Serdica J. Computing **1** (2007), 443–454

Serdica
Journal of Computing

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ALGEBRAIC COMPUTATIONS WITH HAUSDORFF CONTINUOUS FUNCTIONS*

Roumen Anguelov

Abstract. The set of Hausdorff continuous functions is the largest set of interval valued functions to which the ring structure of the set of continuous real functions can be extended. The paper deals with the automation of the algebraic operations for Hausdorff continuous functions using an ultra-arithmetical approach.

1. Introduction. CAS typically deal with computations, operations and manipulations of functions from the set of what is considered elementary functions. These functions are all analytic on their domains of definition. To the author's best knowledge discontinuous functions have not yet been considered in this context. In this paper we consider algebraic computations with Hausdorff continuous (H-continuous) interval valued functions. The H-continuous functions

ACM Computing Classification System (1998): I.1.2
Key words: Hausdorff continuous function, ultra-arithmetical functoid.

*The paper has been presented at the 12th International Conference on Applications of Computer Algebra, Varna, Bulgaria, June, 2006.

are useful in representing discontinuities of real functions through interval values. They were originally defined within the realm of Approximation Theory but have been applied since in many other areas. In particular, recent results have shown that they are also a powerful tool in the Analysis of PDEs since the solution of large classes of nonlinear PDEs can be assimilated with H-continuous functions, [6]. It was also shown recently, see [5], that the algebraic operations for continuous functions can be extended to H-continuous functions in such a way that the set of H-continuous functions is a commutative ring. This result is particularly significant in view of the fact that interval structures typically do not form linear spaces. In fact, the set of H-continuous functions is the largest set of interval functions which is a linear space, [4]. The present paper deals with the automation of the algebraic operations with H-continuous functions within the structure of a functoid.

2. The algebraic operations with H-continuous functions. The real line is denoted by $\mathbb{R}$ and the set of all finite real intervals by $\mathbb{IR} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R},\ \underline{a} \leq \overline{a}\}$. Given an interval $a = [\underline{a}, \overline{a}] \in \mathbb{IR}$, $w(a) = \overline{a} - \underline{a}$ is the width of a. An interval a is called a proper interval if $w(a) > 0$ and a point interval if $w(a) = 0$. Identifying $a \in \mathbb{R}$ with the point interval $[a, a] \in \mathbb{IR}$, we consider $\mathbb{R}$ as a subset of $\mathbb{IR}$. Let $\Omega \subseteq \mathbb{R}$ be open. We recall, [10], that an interval function $f : \Omega \to \mathbb{IR}$ is S-continuous if its graph is a closed subset of $\Omega \times \mathbb{R}$. An interval function $f : \Omega \to \mathbb{IR}$ is Hausdorff continuous (H-continuous) if it is an S-continuous function which is minimal with respect to inclusion, that is, if $\varphi : \Omega \to \mathbb{IR}$ is an S-continuous function, then $\varphi \subseteq f$ implies $\varphi = f$. Here the inclusion is considered in a pointwise sense. We denote by $\mathbb{H}(\Omega)$ the set of H-continuous functions on Ω. The following theorem states an essential property of the continuous functions which is preserved by the H-continuity [1].

Theorem 1. *Let $f, g \in \mathbb{H}(\Omega)$. If there exists a dense subset D of Ω such that $f(x) = g(x)$, $x \in D$, then $f(x) = g(x)$, $x \in \Omega$.*

H-continuous functions are also similar to the usual continuous real functions in that they assume point values on a residual subset of Ω. More precisely, it is shown in [1] that for every $f \in \mathbb{H}(\Omega)$ the set $W_f = \{x \in \Omega : w(f(x)) > 0\}$ is of first Baire category and f is continuous on $\Omega \setminus W_f$. Since a finite or countable union of sets of first Baire category is also a set of first Baire category we have:

Theorem 2. *Let $\mathcal{F}$ be a finite or countable set of H-continuous functions. Then the set $D_{\mathcal{F}} = \{x \in \Omega : w(f(x)) = 0,\ f \in \mathcal{F}\} = \Omega \setminus \bigcup_{f\in\mathcal{F}} W_f$ is dense in Ω and all functions $f \in \mathcal{F}$ are continuous on $D_{\mathcal{F}}$.*

For every S-continuous function g we denote by $[g]$ the set of H-continuous functions contained in g, that is,

$$[g] = \{f \in \mathbb{H}(\Omega) : f \subseteq g\}.$$

Identifying $\{f\}$ with f we have $[f] = f$ whenever f is H-continuous. The S-continuous functions g such that the set $[g]$ is a singleton, that is, it contains only one function, play an important role in the sequel. In analogy with the H-continuous functions which are minimal S-continuous functions, we call these functions quasi-minimal. The following characterization of the quasi-minimal S-continuous functions is an easy consequence of Theorem 1.

Theorem 3. *If f is an S-continuous function on Ω which assumes point values on a dense subset of Ω, then f is quasi-minimal S-continuous function.*

The familiar operations of addition, scalar multiplication and multiplication on the set of real intervals are defined for $[\underline{a},\overline{a}], [\underline{b},\overline{b}] \in \mathbb{I}\mathbb{R}$ and $\alpha \in \mathbb{R}$ as follows:

$$[\underline{a},\overline{a}] + [\underline{b},\overline{b}] = \{a+b : a \in [\underline{a},\overline{a}], b \in [\underline{b},\overline{b}]\} = [\underline{a}+\underline{b}, \overline{a}+\overline{b}],$$

$$\alpha \cdot [\underline{a},\overline{a}] = \{\alpha a : a \in [\underline{a},\overline{a}]\} = [\min\{\alpha\underline{a}, \alpha\overline{a}\}, \max\{\alpha\underline{a}, \alpha\overline{a}\}],$$

$$[\underline{a},\overline{a}] \times [\underline{b},\overline{b}] = \{ab : a \in [\underline{a},\overline{a}], b \in [\underline{b},\overline{b}]\} = [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}].$$

Pointwise operations for interval functions are defined in the usual way:

$$(1) \quad (f+g)(x) = f(x) + g(x),\ (\alpha \cdot f)(x) = \alpha \cdot f(x),\ (f \times g)(x) = f(x) \times g(x).$$

It is easy to see that the set of the S-continuous functions is closed under the above pointwise operations while the set of H-continuous functions is not, see [2], [4]. Hence the following result is significant.

Theorem 4. *For any $f, g \in \mathbb{H}(\Omega)$ and $\alpha \in \mathbb{R}$ the functions $f+g$, $\alpha \cdot f$ and $f \times g$ are quasi-minimal S-continuous functions.*

P r o o f. Denote by D_{fg} the subset of Ω where both f and g assume point values. Then $f+g$ assumes point values on D_{fg}. According to Theorem 2 the set D_{fg} is dense in Ω which in terms of Theorem 3 implies that $f+g$ is quasi-minimal. The quasi-minimality of $\alpha \cdot f$ and $f \times g$ is proved in a similar way. $\square$

We define the algebraic operations on $\mathbb{H}(\Omega)$ using Theorem 4. We denote these operations respectively by $\oplus$, $\odot$ and $\otimes$ so that distinction from the pointwise operations can be made.

Definition 5. *Let $f, g \in \mathbb{H}(\Omega)$ and $\alpha \in \mathbb{R}$. Then*

$$f \oplus g = [f+g],\ \alpha \odot f = [\alpha \cdot f],\ f \otimes g = [f \times g]. \tag{2}$$

Theorem 6. *The set $\mathbb{H}(\Omega)$ is a commutative algebra with respect to the operations $\oplus$, $\odot$ and $\otimes$ given in* (2)*.*

The proof is partially discussed in [5] and, since it involves standard techniques, will be omitted here.

3. The concept of ultra-arithmetical functoid. Functoid is a structure resulting from the ultra-arithmetical approach to the solution of problems in function spaces. The aim of the ultra-arithmetic is the development of structures, data types and operations corresponding to functions for direct digital implementation. On a digital computer equipped with ultra-arithmetic, problems associated with functions will be solvable, just as now we solve algebraic problems [8]. Ultra-arithmetic is developed in analogy with the development of computer arithmetic.

Let $\mathcal{M}$ be a space of functions and let M be a finite dimensional subspace spanned by $\Phi_N = \{\varphi_k\}_{k=0}^N$. Every function $f \in \mathcal{M}$ is approximated by $\tau_N(f) \in M$. The mapping τ_N is called rounding (in analogy with the rounding of numbers) and the space M is called a screen of $\mathcal{M}$. Every rounding must satisfy the requirement (invariance of rounding on the screen): $\tau_N(f) = f$ for every $f \in M$. Every function $f = \sum_{i=0}^N \alpha_i \varphi_i \in M$ can be represented by its coefficient vector $\nu(f) = (\alpha_0, \alpha_1, \ldots, \alpha_N)$. Therefore the approximation of the functions in $\mathcal{M}$ is realized through the mappings $\mathcal{M} \xrightarrow{\tau_N} M \xleftrightarrow{\nu} K^{N+1}$, where K is the scalar field of $\mathcal{M}$ (i.e. $K = \mathbb{R}$ or $K = \mathbb{C}$). Since ν is a bijection we can identify M and K^{N+1} and consider only the rounding τ_N.

In $\mathcal{M}$ we consider the operations addition $(+)$, scalar multiplication $(.)$, multiplication of functions $(\times)$ and integration $(\int)$ defined in the conventional way. By the semimorphism principle τ_N induces corresponding operations in M:

$$f \boxed{\circ} g = \tau_N(f \circ g)\ , \quad \circ \in \{+, ., \times\};$$

$$\boxed{\int} f = \tau_N\left(\int f\right).$$

The structure $(M, \boxplus, \boxdot, \boxtimes, \boxed{\int})$ is called an (ultra-arithmetical) functoid [9].

4. A functoid in $\mathbb{H}(\Omega)$. To simplify the matters we will consider the function space of all bounded H-continuous functions on $\Omega = (-1, 1)$. Furthermore, since we will often use a shift of the argument, we assume that all functions are produced periodically (period 2) over $\mathbb{R}$. Hence we denote the space under consideration by $\mathbb{H}_{per}(-1, 1)$. All algebraic operations on $\mathbb{H}_{per}(-1, 1)$ are considered in terms of Definition 5. For simplicity we denote them as the operations for reals. Namely, addition is "+" and a space is interpreted as multiplication, where the context shows whether this is a scalar multiplication or a product of functions. In particular, note that indicating the argument of a function in a formula does not mean pointwise operation. Denote by s_1 the H-continuous function given by

$$s_1(x) = \begin{cases} x, & \text{if} \quad x \in (-1, 1), \\ [-1, 1], & \text{if} \quad x = \pm 1; \end{cases}$$

and produced periodically over the real line. Since the integrals of $\underline{s}_1$ and $\overline{s}_1$ are equal over any interval the integral of s_1 is a regular real function. We construct iteratively the sequence of periodic splines $s_1, s_2, s_3, \ldots$ using

$$s_{j+1} = \int s_j(x)dx + c,$$

$$\int_{-1}^{1} s_{j+1}(x)dx = s_{j+2}(1) - s_{j+2}(-1) = 0.$$

The first few splines after s_1 are given on the interval $[-1,1]$ as follows

$$s_2(x) = \frac{x^2}{2!} - \frac{1}{6}\ ,$$

$$s_3(x) = \frac{x^3}{3!} - \frac{1}{6}x\ ,$$

$$s_4(x) = \frac{x^4}{4!} - \frac{1}{6}\frac{x^2}{2!} + \frac{7}{360}\ ,$$

$$s_5(x) = \frac{x^5}{5!} - \frac{1}{6}\frac{x^3}{3!} + \frac{7}{360}x\ .$$

It should be emphasized that the above splines are periodic functions over $\mathbb{R}$. Furthermore, for any $j > 2$ the spline s_j is a $j-2$ times continuously differentiable function on $\mathbb{R}$ and the $j-1$ derivative is discontinuous only at the odd integers.

Theorem 7. *Let $f \in \mathbb{H}_{per}(-1,1)$ be given. Assume that there exists a finite set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \subset (-1,1]$ such that f assumes real values and is p times differentiable on $(-1,1] \setminus \Lambda$ with the pth derivative in $L^2(-1,1)$. Then f has a unique representation in the form*

$$f(x) = a_0 + \sum_{l=1}^{m}\sum_{j=1}^{p} a_{jl}s_j(x+1-\lambda_l) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} b_k e^{ik\pi x}, \tag{3}$$

where $\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} b_k e^{ik\pi x}$ is p times differentiable with its pth derivative in $L^2(-1,1)$. Furthermore, the coefficients are given by:

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-1}^{1} f(x)dx \\ a_{jl} &= \frac{1}{2}\left(\frac{d^{j-1}f}{dx^{j-1}}(\lambda_l - 0) - \frac{d^{j-1}f}{dx^{j-1}}(\lambda_l + 0)\right)\ ,\quad \begin{array}{l} j = 1, \ldots, p \\ l = 1, \ldots, m \end{array} \\ b_k &= \frac{1}{2(ik\pi)^p}\int_{-1}^{1} \frac{d^p f(x)}{dx^p} e^{-ik\pi x}dx\ ,\ k = \pm 1, \pm 2, \ldots \end{aligned}$$

The proof is carried out by considering the Fourier series of the function

$$g(x) = f(x) - a_0 - \sum_{l=1}^{m}\sum_{j=1}^{p} a_{jl}s_j(x+1-\lambda_l)$$

and uses standard techniques. Hence it will be omitted. It should be noted that expansion (3) is unique for a given value of p but for different values of p one may have different representations of the function f in the form (3).

Function f is approximated by

$$\rho_{Np}(f;x) = a_0 + \sum_{l=1}^{m}\sum_{j=1}^{p} a_{jl}s_j(x+1-\lambda_l) + \sum_{\substack{k=-N\\ k\neq 0}}^{N} b_k e^{ik\pi x} \tag{4}$$

with a rounding error

$$|f(x)-\rho_{Np}(f;x)| \;=\; \left|\sum_{|k|>N} b_k e^{ik\pi x}\right| \;\leq\; \sum_{|k|>N} |b_k|$$

$$\leq \left(\sum_{|k|>N}(k\pi)^{2p}|b_k|^2\right)^{\frac{1}{2}} \left(\sum_{|k|>N}\frac{1}{(k\pi)^{2p}}\right)^{\frac{1}{2}} \tag{5}$$

$$\leq \left(\frac{1}{2}\int_{-1}^{1}\left(\frac{d^p f(x)}{dx^p}\right)^2 dx - \left(\sum_{l=1}^{m} a_{pl}\right)^2 - \sum_{\substack{k=-N\\ k\neq 0}}^{N}(k\pi)^{2p}|b_k|^2\right)^{\frac{1}{2}} \left(\frac{2}{(2p-1)\pi^{2p}N^{2p-1}}\right)^{\frac{1}{2}}$$

$$= o\left(\frac{1}{N^{p-\frac{1}{2}}}\right)$$

Motivated by the above we consider a screen in $\mathbb{H}_{per}(-1,1)$ comprising the subspace M spanned by the basis

$$\{s_0(x)\}\cup\{s_j(x+1-\lambda_l) : j=0,1,\ldots,p,\ l=1,\ldots,m\}\cup$$

$$\cup\{e^{ik\pi x} : k=0,\pm 1,\ldots,\pm N\},$$

where $p,m,N\in\mathbb{N}$ and $\{\lambda_1,\lambda_2,\ldots,\lambda_m\}\subset(-1,1]$ are parameters with arbitrary but fixed values. Here s_0 is the function which is constant 1 on $\mathbb{R}$. Defining a rounding from $\mathbb{H}_{per}(-1,1)$ to M is still an open problem. However, for functions of the kind described in Theorem 7 the rounding is defined through ρ_{Np}. Furthermore, to define a functoid we only need to know how to round the functions resulting from operations in M. For this purpose the rounding ρ_{Np} is sufficient.

Naturally, since M is a subspace it is closed under the operations addition and scalar multiplication. Furthermore, to define multiplication of functions and integration we only need to define these operations on the elements of the basis. The products of the functions in the basis are given by

$$
\begin{aligned}
&s_{q_1}(x+1-\lambda_{l_1})s_{q_2}(x+1-\lambda_{l_2})\\
&\quad=\sum_{j=q_1}^{q_1+q_2}\binom{j-1}{q_1-1}s_{q_1+q_2-j}(1+\lambda_{l_1}-\lambda_{l_2})s_j(x+1-\lambda_{l_1})\\
&\qquad+\sum_{j=q_2}^{q_1+q_2}\binom{j-1}{q_2-1}s_{q_1+q_2-j}(1-\lambda_{l_1}+\lambda_{l_2})s_j(x+1-\lambda_{l_2})
\end{aligned} \tag{6}
$$

$$
e^{ik_1\pi x}e^{ik_2\pi x}=e^{i(k_1+k_2)\pi x} \tag{7}
$$

$$
\begin{aligned}
&s_q(x+1-\lambda_l)e^{in\pi x}\\
&=\sum_{j=q}^{p}(-1)^n e^{i(\lambda_l-1)\pi}\binom{j-1}{q-1}(in\pi)^{j-q}s_j(x+1-\lambda_l)+\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\beta_k e^{ik\pi x}
\end{aligned} \tag{8}
$$

where in the formula (8) the coefficients β_k are given by

$$
\begin{aligned}
\beta_k &= \frac{(-1)^{k-n-1}n^{p-q}}{k^p(i\pi)^q}\sum_{r=0}^{q-1}\binom{p}{r}\left(\frac{n}{k-n}\right)^{q-r} \quad \text{if } \ k\neq 0,n\\
\beta_n &= \binom{p}{q}(in\pi)^{-q}
\end{aligned}
$$

For the respective integrals we have

$$
\int s_j(x)dx = s_{j+1}(x)\ , \quad j=1,\ldots,p \tag{9}
$$

$$
\int e^{ik\pi x}dx = \frac{1}{ik\pi}e^{ik\pi x}\ , \quad k=0,\pm 1,\ldots,\pm N \tag{10}
$$

Obviously in the formulas (6)–(9) we obtain splines s_j with $j>p$ and exponents

$e^{ik\pi x}$ with $|k| > N$ which need to be rounded. Using that

$$s_j(x+1-\lambda_l) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{(-1)^{k-1} e^{i(1-\lambda_l)\pi}}{(ik\pi)^j} e^{ik\pi}$$

all that needs to be rounded is a Fourier series which is done by truncation. Note that each time we truncate a Fourier series of a function which is at least p times differentiable with its pth derivative in $L^2(-1,1)$. Hence the uniform norm of the error is $o\left(\frac{1}{N^{p-\frac{1}{2}}}\right)$. The integration of s_0, when it arises in practical problems, should be handled with special care since $\int s_0(x)dx = s_1(x)$ holds only on $(-1,1)$.

5. Applications. As an illustrative example we consider the wave equation in the form

$$u_{tt}(x,t) - u_{xx}(x,t) = \phi(x,t) \tag{11}$$

$$u(x,0) = g_1(x),\ u_t(x,0) = g_2(x) \tag{12}$$

with periodic boundary conditions at $x = -1$ and $x = 1$, assuming that g_1, g_2, ϕ or some of their space derivatives may be discontinuous but the functions can be represented as a spline-Fourier series (3) of the space variable. More precisely, for the function ϕ we assume a representation of the form

$$\phi(x,t) = \alpha_0(t) + \sum_{l=1}^{m} \sum_{j=1}^{p} \sum_{\delta=-1}^{1} \alpha_{lj\delta}(t) s_j(x+\delta t+1-\alpha_l) + \sum_{\substack{k=-N \\ k\neq 0}}^{N} \beta_k(t) e^{ik\pi x} \tag{13}$$

The solution is

$$u(x,t) = \frac{1}{2}\left(g_1(x+t) + g_1(x-t) + \int_{x-t}^{x+t} g_2(\theta)d\theta + \iint\limits_{G(x,t)} \phi(y,\theta)dyd\theta \right),$$

where $G(x,t)$ is the triangle in $\mathbb{R}^2$ with vertices $(x-t,0)$, $(x+t,0)$ and (x,t). Note that the above form of the solution is an explicit representation of the well known property that the value of the solution at (x,t) depends only on the values of g_1 and g_2 on the line segment connecting $(x-t,0)$ and $(x+t,0)$ and on the values of ϕ in the triangle $G(x,t)$. The essential part of the above computation is evaluating the integral over $G(x,t)$. For that purpose we need to assume some sort of representation of the coefficients in (13) as elementary functions of t. For simplicity we assume here that they are polynomials of t. Then the integral over $G(x,t)$ is evaluated through the following formulas:

$$(14)\iint\limits_{G(x,t)} \frac{\theta^q}{q!} s_j(y)\mathrm{d}y\mathrm{d}\theta = s_{j+q+2}(x+t)+(-1)^q s_{j+q+2}(x-t)-2\sum_{\substack{l=0\\ l-\text{even}}}^{q} \frac{t^{q-l}}{(q-l)!} s_{j+l+2}(x)$$

$$(15)\iint\limits_{G(x,t)} \frac{\theta^q}{q!} s_j(y+\theta)\mathrm{d}y\mathrm{d}\theta = \sum_{l=0}^{q+1}\left(-\frac{1}{2}\right)^l \frac{t^{q+1-l}}{(q+1-l)!} s_{j+l+1}(x+t)-\left(-\frac{1}{2}\right)^{q+1} s_{j+q+2}(x-t)$$

$$(16)\iint\limits_{G(x,t)} \frac{\theta^q}{q!} s_j(y-\theta)\mathrm{d}y\mathrm{d}\theta = \left(\frac{1}{2}\right)^{q+1} s_{j+q+2}(x+t) - \sum_{l=0}^{q+1}\left(\frac{1}{2}\right)^l \frac{t^{q+1-l}}{(q+1-l)!} s_{j+l+1}(x-t)$$

$$(17)\iint\limits_{G(x,t)} \frac{\theta^q}{q!} e^{ik\pi y}\mathrm{d}y\mathrm{d}\theta = \frac{1}{(ik\pi)^{q+2}}\left(e^{ik\pi(x+t)}+(-1)^q e^{ik\pi(x-t)}-2\sum_{\substack{l=0\\ l-\text{even}}}^{q}\frac{t^{q-l}}{(ik\pi)^{q-l}(q-l)!}e^{ik\pi x}\right)$$

$$= 2\sum_{\substack{l=0\\ l-\text{even}}}^{\infty} (ik\pi)^l \frac{t^{q+l+2}}{(q+l+2)!}\, e^{ik\pi x}$$

Hence the solution is obtained in the form

$$(18)\qquad u(x,t) = a_0(t) + \sum_{l=1}^{m}\sum_{j=1}^{p}\sum_{\delta=-1}^{1} a_{lj\delta}(t) s_j(x+\delta t+1-\alpha_l) + \sum_{\substack{k=-N\\ k\neq 0}}^{N} b_k(t) e^{ik\pi x}$$

where all coefficients are polynomials of t.

Let us consider the solution of the problem (11–12) where

$$(19)\qquad g_1(x) = 0.05 + 0.1 s_1(x+0.75) - 0.1 s_1(x+1.25)$$

(a square wave, see Fig. 1)

$$(20)\qquad g_2(x) = s_1(x)$$

$$(21)\qquad \phi(x,t) = t s_1(x)$$

Using the above method the solution is obtained in the form (18) and it is plotted on Fig. 2. One can observe the accurate representation of the discontinuities of the solution and its derivatives.

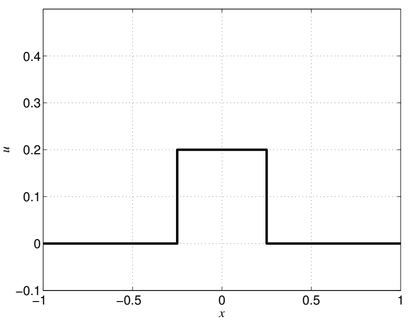

Fig. 1. Function g_1 in (19).

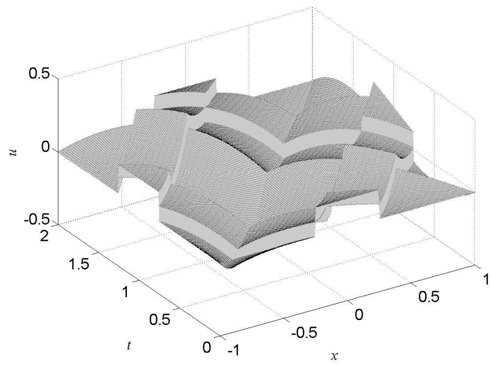

Fig. 2. The exact solution of (11)–(12) with data given in (19)–(21).

When the right hand side of the equation depends on u, one can use the techniques derived above to establish an iterative procedure converging to the exact solution. For example, for the equation

$$u_{tt}(x,t) - u_{xx}(x,t) = \rho(t)u(x,t) + \phi(x,t)$$

the following iterative procedure can be used:

$$u^{(r+1)} = (1-\lambda)u^{(r)} + \lambda\left(g + \frac{1}{2}\iint\limits_{G(x,t)} \rho u^{(r)}\right),\ r = 0,1,2,\dots, \tag{22}$$

where

$$g(x,t) = \frac{1}{2}\left(g_1(x+t) + g_1(x-t) + \int_{x-t}^{x+t} g_2(\theta)d\theta + \iint\limits_{G(x,t)} \phi(y,\theta)dyd\theta\right).$$

Naturally, successive iterations through (22) will produce functions s_j with large indexes. The obtained expressions can be simplified using the rounding discussed in the preceding section.

REFERENCES

[1] Anguelov R. Dedekind order completion of $C(X)$ by Hausdorff continuous functions. *Quaestiones Mathematicae* **27** (2004) 153–170.

[2] Anguelov R., S. Markov. Extended Segment Analysis. Freiburger Intervall-Berichte, **10**, 1981, 1–63.

[3] Anguelov R., S. Markov, B. Sendov. On the Normed Linear Space of Hausdorff Continuous Functions. Lect. Notes in Comput. Sci. **3743**, 2006, 281–288.

[4] Anguelov R., S. Markov, B. Sendov. The Set of Hausdorff Continuous Functions – the Largest Linear Space of Interval Functions. *Reliable Computing* **12** (2006), 337–363.

[5] Anguelov R., S. Markov, B. Sendov. Algebraic operations for H-continuous functions. In: Constructive Theory of Functions (Ed. B. Bojanov) Varna 2005, Marin Drinov, Sofia, 2006, 35–44.

[6] Anguelov R., E. E. Rosinger. Hausdorff Continuous Solutions of Nonlinear PDEs through the Order Completion Method. *Quaestiones Mathematicae* **28** (2005), 271–285.

[7] Anguelov R., J. H. van der Walt. Order Convergence Structure on $C(X)$. *Quaestiones Mathematicae* **28** (2005), 425–457.

[8] Epstein C., W. L. Miranker, T. J. Rivlin. Ultra Arithmetic, Part I: Function Data Types; Part 2: Intervals of Polynomials. Mathematics and Computers in Simulation, vol. **XXIV**, 1982, 1–18.

[9] Kaucher E., W. Miranker. Self-Validating Numerics for Function Space Problems. Academic Press, New York, 1984.

[10] Sendov B. Hausdorff Approximations, Kluwer, 1990.

University of Pretoria
Pretoria 0002, South Africa
e-mail: `roumen.anguelov@up.ac.za`

Received September 25, 2006
Final Accepted October 18, 2007