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# BOUNDARY-VALUE PROBLEMS FOR PDES ARISING IN THE VALUATION OF STRUCTURED FINANCIAL PRODUCTS

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We explicitly solve some mixed initial/boundary value problems for generalized Black-Scholes PDEs with financially relevant boundary conditions. As an illustration, new pricing formulas are obtained for convertible and reverse convertible bonds under credit risk.

## Introduction

Boundary-value problems for PDEs of Black-Scholes type or their generalizations often arise in a number of pricing problems for complex financial products. The first example was Merton (1973), where an exact pricing formula was found for a down-and-out European call as the solution of a Black-Scholes equation in a half-space with zero boundary condition. Further examples are provided – among others – in Rubinstein and Reiner [15], Zhang [17], Buchen [8] where several types of barrier options are studied, and in Kwok, Wu and Yu [12] for a multi-dimensional setting. Another substantial body of financial problems results in mixed initial/boundary value problems for parabolic PDEs, for example, several credit-risk models assume that the default event is triggered when a signalling variable hits a pre-specified boundary value (See Ericsson and Reneby [9]). On the other hand, a good many economic problems can be formulated as boundary value problems for second-order PDEs (see Agliardi, Popivanov and

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Slavova [3] for a mathematical framework). In the recent decades a huge variety of complex financial products have been designed to tailor issuers' and investors' requirements, a good deal demanding a proper valuation method that can be accomplished throughout the solution of a Dirichlet problem for (generalized) Black-Scholes equations. Despite the large number of such financial products, the suitable differential equation method has not been as widely published and the existing financial literature confines itself to the classical Black-Scholes equation and boundary conditions with a very specific and simple shape. At the same time, a theoretical framework for existence and regularity of degenerate parabolic equations is available in Il'in [11] and Oleinik [14]. In this paper we aim at bridging the theoretical and applied literature by providing a ready-to-use general framework allowing to find closed-form solutions for a large variety of financial products. In particular, we show how to incorporate also the credit-riskiness of a product and, more importantly, we can handle also some more complicated features of the financial contract. As an illustration, we show how convertible bonds and reverse convertible bonds can be properly priced. Despite their popularity, the common valuation method for these financial instruments relies on known formulas that add up a standard bond and a plain option, thus neglecting the interaction between the two embedded products and the underlying credit-risk as well.

Our framework could be applied to many more financial products in a very straightforward manner, whenever the payoff function is a linear combination of some functions alike the ones we consider in the Propositions 1 and 2. This is actually the case for most situations of practical interest.

In Section 1 some motivating financial examples are introduced. Section 2 presents the solution method and Section 3 provides some pricing formulas, as an illustration.

## 1. Pricing models as boundary value problems for a PDE

A few financial problems taking the form of boundary value problems have been often solved via reflection principle, while the alternative method of images in PDE has been employed to obtain analytical formulas for some simple cases of barrier options. A first example is offered in Merton [13] where down-and-out barrier options with zero-rebate are priced. We briefly recall the notation for future reference.

Let $S_t$ denote the underlying asset of a derivative product at time $t$ and assume that

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

where $W_t$ is a Wiener process, $\mu \in \mathbb{R}$, $\sigma > 0$. (Usually, $\mu = r$, the risk-free interest rate).

Consider an option with final (at time $T$) payoff $g(S)$. Assume that the option is a down-and-out barrier option (with barrier $S^* > 0$), that is $S_0 > S^*$ and the option is knocked-out (i.e. its value becomes 0) if $S_t$ hits the barrier $S^*$.

Let $f(S,t)$ denote the option value (at time $t < T$). Then $f(S,t)$ is found as the solution of:

$\mathfrak{L}(f) = 0$ for $S > S^*$ and $0 \le t < T$
$f(S,T) = g(S)$
$f(S,t) = 0$ when $S \le S^*$,

where $\mathfrak{L}$ is the Black-Scholes operator:

$$\mathfrak{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + rS\frac{\partial}{\partial S} - r. \tag{2}$$

An explicit closed-form expression for the solution is known for some relevant financial products. For example, if $g(S) = 1_{S>K}$ (cash-or-nothing option) the value of the barrier option is:

$$f(S,t) = e^{-r(T-t)}\left[N\left(\frac{\ln(S/K) + (\mu - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) - \left(\frac{S^*}{S}\right)^{\alpha} N(\frac{\ln((S^*)^2/(KS)) + (\mu - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}})\right]$$

with $\alpha = \dfrac{2r}{\sigma^2} - 1$.

If $g(S) = \max(S-K,0)$ (standard call option), then the value of the down-and-out barrier option is:

$$\begin{aligned} Se^{(\mu-r)(T-t)} N\left(\frac{\ln(S/K) + (\mu + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\ - Ke^{-r(T-t)} N\left(\frac{\ln(S/K) + (\mu - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\ - \frac{(S^*)^{\alpha+2}}{S^{\alpha+1}} e^{(\mu-r)(T-t)} N\left(\frac{\ln((S^*)^2/(KS)) + (\mu + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\ + K\left(\frac{S^*}{S}\right)^{\alpha} e^{-r(T-t)} N\left(\frac{\ln((S^*)^2/(KS)) + (\mu - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

Pricing formulas for first generation exotic options under the stochastic process (1) can be found in several books (see Zhang [17], for a most comprehensive presentation) and extensions have been provided even for more general stochastic processes for the underlying assets (Agliardi [2]).

On the other hand, the market for structured financial instruments has been dramatically expanding in the last decades and a good many demand complex formulas to properly evaluate such products. As an illustration, we focus on some structured bonds that can be prices throughout a generalized Black-Scholes PDE affording the incorporation of the default risk. Consider the pricing problem for a corporate bond with maturity $T$ which has been issued by a corporation having a non-zero default risk and whose stock has a price $S_t$, at time $t$, following (1) and paying a proportional dividend $q$. Assume that the probability of default in the time period from $t$ to $tdt$, conditional on no-default before $t$, be $pdt$, i.e. $p$ is a deterministic hazard rate. Let $R \in [0,1]$ denote the recovery rate in case of default. Let $f(S,t)$, $0 \le t \le T$, denote the value of this risky bond. Then Ito's formula and a standard hedging argument yields the partial differential equation $\mathfrak{L}(f) = 0$, where:

$$\mathfrak{L} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r+p-q)S\frac{\partial}{\partial S} - r - p(1-R). \tag{3}$$

The amount $\rho = r + p(1-R)$ is the credit risk-adjusted interest rate, embedding the risk premium to investors due to the possibility of defaulting by the issuer.

The terminal condition is of the form $f(S,T) = F$, where $F$ represent the face value of the bond or, more generally, the face value accreted with the accrued interest payments. While the price of a straight bond, where cash amounts are paid back to investors at fixed dates, can be obtained by solving a Cauchy problem for (3) with only a terminal condition, some contracts have clauses that turn the problem into a Dirichlet problem. For example, the investor on a convertible bond has the option to convert it to a number, $k$, of shares of a specified stock; conversely, the issuer of a reverse convertible bond can pay back the nominal amount in a prespecified number of shares of a stock, should this stock be trading below a stated limit, the so-called conversion limit ($S^*$). Otherwise, the nominal amount is paid back in cash just as in the case of a straight bond.

The pricing problem for such structured bonds can be written as follows:

$$\begin{cases} \mathfrak{L}(f) = 0 & 0 < S < S^*,\ 0 \le t < T \\ f(S,T) = \max(F, kS) & \\ f(S,t) = kS & \text{when } S \ge S^* \end{cases} \tag{4}$$

for a convertible bond, and

$$(5)\qquad \begin{cases} \mathfrak{L}(f) = 0 & S > S^*,\ 0 \le t < T \\ f(S,T) = \min(F, kS) & \\ f(S,t) = kS & \text{when } 0 < S \le S^* \end{cases}$$

for a reverse convertible one, where $\mathfrak{L}$ is defined in (3). Further common clauses (e.g. callability, puttability, etc.) can be accommodated in this setting as well. In the next section, we provide a general framework where all the problems above can be solved explicitly and, at the same time, more complicated financial products can be priced in a straightforward way.

## 2. Solving a generalized Black-Scholes PDE with initial and boundary conditions

Consider the following differential equation:

$$(6)\qquad \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (\rho - \delta) S \frac{\partial f}{\partial S} - \rho f = 0, \quad S > 0, 0 \le t < T,$$

with $\rho \ge 0$, $\delta \in \mathbb{R}$. Changing to variables:

$$f(S,t) = \exp(-\alpha x - \beta\tau) H(x,\tau), \quad S = \exp(x), \quad \tau = \frac{\sigma^2 (T-t)}{2},$$

and choosing $\alpha = \dfrac{\rho - \delta - \sigma^2/2}{\sigma^2}$, $\beta = \dfrac{2\rho}{\sigma^2} + \alpha^2$, the equation (6) is turned into the heat equation:

$$(7)\qquad (\partial_\tau - \partial_x^2) H(x,\tau) = 0,\ x \in \mathbb{R}, \quad 0 < \tau \le T' = \sigma^2 T/2.$$

Having the problems (4) and (5) in mind, we need to solve a Dirichlet problem – both in an interior and in an exterior domain – with boundary and initial conditions allowing to accommodate the most common functions that are prescribed by the financial problems. As a first step, the following preliminary results are obtained.

**Proposition 1.** *The solution to*

$$\begin{cases} (\partial_\tau - \partial_x^2) H = 0, & x < x^*,\ 0 < \tau \le T' \\ H(x, 0+) = e^{\gamma x} 1_{(-\infty, c)}(x) & x < x^*,\ x \ne c, \text{ for some } c \le x^* \\ H(x^*, \tau) = 0 & 0 < \tau \le T' \end{cases}$$

*is* $H(x,\tau) = e^{\gamma x+\gamma^2\tau}\left[N\left(-\frac{x-c+2\gamma\tau}{\sqrt{2\tau}}\right) - e^{2\gamma(x^*-x)}N\left(-\frac{2x^*-x-c+2\gamma\tau}{\sqrt{2\tau}}\right)\right]$, *where* $N(\dots)$ *is the distribution function for the standard Gaussian distribution.*

Proof. Let $E^{x^*}(\tau,x,\cdot)$ denote the fundamental solution for this problem, that is

(8)
$$E^{x^*}(\tau,x,y) = \frac{1}{\sqrt{4\pi\tau}}[\exp(-(x-y)^2/(4\tau)) - \exp(-(2x^*-x-y)^2/(4\tau))].1_{(-\infty,x^*]}(y)$$

Then

$$\begin{aligned} H(x,\tau) &= \int_{-\infty}^{c} E^{x^*}(\tau,x,y)e^{\gamma y}dy \\ &= \int_{-\infty}^{c} \frac{\exp(-(x-y)^2/(4\tau))}{\sqrt{4\pi\tau}}e^{\gamma y}dy - \int_{-\infty}^{c} \frac{\exp(-(2x^*-x-y)^2/(4\tau))}{\sqrt{4\pi\tau}}e^{\gamma y}dy. \end{aligned}$$

Changing to variables $\frac{x-y}{\sqrt{2\tau}} = -y'$ in the first integral, it becomes:

$$\frac{e^{\gamma x+\gamma^2\tau}}{\sqrt{2\pi}}\int_{-\infty}^{(c-x)/\sqrt{2\tau}} \exp(-\frac{(y'-\sqrt{2\tau}\gamma)^2}{2})dy'$$

and setting $y'-\sqrt{2\tau}\gamma = u$, the expression $e^{\gamma x+\gamma^2\tau}N\left(-\frac{x-c+2\gamma\tau}{\sqrt{2\tau}}\right)$ obtains. The other integral is written in the form

$$\frac{e^{\gamma(2x^*-x)+\gamma^2\tau}}{\sqrt{2\pi}}\int_{-\infty}^{(c+x-2x^*)/\sqrt{2\tau}} \exp\left(-\frac{(y'-\sqrt{2\tau}\gamma)^2}{2}\right)dy'$$

throughout the change $\frac{2x^*-x-y}{\sqrt{2\tau}} = -y'$. Changing to $y'-\sqrt{2\tau}\gamma = u$, one gets the expression $e^{\gamma(2x^*-x)+\gamma^2\tau}N\left(-\frac{2x^*-x-c+2\gamma\tau}{\sqrt{2\tau}}\right)$. □

**Proposition 2.** *The solution to*

$$\begin{cases} (\partial_\tau - \partial_x^2)H = 0, & x > x^*,\ 0 < \tau \le T' \\ H(x,0+) = e^{\gamma x}1_{(c,+\infty)}(x) & x > x^*,\ x \ne c, \textit{ for some } c \ge x^* \\ H(x^*,\tau) = 0 & 0 < \tau \le T' \end{cases}$$

*is* $H(x,\tau) = e^{\gamma x+\gamma^2\tau}\left[N\left(\frac{x-c+2\gamma\tau}{\sqrt{2\tau}}\right) - e^{2\gamma(x^*-x)}N\left(\frac{2x^*-x-c+2\gamma\tau}{\sqrt{2\tau}}\right)\right]$, *where* $N(\dots)$ *is the distribution function for the standard Gaussian distribution.*

P r o o f. Let $E_{x^*}(\tau, x, \cdot)$ denote the fundamental solution for this problem, that is

$$\text{(9)} \quad E_{x^*}(\tau, x, y) = \frac{1}{\sqrt{4\pi\tau}}[\exp(-(x-y)^2/(4\tau)) - \exp(-(2x^* - x - y)^2/(4\tau))]1_{[x^*,+\infty)}(y)$$

Then

$$\begin{aligned} H(x,\tau) &= \int_c^{+\infty} E_{x^*}(\tau, x, y)e^{\gamma y}dy \\ &= \int_c^{+\infty} \frac{\exp(-(x-y)^2/(4\tau))}{\sqrt{4\pi\tau}}e^{\gamma y}dy \\ &\quad - \int_c^{+\infty} \frac{\exp(-(2x^*-x-y)^2/(4\tau))}{\sqrt{4\pi\tau}}e^{\gamma y}dy. \end{aligned}$$

Changing to variables $\frac{x-y}{\sqrt{2\tau}} = y'$ in the first integral, it becomes:

$$\frac{e^{\gamma x+\gamma^2\tau}}{\sqrt{2\pi}} \int_{-\infty}^{(x-c)/\sqrt{2\tau}} \exp\left(-\frac{(y'+\sqrt{2\tau}\gamma)^2}{2}\right) dy'$$

and setting $y'+\sqrt{2\tau}\gamma = u$, the expression $e^{\gamma x+\gamma^2\tau}N\left(\frac{x-c+2\gamma\tau}{\sqrt{2\tau}}\right)$ obtains. The other integral becomes

$$\frac{e^{\gamma(2x^*-x)+\gamma^2\tau}}{\sqrt{2\pi}} \int_{-\infty}^{(2x^*-x-c)/\sqrt{2\tau}} \exp\left(-\frac{(y'+\sqrt{2\tau}\gamma)^2}{2}\right) dy'$$

under the change $\frac{2x^*-x-y}{\sqrt{2\tau}} = y'$, which eventually yields the result. □

Let us now turn to problems with non-vanishing boundary conditions that are of interest in the financial applications.

**Proposition 3.** *Consider the problem*

$$\text{(10)} \qquad \begin{cases} (\partial_\tau - \partial_x^2)H = 0, & x < x^*,\ 0 < \tau \le T' \\ H(x, 0+) = h(x) & \textit{for } x < x^* \\ H(x^*, \tau) = H^* e^{\beta\tau} & 0 < \tau \le T' \end{cases}$$

*where* $\beta > 0$ *and* $h \in L^1_{loc}((-\infty, x^*))$ *with* $\sup_{x<x^*} |e^{\gamma x} h(x)| < \infty$ *for some* $\gamma \geq 0$. *Then the solution of* (10) *is of the form*

$$H(x,\tau) = \int_{-\infty}^{x^*} E^{x^*}(\tau, x, y) h(y) dy$$
$$+ H^* e^{\beta\tau + \sqrt{\beta}(x-x^*)} \left[ N\left(\frac{x - x^* + 2\tau\sqrt{\beta}}{\sqrt{2\tau}}\right) + e^{2\sqrt{\beta}(x^*-x)} N\left(\frac{x - x^* - 2\tau\sqrt{\beta}}{\sqrt{2\tau}}\right)\right]$$

*where* $E^{x^*}$ *is defined in* (8). *Moreover,* $\sup_{x<x^*, t\in[0,T']} \left| H(x,\tau) e^{\max\{\gamma, \sqrt{\beta}\} x} \right|$ *is finite.*

P r o o f. Let $H_1(x,\tau) = H^* e^{\beta\tau + \sqrt{\beta}(x-x^*)}$. Then $H_1$ solves the PDE and $H_1(x^*, \tau) = H^* e^{\beta\tau}$. Now consider the initial/boundary value problem:

$$\begin{cases} (\partial_\tau - \partial_x^2) H_2 = 0, & x < x^*,\ 0 < \tau \leq T' \\ H_2(x, 0) = h(x) - H^* e^{\sqrt{\beta}(x-x^*)} & x < x^* \\ H_2(x^*, \tau) = 0 & 0 < \tau \leq T' \end{cases}$$

Then, in view of Proposition 1, a solution is found in the form:

$$H_2(x,\tau) = \int_{-\infty}^{x^*} E^{x^*}(\tau, x, y) h(y) dy$$
$$- H^* e^{\sqrt{\beta}(x-x^*)} e^{\beta\tau} \left[ N\left(-\frac{x - x^* + 2\tau\sqrt{\beta}}{\sqrt{2\tau}}\right) - e^{2\sqrt{\beta}(x^*-x)} N\left(\frac{x - x^* - 2\tau\sqrt{\beta}}{\sqrt{2\tau}}\right)\right].$$

Notice that there exist some constants $C_1$, $C_2 \geq 0$ such that

$$\begin{aligned} &\left| e^{\gamma x} \int_{-\infty}^{x^*} E^{x^*}(\tau, x, y) h(y) dy \right| \\ &\qquad \leq C_1 \int_{-\infty}^{x^*} \frac{\exp(-(x-y)^2/(4\tau))}{\sqrt{4\pi\tau}} e^{\gamma(x-y)} dy \\ &\qquad + C_2 \int_{-\infty}^{x^*} \frac{\exp(-(2x^* - x - y)^2/(4\tau))}{\sqrt{4\pi\tau}} e^{\gamma(x-y)} dy \\ &\qquad = (C_1 + C_2) e^{\tau\gamma^2}. \end{aligned}$$

Finally, $H := H_1 + H_2$ is a solution of (10) such that $|H(x,\tau)e^{\gamma x}|$ is uniformly bounded in $(-\infty, x^*) \times [0, T']$. $\square$

**Proposition 4.** *Consider the problem*

$$
\begin{cases} (\partial_\tau - \partial_x^2)H = 0, & x > x^*,\ 0 < \tau \le T' \\ H(x, 0+) = h(x) & \text{for } x > x^* \\ H(x^*, \tau) = H^* e^{\beta\tau} & 0 < \tau \le T' \end{cases} \tag{11}
$$

*where* $\beta > 0$ *and* $h \in L^1_{loc}((x^*, \infty))$ *with* $\sup_{x>x^*} |e^{\gamma x} h(x)| < \infty$ *for some* $\gamma \le 0$. *Then the solution of* (11) *is of the form*

$$
\begin{aligned} H(x,\tau) = {} & \int_{x^*}^{+\infty} E_{x^*}(\tau, x, y) h(y) dy \\ & + H^* e^{\beta\tau - \sqrt{\beta}(x - x^*)} \left[ N\left( \frac{x^* - x + 2\tau\sqrt{\beta}}{\sqrt{2\tau}} \right) + e^{2\sqrt{\beta}(x-x^*)} N\left( \frac{x^* - x - 2\tau\sqrt{\beta}}{\sqrt{2\tau}} \right) \right] \end{aligned}
$$

*where* $E_{x^*}$ *is defined in* (9). *Moreover,* $\sup_{x>x^*, t\in[0,T']} \left| H(x,\tau) e^{\min\{\gamma, -\sqrt{\beta}\}x} \right|$ *is finite.*

P r o o f. Let $H_1(x, \tau) = H^* e^{\beta\tau - \sqrt{\beta}(x-x^*)}$. Then $H_1$ solves the PDE and $H_1(x^*, \tau) = H^* e^{\beta\tau}$. Now consider the initial/boundary problem:

$$
\begin{cases} (\partial_\tau - \partial_x^2)H_2 = 0, & x > x^*,\ 0 < \tau \le T' \\ H_2(x, 0) = h(x) - H^* e^{-\sqrt{\beta}(x-x^*)} & x > x^* \\ H_2(x^*, \tau) = 0 & 0 < \tau \le T' \end{cases}
$$

Then, in view of Proposition 2, a solution can be written in the form:

$$
\begin{aligned} H_2(x,\tau) = {} & \int_{x^*}^{+\infty} E_{x^*}(\tau, x, y) h(y) dy \\ & - H^* e^{-\sqrt{\beta}(x-x^*)} e^{\beta\tau} \left[ N\left( \frac{x - x^* - 2\tau\sqrt{\beta}}{\sqrt{2\tau}} \right) - e^{2\sqrt{\beta}(x-x^*)} N\left( \frac{x^* - x - 2\tau\sqrt{\beta}}{\sqrt{2\tau}} \right) \right]. \end{aligned}
$$

Notice that there exist some constants $C_1$, $C_2 \ge 0$ such that

$$
\begin{aligned} & \left| e^{\gamma x} \int_{x^*}^{+\infty} E_{x^*}(\tau, x, y) h(y) dy \right| \\ & \qquad \le C_1 \int_{x^*}^{+\infty} \frac{\exp(-(x-y)^2/(4\tau))}{\sqrt{4\pi\tau}} e^{\gamma(x-y)} dy \\ & \qquad + C_2 \int_{x^*}^{+\infty} \frac{\exp(-(2x^* - x - y)^2/(4\tau))}{\sqrt{4\pi\tau}} e^{\gamma(x-y)} dy \end{aligned}
$$

$$= (C_1 + C_2)e^{\tau\gamma^2}.$$

Finally, $H := H_1 + H_2$ is a solution of (11) such that $|H(x,\tau)e^{\gamma x}|$ is uniformly bounded in $(x^*, \infty) \times [0, T']$. $\square$

## 3. Application to pricing formulas

In this Section, the framework presented in Section 2 is employed to find the pricing formulas for some financial problems. As a first illustration, we recover two known pricing formulas; then we obtain two new valuation formulas.

**Example 1.** A down-and-out barrier call option without rebate is priced by solving the following problem:

$\mathfrak{L}(f) = 0$ for $S > S^*$ and $0 \le t < T$

$f(S,T) = \max(S - K, 0)$ for $S > S^*$

$f(S,t) = 0$ when $S \le S^*$,

where $\mathfrak{L}$ is the Black-Scholes operator (possibly, with dividends $q$). Changing to variables:

$$f(S,t) = \exp(-\alpha x - \beta\tau)H(x,\tau), \quad S = \exp(x), \quad \tau = \frac{\sigma^2(T-t)}{2},$$

with $\alpha = \dfrac{r-q}{\sigma^2} - \dfrac{1}{2}$, $\beta = \dfrac{2r}{\sigma^2} + \alpha^2$, the problem is turned into

$$\begin{cases} (\partial_\tau - \partial_x^2)H = 0, & x > x^*,\ 0 < \tau \le T' \\ H(x,0) = [e^{(\alpha+1)x} - Ke^{\alpha x}]1_{(\ln K, +\infty)}(x) & x > x^* \\ H(x^*,\tau) = 0 & 0 < \tau \le T' \end{cases}$$

where we suppose $\ln K \ge x^* = \ln S^*$. Employing Proposition 2 twice (with $\gamma = \alpha + 1$ and $\gamma = \alpha$, respectively) and changing back to the variables $S$ and $t$, we immediately recover Merton's formula for a down-and-out call, that is, the expression reported in Section 1.

**Example 2.** Sometimes a rebate is paid when a financial option is knocked-out. For simplicity's sake, assume that a unit rebate is paid anytime a knock-out occurs within the life of a contract embedding a barrier option. In order to find the present value of a rebate option, $f(S,t)$, we need to distinguish between an up and a down barrier, depending on whether the barrier is hit from below or above. If the underlying asset follows (1), possibly with constant dividends $q$, then $f$ satisfies a Black-Scholes equation for $S < S^*$ (up case) or $S > S^*$ (down case); moreover, $f(S,T) = 0$ at the maturity and $f(S^*,t) = 1$. The usual change of variables transforms the problem into:

$$\begin{cases} (\partial_\tau - \partial_x^2)H = 0, & x < x^* \text{ (or } x > x^*), \ 0 < \tau \leq T' \\ H(x,0) = 0 & x < x^* \text{ (or } x > x^*) \\ H(x^*,\tau) = e^{\alpha x^* + \beta\tau} & 0 < \tau \leq T' \end{cases}$$

Using Proposition 3 (Proposition 4, respectively) and changing back to variables $(S,t)$ the following expressions obtain:

$$\left(\frac{S^*}{S}\right)^{\alpha - \sqrt{\beta}} \left[ N\left(\frac{\ln(S/S^*) - \sqrt{\beta}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) + \left(\frac{S^*}{S}\right)^{2\sqrt{\beta}} N\left(\frac{\ln(S/S^*) - \sqrt{\beta}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right]$$

for the up-and-out case, and

$$\left(\frac{S^*}{S}\right)^{\alpha + \sqrt{\beta}} \left[ N\left(\frac{\ln(S^*/S) + \sqrt{\beta}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) + \left(\frac{S}{S^*}\right)^{2\sqrt{\beta}} N\left(\frac{\ln(S^*/S) - \sqrt{\beta}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right]$$

for the down-and-out case.

**Example 3.** Let $f(S,t)$ denote the current value of a reverse convertible bond with maturity $T$, conversion ratio $k$ and conversion limit $S^*$. Let $F$ denote the nominal value of the bond. For easiness of exposition we restrict the computation to zero-coupon bonds. However, the case of coupon-bearing bond can be accommodated, a simplified way being the interpretation of $F$ as the sum of the accrued coupons and the face value.[1] As credit risk is usually a concern, $f(S,t)$ is assumed to satisfy problem (6) with $\rho = r + p(1-R)$, i.e. we need to consider the problem (5). Changing to variables:

$$f(S,t) = \exp(-\alpha x - \beta\tau)H(x,\tau), \quad S = \exp(x), \quad \tau = \frac{\sigma^2(T-t)}{2},$$

[1] A precise handling of the coupon stream would require a very complicated method, such as Agliardi (2011).

with $\alpha = \dfrac{\rho - \delta - \sigma^2/2}{\sigma^2}$ and $\beta = \dfrac{2\rho}{\sigma^2} + \alpha^2$, the pricing problem is transformed into:

$$\begin{cases} (\partial_\tau - \partial_x^2)H = 0, & x > x^*,\ 0 < \tau \le T' \\ H(x,0) = e^{\alpha x}\min(F, ke^x) & \text{for } x > x^* \\ H(x^*,\tau) = ke^{\beta\tau + (\alpha+1)x^*} & 0 < \tau \le T' \end{cases}$$

where $x^* = \ln(S^*)$. Denote $\ln(F/k)$ by $c$ and assume that $c \ge x^*$, i.e. $F \ge kS^*$. Application of Proposition 4 yields:

$$\begin{aligned} H(x,\tau) &= Fe^{\alpha x + \alpha^2\tau}\left[N\left(\frac{x - c + 2\alpha\tau}{\sqrt{2\tau}}\right) - e^{2\alpha(x^*-x)}N\left(\frac{2x^* - x - c + 2\alpha\tau}{\sqrt{2\tau}}\right)\right] \\ &+ke^{(\alpha+1)x + (\alpha+1)^2\tau}\left[N\left(\frac{x - x^* + 2(\alpha+1)\tau}{\sqrt{2\tau}}\right) - N\left(\frac{x - c + 2(\alpha+1)\tau}{\sqrt{2\tau}}\right)\right. \\ &\left.-e^{2(\alpha+1)(x^*-x)}\left(N\left(\frac{x^* - x - c + 2(\alpha+1)\tau}{\sqrt{2\tau}}\right) - N\left(\frac{2x^* - x - c + 2(\alpha+1)\tau}{\sqrt{2\tau}}\right)\right)\right] \\ &+ke^{\beta\tau + (\alpha+1)x^* - \sqrt{\beta}(x-x^*)}\left[N\left(\frac{x - x^* + 2\tau\sqrt{\beta}}{\sqrt{2\tau}}\right) + e^{2(x-x^*)\sqrt{\beta}}N\left(\frac{x^* - x - 2\tau\sqrt{\beta}}{\sqrt{2\tau}}\right)\right]. \end{aligned}$$

Note that $(\alpha^2 - \beta)\tau = -\rho(T-t)$ and $((\alpha+1)^2 - \beta)\tau = (pR - q)(T-t)$. Then substituting back the variables one gets the following valuation formula:

$$\begin{aligned} f(S,t) = Fe^{-\rho(T-t)}&\left[N\left(\frac{\ln(kS/F) + \alpha\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right. \\ &\left.-\left(\frac{S^*}{S}\right)^{2\alpha}N\left(\frac{\ln(k(S^*)^2/(SF)) + \alpha\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right] \\ +kSe^{(pR-q)(T-t)}&\left(N\left(\frac{\ln(S/S^*) + (\alpha+1)\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right. \\ &-N\left(\frac{\ln(kS/F) + (\alpha+1)\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \\ -\left(\frac{S^*}{S}\right)^{2(\alpha+1)}&\left[N\left(\frac{\ln(S^*/S) + (\alpha+1)\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right. \\ &\left.\left.-N\left(\frac{\ln(k(S^*)^2/(SF)) + (\alpha+1)\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right]\right) \end{aligned}$$

$$+kS\left(\frac{S^*}{S}\right)^{\alpha+1+\sqrt{\beta}}\left[N\left(\frac{\ln(S^*/S)+\sqrt{\beta}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right.$$

$$\left.+\left(\frac{S^*}{S}\right)^{2\sqrt{\beta}}N\left(\frac{\ln(S^*/S)-\sqrt{\beta}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right].$$

In contrast to the valuation formulas of practical use, the embedded credit risk and its interaction with the conversion option is properly modelled in our setting. Thus we obtain a novel pricing formula for RCs in a credit risk perspective, along the lines of Agliardi (2016) where, however, an infinite maturity was considered, that is, an ODE was solved.

**Example 4.** Consider the pricing problem (4) for a convertible bond. Changing to variables:

$$f(S,t)=\exp(-\alpha x-\beta\tau)H(x,\tau),\quad S=\exp(x),\quad \tau=\frac{\sigma^2(T-t)}{2},$$

with $\alpha=\dfrac{\rho-\delta-\sigma^2/2}{\sigma^2}$, $\beta=\dfrac{2\rho}{\sigma^2}+\alpha^2$, the pricing problem is transformed into:

$$\begin{cases}(\partial_\tau-\partial_x^2)H=0, & x<x^*,\ 0<\tau\le T'\\ H(x,0)=e^{\alpha x}\max(F,ke^x) & \text{for } x<x^*\\ H(x^*,\tau)=ke^{\beta\tau+(\alpha+1)x^*} & 0<\tau\le T'\end{cases}$$

where $x^*=\ln(S^*)$. Denote $\ln(F/k)$ by $c$ and assume that $c\le x^*$, i.e. $F\le kS^*$. Then applying Proposition 3 and substituting back the variables one has the pricing formula:

$$f(S,t)=Fe^{-\rho(T-t)}\left[N\left(\frac{\ln(F/kS)-\alpha\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right.$$

$$\left.-\left(\frac{S^*}{S}\right)^{2\alpha}\left(\frac{\ln((SF)/(k(S^*)^2))-\alpha\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right]$$

$$+kSe^{(pR-q)(T-t)}\left(N\left(\frac{\ln(S^*/S)-(\alpha+1)\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right.$$

$$-N\left(\frac{\ln(F/(kS))-(\alpha+1)\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$
\begin{aligned}
&-\left(\frac{S^*}{S}\right)^{2(\alpha+1)}\left[N\left(\frac{\ln(S/S^*)-(\alpha+1)\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right.\\
&\qquad\qquad\left.\left.-N\left(\frac{\ln((SF)/(k(S^*)^2))-(\alpha+1)\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right]\right)\\
&+kS\left(\frac{S^*}{S}\right)^{\alpha+1-\sqrt{\beta}}\left[N\left(\frac{\ln(S/S^*)+\sqrt{\beta}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right.\\
&\qquad\qquad\left.+\left(\frac{S^*}{S}\right)^{2\sqrt{\beta}}N\left(\frac{\ln(S/S^*)-\sqrt{\beta}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right].
\end{aligned}
$$

In view of the maximum principle for PDEs of Black-Scholes type (see Agliardi, Popivanov and Slavova (2011), for example) one has $Fe^{-\rho(T-t)} \leq f(S,t) \leq kS^*$ in the continuation region, that is, a convertible bond is more worthy than a straight bond and reaches the value of the underlying stock (times the conversion ratio) only at the conversion threshold. On the other hand, empirical evidence shows that a convertible bond has a higher price than a comparable straight bond with similar characteristics (maturity, coupon flow, etc) and, consequently, it provides a lower premium to investors. The reverse is true for reverse convertibles.

## 4. Conclusion

We revisit the classical solution method of images in order to provide a ready-to-use setting for several financial problems that take the form of initial/boundary value problems for generalized Black-Scholes equations. In the presentation, PDEs with constant coefficients are considered, as this is the common situation of practical interest. However, the case of time-dependent coefficients can be easily accommodated in our setting.

A few examples of pricing formulas are provided as an illustration. Some of them are new the financial literature and contribute to a more accurate modeling of some popular structured financial products.

Finally, in recent decades, a stream of financial literature has replaced the classical model for the underlying asset with more general stochastic processes, in particular Lévy processes. A comprehensive presentation of this approach is found, for example, in the monographs by Boyarchenko and Levendorskiĭ (2002) and Schoutens (2003). In such a framework the Black-Merton-Scholes PDE takes the form of a pseudo-differential equations and some boundary-value problems of relevance to finance have been solved throughout Wiener-Hopf factorisation in Boyarchenko and Levendorskiĭ (2002). For example, the price of a first-touch digital is obtained solving the problem:

$(\partial_t - r - \psi(D_x))u(t,x) = 0 \quad x > 0,\, t < T$
$u(t,x) = 1 \;\; x \leq 0,\, t \leq T$
$u(T,x) = 0 \quad x > 0$

where the symbol of the pseudo-differential operator $\psi$ is the characteristic exponent of the assumed Lèvy process $X_t$, and the price of the underlying stock is modelled as $S_t = e^{X_t}$. Then the option price takes the form:

$$f(t,S) = \frac{1}{(2\pi)^2}\int_{-\infty+i\nu}^{+\infty+i\nu}\int_{-\infty+i\omega}^{+\infty+i\omega} e^{i[(T-t)\lambda+\ln(\frac{S}{H})\xi]}\Phi_-(\lambda,\xi)\frac{d\xi d\lambda}{\lambda\xi}$$

where $\Phi_-$ is a Wiener-Hopf factor, i.e. $\dfrac{i\lambda+r}{i\lambda+r+\psi(\xi)} = \Phi^+(\xi)\Phi^-(\xi)$ and $\Phi^\pm$ admit the analytic continuation in the upper (lower) half-space $\mathrm{Im}\xi \gtrless 0$. On one hand, this method is a powerful tool that lends itself to a nice interpretation in terms of the underlying stochastic processes and the fluctuation theory; on the other hand, the Wiener-Hopf factors are not available in explicit form for any generic Lévy process. From the numerical perspective, the pricing formulas require double integration along some lines in the complex plane, which is not very effective in practice. Therefore, there is a scope for future research aiming at developing some simple ready-to-use pricing formulas, in the spirit of our contribution.

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