\beta}{\alpha}\right) + \frac{I_0\beta}{\alpha}}$$

and

$$S(t) = \kappa - \frac{\kappa - S_0}{e_{\ominus\alpha}(t, t_0)\left(\frac{S_0\beta - (\gamma+\nu)}{\alpha}\right) + \frac{(\kappa - S_0)\beta}{\alpha}}.$$

The proof is complete. □

Theorem 4.2. (Stability Theorem) *The steady states of* (4.1) *are*

$$(S_1^*, I_1^*) = (\kappa, 0) \quad \textit{and} \quad (S_2^*, I_2^*) = \left(\frac{\gamma+\nu}{\beta}, \kappa - \frac{\gamma+\nu}{\beta}\right).$$

Let $R_0 = \dfrac{\kappa\beta}{\gamma+\nu}$ *be the reproduction rate. Then the following holds:*

a) If $R_0 < 1$*, then the steady state* (S_1^*, I_1^*) *is asymptotically stable for solutions with initial conditions* $(S_0, I_0) \in \mathbb{R}^+ \times \mathbb{R}_0^+$*.*

b) If $R_0 > 1$*, then the steady state* (S_2^*, I_2^*) *is asymptotically stable for solutions with initial conditions* $(S_0, I_0) \in \mathbb{R}^+ \times \mathbb{R}^+$*.*

Proof. It is easy to verify that (S_1^*, I_1^*) and (S_2^*, I_2^*) yield $S^\Delta = 0$ and $I^\Delta = 0$. Assume $R_0 < 1$. Then $\alpha = \beta\kappa - (\gamma + \alpha) < 0$ and $e_\alpha(t, t_0) \to 0$, i.e., $e_{\ominus\alpha}(t, t_0) \to \infty$ as $t \to \infty$. We then have

$$I(t) \to 0, \quad S(t) \to \kappa \quad \text{as} \quad t \to \infty.$$

Assume now $R_0 > 1$. Then $\alpha = \beta\kappa - (\gamma+\nu) > 0$ and $e_{\ominus\alpha}(t,t_0) \to 0$ as $t \to \infty$. We then have

$$I(t) \to \frac{\alpha}{\beta} = \frac{\beta\kappa - (\gamma+\nu)}{\beta}, \quad S(t) \to \frac{\gamma+\nu}{\beta} \quad \text{as} \quad t \to \infty,$$

which completes the proof. $\square$

Example 4.3. Consider the time scale $\mathbb{T} = [0,3] \cup \{4,5,6\} \cup [7,9]$ and $t_0 = 0$. Then the solution to (4.1) is given by

$$I(t) = \begin{cases} I_0 \dfrac{\alpha e^{\alpha t}}{I_0\beta\left(e^{\alpha t}-1\right)+\alpha} & \text{if} \quad t \in [0,3] \cap [7,9], \\ I_0 \dfrac{\alpha\left(1+\alpha\right)^t}{I_0\beta\left(\left(1+\alpha\right)^t - 1\right)+\alpha} & \text{if} \quad t \in \{4,5,6\}. \end{cases}$$

For Figure 1, the values are

$$\kappa = 1, \ (S_0, I_0) = (0.8, 0.2), \ \beta = 0.6, \ \gamma = 0.3, \ \nu = 0.4.$$

Then $\alpha = -0.1$ has been chosen. We can clearly see that the class of infected individuals is vanishing. If we slightly change $\nu = 0.1$, then $\alpha = 0.2$, and we obtain the values presented in Figure 2. The class of infected individuals is increasing to a positive value. Note that the values are $R_0 < 1$ for Figure 1 and $R_0 > 1$ for Figure 2.

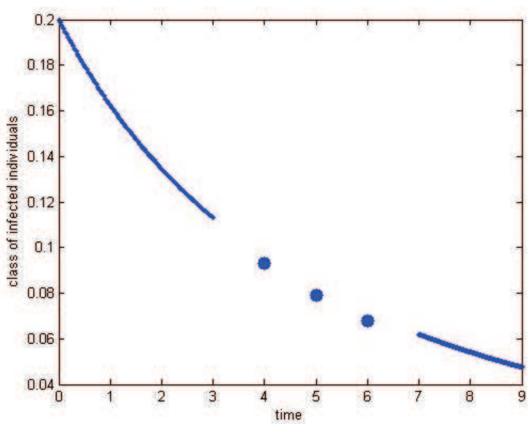

Figure 1: The class of susceptible I for $\alpha < 0$, i.e., $R_0 < 1$

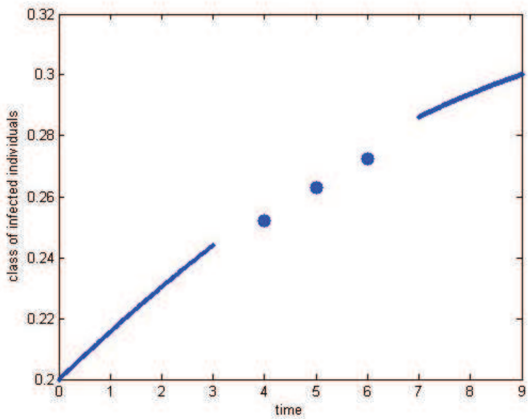

Figure 2: The class of susceptible I for $\alpha > 0$, i.e., $R_0 > 1$

Similarly, this procedure can be applied to a modified version of (3.4), which gives the following theorem.

Theorem 4.4. *The unique solution to*

$$S^\Delta = -\beta S I^\sigma + \gamma I^\sigma + \nu\kappa - \nu S, \quad I^\Delta = \beta S I^\sigma - \gamma I^\sigma - \nu I \tag{4.4}$$

with $I(t_0) + S(t_0) = I_0 + S_0 = \kappa$ *is given by*

$$I(t) = \frac{I_0}{e_{\tilde{\alpha}}(t, t_0)\left(1 + \frac{I_0\beta}{\gamma+\nu-\beta\kappa}\right) - \frac{I_0\beta}{\gamma+\nu-\beta\kappa}}$$

and

$$S(t) = \kappa - \frac{I_0}{e_{\tilde{\alpha}}(t, t_0)\left(1 + \frac{I_0\beta}{\gamma+\nu-\beta\kappa}\right) - \frac{I_0\beta}{\gamma+\nu-\beta\kappa}},$$

where $\tilde{\alpha} = \dfrac{\gamma + \nu - \beta\kappa}{1 - \mu\nu} = ((\gamma - \beta\kappa) \ominus (-\nu)) \in \mathcal{R}^+$ *is bounded and* $\kappa = I_0 + S_0$.

P r o o f. Clearly, $(S + I)^\Delta = 0$, so $S(t) + I(t) = \kappa$, and the second equation becomes

$$I^\Delta = \beta(\kappa - I)I^\sigma - \gamma I^\sigma - \nu I,$$

i.e.,

$$I^{\Delta} = \frac{\beta}{1-\mu\nu}(\kappa - I)I^{\sigma} - \frac{\gamma+\nu}{1-\mu\nu}I^{\sigma}.$$

Applying the substitution $y = 1/I$ yields

$$y^{\Delta} = \left(\frac{\gamma+\nu-\beta\kappa}{1-\mu\nu}\right)y + \frac{\beta}{1-\mu\nu},$$

which has the solution

$$y(t) = e_{\tilde{\alpha}}(t,t_0)y(t_0) + \int_{t_0}^{t} e_{\tilde{\alpha}}(t,\sigma(s))\frac{\beta}{1-\mu(s)\nu}\Delta s,$$

where $\tilde{\alpha} = ((\gamma - \beta\kappa) \ominus (-\nu))$. The solution is equivalent to

$$y(t) = e_{\tilde{\alpha}}(t,t_0)y(t_0) + \frac{\beta}{\gamma+\nu-\beta\kappa}\int_{t_0}^{t} e_{\tilde{\alpha}}(t,\sigma(s))\tilde{\alpha}\Delta s,$$

i.e.,

$$y(t) = e_{\tilde{\alpha}}(t,t_0)y(t_0) + \frac{\beta}{\gamma+\nu-\beta\kappa}\left(e_{\tilde{\alpha}}(t,t_0) - 1\right).$$

Resubstituting $I = 1/y$ yields

$$I(t) = \frac{I_0}{e_{\tilde{\alpha}}(t,t_0)\left(1 + \frac{I_0\beta}{\gamma+\nu-\beta\kappa}\right) - \frac{I_0\beta}{\gamma+\nu-\beta\kappa}}$$

and therefore

$$S(t) = \kappa - \frac{I_0}{e_{\tilde{\alpha}}(t,t_0)\left(1 + \frac{I_0\beta}{\gamma+\nu-\beta\kappa}\right) - \frac{I_0\beta}{\gamma+\nu-\beta\kappa}},$$

completing the proof. □

We can further discuss the stability of the steady states for this model, summarized in the following theorem.

Theorem 4.5. (Stability Theorem) *Assume* $\gamma - \beta\kappa \in \mathcal{R}^+$ *and* $-\nu \in \mathcal{R}^+$. *The steady states of* (4.4) *are*

$$(S_1^*, I_1^*) = (\kappa, 0) \quad \textit{and} \quad (S_2^*, I_2^*) = \left(\frac{\gamma+\nu}{\beta}, \kappa - \frac{\gamma+\nu}{\beta}\right).$$

Let $R_0 = \dfrac{\kappa\beta}{\gamma+\nu}$ *be the reproduction rate. Then the following holds:*

a) If $R_0 < 1$*, then the steady state* (S_1^*, I_1^*) *is asymptotically stable for solutions with initial conditions* $(S_0, I_0) \in \mathbb{R}^+ \times \mathbb{R}_0^+$*.*

b) If $R_0 > 1$*, then the steady state* (S_2^*, I_2^*) *is asymptotically stable for solutions with initial conditions* $(S_0, I_0) \in \mathbb{R}^+ \times \mathbb{R}^+$*.*

Proof. It is easy to verify that if $\beta S - \gamma \in \mathcal{R}$, (S_1^*, I_1^*) and (S_2^*, I_2^*) yield $S^\Delta = 0$ and $I^\Delta = 0$. Assume $R_0 < 1$. Then $\tilde{\alpha} = \dfrac{-\beta\kappa + (\gamma+\nu)}{1+\mu(-\nu)} > 0$ and $e_{\tilde{\alpha}}(t, t_0) \to \infty$ as $t \to \infty$. We then have

$$I(t) \to 0, \quad S(t) \to \kappa \quad \text{as} \quad t \to \infty.$$

Assume now $R_0 > 1$. Then $\tilde{\alpha} = \dfrac{\gamma+\nu-\beta\kappa}{1+\mu(-\nu)} < 0$ and therefore $e_{\tilde{\alpha}}(t, t_0) \to 0$ as $t \to \infty$. We then have

$$I(t) \to I_0 \frac{\gamma+\nu-\beta\kappa}{-I_0\beta} = \kappa - \frac{\gamma+\nu}{\beta}, \quad S(t) \to \frac{\gamma+\nu}{\beta} \quad \text{as} \quad t \to \infty,$$

which completes the proof. □

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Martin Bohner
e-mail: `bohner@mst.edu`
Sabrina H. Streipert
e-mail: `shsbrf@mst.edu`
Missouri University of Science and Technology
Department of Mathematics and Statistics
400 West 12th Street, Rolla, MO 65409-0020, USA