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CONTROL OF CHAOTIC BEHAVIOR OF INTEGRO-DIFFERENTIAL CNN MODEL ARISING IN PIEZOELECTRIC MATERIAL WITH NANO-HETEROGENEITIES*

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Piezoelectrical material with heterogeneities of nano-holes or nano-inclusions is considered in the case when it is subjected to time harmonic electro-mechanical load. The model is reduced to a system of integro-differential equations (IDE). We construct Cellular Nonlinear/Nanoscale Network (CNN) architecture for the boundary value IDE problem under consideration. For such IDE CNN model we shall determine "edge of chaos" region of the parameter set.Validation will be provided as well. Feedback control will be applied in order to stabilize the model. The computer simulations will illustrate the obtained theoretical results.

1. Introduction

The aim of this study is to propose an efficient Cellular Nonlinear/Nanoscale Network (CNN) method for studying of $2D$ anti-plane dynamic problem of piezoelectric solids with heterogeneities of different type and size as nano-holes or nano-inclusions. The modeling approach is in the frame of continuum mechanics

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of coupled fields, wave propagation theory, piezoelectricity and surface/interface elasticity theory of Gurtin and Murdoch [5]. The numerical modeling via CNN has the potential to reveal the dependence of the scattered wave far-field and stress concentration near field on the electromechanical coupling, on the type and characteristics of the dynamic load, on the material characteristics and on the geometric shape and size of the heterogeneities [1].

There are no numerical results for dynamic behavior of bounded piezoelectric domain with heterogeneities under anti-plane load. Validation is done in [3] for infinite piezoelectric plane with a hole, in [4] for isotropic bounded domain with holes and inclusions and in [6] for piezoelectric plane with nano-hole or nano-inclusion. In Section 2 we shall state the model of piezoelectric solid with heterogeneities under time-harmonic anti-plane load. Section 3 will deal with the discretization of the integro-differential equations (IDE) model by Cellular Nonlinear/Nanoscale Network (CNN) architecture. We shall determine "edge of chaos" region of the IDE CNN model in Section 4 and we shall present simulations and validation results. Section 5 will present feedback control of the IDE CNN model under investigation. Discussions will be provided in the conclusions.

2. Statement of the problem

Let $G \in R^2$ is a bounded piezoelectric domain with a set of inhomogeneities $I = \cup I_k \in G$ (holes, inclusions, nano–holes, nano–inclusions) subjected to time–harmonic load on the boundary ∂G, see Figure 1. Note that heterogeneities are of macro size if their diameter is greater than 10^{-6} m, while heterogeneities are of nano–size if their diameter is less than 10^{-7} m. The aim is to find the field in every point of $M = G \backslash I$, I and to evaluate stress concentration around the inhomogeneities.

Using the methods of continuum mechanics the problem can be formulated in terms of boundary value problem for a system of 2-nd order differential equations, see [3], Chapter 2.

$$(1) \qquad \left| \begin{array}{l} \rho^N \dfrac{\partial^2 u_3}{\partial t^2} = c_{44}^N \Delta u_3^N + e_{15}^N \Delta u_4^N, \\ e_{15}^N \Delta u_3^N - \varepsilon_{15}^N \Delta u_4^N = 0, \end{array} \right.$$

where $x = (x_1, x_2)$, $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}$ is Laplace operator with respect to t, $N = M$ for $x \in M$ and $N = I$ for $x \in I$; u_3^N is mechanical displacement, u_4^N is electric potential, ρ^N is the mass density, $c_{44}^N > 0$ is the shear stiffness, $e_{15}^N \neq 0$ is the piezoelectric constant and $\varepsilon_{11}^N > 0$ is the dielectric permittivity.

Let us define generalized stress $\sigma_{kj}, k = 1,2; j = 3,4$ as

$$(2)\qquad \left| \begin{array}{l} \sigma^N_{k3} = c^N_{44}\dfrac{\partial u^N_3}{\partial x_k} + e^N_{15}\dfrac{\partial u^N_4}{\partial x_k}, \\ \sigma^N_{k4} = e^N_{15}\dfrac{\partial u^N_3}{\partial x_k} - \varepsilon^N_{11}\dfrac{\partial u^N_4}{\partial x_k}, \end{array} \right.$$

Note that σ^N_{i3} is called mechanical stress, while σ^N_{i4} is called electrical displacement(the usual notation in mechanics is $D^N_i = \sigma^N_{i4}, i = 1,2$ [3]).

Generalized traction at the point x on the line segment with normal vector $n = (n_1, n_2)$ is defined as

$$(3)\qquad \left| \begin{array}{l} \tau^N_3 = \sigma^N_{13}n_1 + \sigma^N_{23}n_2, \\ \tau^N_4 = \sigma^N_{14}n_1 + \sigma^N_{24}n_2, \end{array} \right.$$

At every point $x \in S = \partial I$ we can define normal vector n and unit tangential vector l such that (l, n) forms right coordinate system (see Fig. 1).

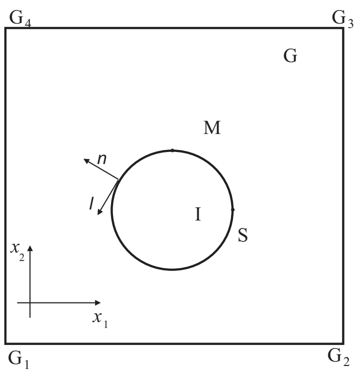

Figure 1: Rectangular PEM matrix with circle inhomogeneity

We shall state the boundary conditions for two cases:

A) In the case when I is a nano–hole, formally we can consider that the constants in I are $c^I_{44} = 0$, $e^I_{15} = 0$, $\varepsilon^I_{11} = 0$ and boundary conditions on S are

$$(4)\qquad \tau^M_J = \frac{\partial \sigma^S_{lJ}}{\partial l} \quad \text{on } S,$$

Then we consider BVP undr consideration is equation (1) and boundary conditions (3), (4).

Boundary conditions (4) can be written in the following form for the mechanical and electrical part correspondingly:

$$\tau_3^M = \sigma_{n3}^M = \frac{\partial \sigma_{l3}^S}{\partial l}, \quad \tau_4^M = \sigma_{n4}^M = \frac{\partial \sigma_{l4}^S}{\partial l}$$

where τ_3^M and τ_4^M are the normal component of mechanical stress and electrical displacement (see(3)) in the matrix, while $\frac{\partial \sigma_{l3}^S}{\partial l}$ and $\frac{\partial \sigma_{l4}^S}{\partial l}$ are tangential derivatives of tangential components of stress σ_{l3}^S and tangential electrical displacement σ_{l4}^S along the nano–hole boundary S.

B) In the case when I is a nano–inclusion [4,6], the constants in I are $c_{44}^I > 0$, $e_{15}^I \neq 0$, $\varepsilon_{11}^I > 0$; the constants in M are $c_{44}^M > 0$, $e_{15}^M \neq 0$, $\varepsilon_{11}^M > 0$.

On the heterogeneity boundary S where are defined constants $c_{44}^S, e_{15}^S, \varepsilon_{11}^S$ and with the notation for generalized displacement u^S along S the generalized tangential stress on S is defined as:

$$\left| \begin{array}{l} \sigma_{l3}^S = c_{44}^S \dfrac{\partial u_3^S}{\partial l} + e_{15}^S \dfrac{\partial u_4^S}{\partial l}, \\ \sigma_{l4}^S = e_{15}^S \dfrac{\partial u_3^S}{\partial l} - \varepsilon_{11}^S \dfrac{\partial u_4^S}{\partial l}, \end{array} \right. \tag{5}$$

Then boundary conditions on S are

$$\left| \begin{array}{l} u_j^M = u_j^I \quad \text{on } S, \\ \tau_j^I + \tau_j^M = \dfrac{\partial \sigma_{lj}^S}{\partial l} \quad \text{on } S, \end{array} \right. \tag{6}$$

The BVP in this case is equation (1) and boundary conditions (3), (6).

Boundary conditions (6) can be written in the following form for the mechanical and electrical part correspondingly:

$$\tau_3^I + t_3^M = \frac{\partial \sigma_{l3}^S}{\partial l}, \quad \tau_4^I + t_4^M = \frac{\partial \sigma_{l4}^S}{\partial l}$$

where $\tau_3^N, \tau_4^N, N = I, M$ are the normal component of mechanical stress and electrical displacement (see (3)) in the inclusion and in the matrix, while $\frac{\partial \sigma_{l3}^S}{\partial l}$ and $\frac{\partial \sigma_{l4}^S}{\partial l}$ are tangential derivatives of tangential components of stress σ_{l3}^S and tangential electrical displacement σ_{l4}^S along the interface boundary S. Here, it is take into consideration that $n_i^M = -n_i^I = -n_i, i = 1, 2$. Note that for the mechanical displacement u_3^N and for the potential of the electric field $u_4^N = \phi$ continuity conditions are satisfied, see first row of (6).

3. Cellular Nonlinear/Nanoscale Network (CNN) model of the BVP

In [3] fundamental solutions of the BVP are found using the Fourier transform. Then using the Gauss theorem [9] and proceeding as in [3] from the BVP a system of integro-differential equations (IDE) is obtained for the unknowns $u_{3,4}$ on S. In this paper we shall study the general form of IDE obtained in [3]. Let us consider the following system of IDE [8]:

$$\frac{\partial^2}{\partial t^2} = D\frac{\partial^2 u}{\partial x^2} - C_1 \int_S G(u(x))dx + f(u), \tag{7}$$

where C_1 is a constant depending on the ρ^M, $c_{44}^M > 0$, $e_{15}^M \neq 0$ and $\varepsilon_{11}^M > 0$, D is diffusion coefficient, $u = (u_3, u_4)$, function $G(x)$ is a function of the displacement vectors $u_{3,4}$ and the traction $\tau_{3,4}$, function $f(u)$ is monotonically increasing function.

Cellular Nonlinear/Nanoscale Networks (CNN) are analogue dynamic processor arrays, which are made of cells. Let us consider a two–dimensional grid with 3×3 neighborhood system shown on Figure 2.

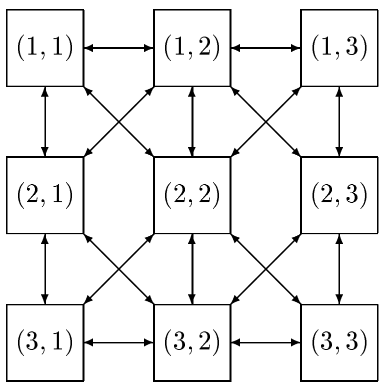

Figure 2: 3×3 neighborhood CNN

One of the key features of CNN is that the individual cells are nonlinear dynamical systems, but the coupling between them is linear. Roughly speaking, one could say that these arrays are nonlinear but have a linear spatial structure, which makes the use of techniques for their investigation common in engineering or physics attractive.

We will give general definition of a CNN which follows the original one [7]:

Definition 1. *An $M \times M$ cellular nanoscale network is defined mathematically by four specifications:*

1). *CNN cell dynamics;*

2). *CNN synaptic law which represents the interactions (spatial coupling) within the neighbor cells;*

3). *Boundary conditions;*

4). *Initial conditions.*

In terms of the definition we can present the dynamical systems describing CNN. For general CNN whose cells are made of time-invariant circuit elements, each cell $C(ij)$ is characterized by its CNN cell dynamics :

$$\dot{x}_{ij} = -g(x_{ij}, u_{ij}, I^s_{ij}), \tag{8}$$

where $x_{ij} \in \mathbf{R}^m$, u_{ij} is usually a scalar. In most cases, the interactions (spatial coupling) with the neighbor cell $C(i+k, j+l)$ are specified by a CNN synaptic law:

$$\begin{aligned} I^s_{ij} &= A_{ij,kl} x_{i+k,j+l} + \\ &+ \tilde{A}_{ij,kl} * f_{kl}(x_{ij}, x_{i+k,j+l}) + \\ &+ \tilde{B}_{ij,kl} * u_{i+k,j+l}(t). \end{aligned} \tag{9}$$

The first term $A_{ij,kl} x_{i+k,j+l}$ of (9) is simply a linear feedback of the states of the neighborhood nodes. The second term provides an arbitrary nonlinear coupling, and the third term accounts for the contributions from the external inputs of each neighbor cell that is located in the N_r neighborhood.

Then the CNN model [7] for the IDE (7) can be written as:

$$\frac{du_{ij}}{dt} = DA_1 * u_{ij} - C_1 \int_S G(u_{ij}))dt + f(u_{ij}), 1 \le i \le n, j = 3, 4, \tag{10}$$

where A_1 is 1-dimensional discretized Laplacian template [7]

$$A_1 : (1, -2, 1),$$

$*$ is convolution operator, $n = M \times M$ is the number of cells of the CNN architecture.

CNN is an excellent candidate for both analog and digital applications because of its structural simplicity, relative ease of fabrication, inherent speed and design flexibility. Many methods used in image processing and pattern recognition can be easily implemented by CNN approach.

4. Dynamics of our CNN model

In this section we shall identify the values of the cell parameters for which the IDE CNN model (7) may exhibit complexity. The necessary condition for a nonconservative system to exhibit complexity is to have its cell locally active [2]. The theory which will be presented below offers a constructive analytical method for uncovering local activity. The precisely defined parameter domain in which the model can exhibit complex behavior is called the edge of chaos. In particular, constructive and explicit mathematical inequalities can be obtained for identifying the region in the CNN parameter space where complexity phenomena may emerge, as well as for localizing in further into a relatively small parameter domain called edge of chaos where the potential for emergency is maximized. By restricting the cell parameter space to the local activity domain, a major reduction in the computing time required by the parameter search algorithms is achieved [2].

We develop the following constructive algorithm for determining the edge of chaos domain:

1. Map the IDE (7) into its discrete-space version which will be called IDE CNN:

$$\frac{du_{ij}}{dt} = D(u_{i-1j} - 2u_{ij} + u_{i+1j}) - C_1 \int_S G(u_{ij}))dt + f(u_{ij}) = F(u_{ij}), \tag{11}$$

where $1 \le i \le n$, $j = 3, 4, C_1$ is depending on the cell parameters ρ, c_{44}, e_{15} and ε.

2. Find the equilibrium points of the IDE CNN model (11). According to the theory of dynamical systems equilibrium points u^* are these for which

$$F(u^*) = 0. \tag{12}$$

In general, the above system may has one, two or three real roots and these roots are functions of the cell parameters ρ, c_{44}, e_{15} and ε.

3. Calculate now the cell coefficients of the Jacobian matrix of (12) about the system equilibrium points E_k, $k = 1, 2, 3$.

4. Calculate the trace $Tr(E_k)$ and the determinant $\Delta(E_k)$ of the Jacobian matrix of (12) for each equilibrium point.

5. Define stable and locally active region $SLAR(E_k)$ according to the following definition:

Definition 2. *Stable and Locally Active Region $SLAR(E_k)$ at the equilibrium points E_k for IDE CNN model* (11) *is such that $Tr < 0$ and $\Delta > 0$.*

6. Edge of chaos

In the literature, the so-called edge of chaos (EC) means a region in the parameter space of a dynamical system where complex phenomena and information processing can emerge. We shall try to define more precisely this phenomena till now known only via empirical examples.

Definition 3. *IDE CNN* (11) *is said to be operating on the edge of chaos EC iff there is at least one equilibrium point which is both locally active and stable.*

Following the above algorithm we have proved the following theorem::

Theorem 1. *The IDE CNN model* (11) *is operating in the EC regime iff the following conditions for the parameters are satisfied:* $\dfrac{\varepsilon_{15}c_{44}+e_{15}^2}{\rho\varepsilon_{15}} > 0$. *In this parameter set there is at least one equilibrium point E_k which is both locally active and stable.*

After simulating our IDE CNN model (11) we obtain the results on Figure 3.

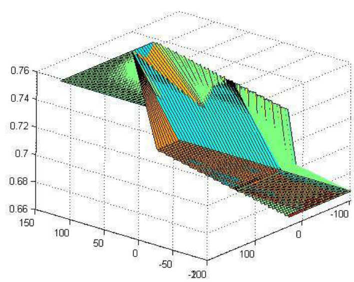

Figure 3: Simulation of IDE CNN model (11)

Remark 1. In order to simulate the model (11), the parameters have to be determined in an optimization process. During the optimization process the mean square error

$$e_{mse} = \sum_i \sum_j \frac{(u_{ij} - \tilde{u}_{ij})^2}{N} \tag{13}$$

can be minimized using Powells method and Simulated Annealing [8]. In each step e_{mse} is calculated by taking the reference $u_{ij}(t)$ and the solution $\tilde{u}_{ij}$ of

IDE CNN model obtained by simulation system MATCNN applying 4th- order Runge-Kutta integration. In order to minimize the computational complexity and to maximize the significance of the mean square error only outputs of 10 cells are taken into account.

4.1. Validation

Let us consider the square domain $G_1G_2G_3G_4$ with a side a, containing a single circular inhomogeneity with a radius $c = \beta a$ and center at the square center, see Figure 1. Note that if $\beta < 0.05$ the influence of the exterior boundary ∂G on the solution is expected to be small, while if $\beta > 0.2$ it is expected significant influence.

Material parameters of the matrix are for transversely isotropic piezoelectric material PZT4:

- elastic stiffness: $c_{44}^M = 2.56 \times 10^{10}$ N/m^2;
- piezoelectric constant: $e_{15}^M = 12.7$ C/m^2;
- dielectric constant: $\varepsilon_{11}^M = 64.6 \times 10^{-10}$ C/Vm;
- density: $\rho^M = 7.5 \times 10^3$ kg/m^3.

The applied load is time harmonic uni-axial along vertical direction uniform mechanical traction with frequency ω and amplitude $\sigma_0 = 400 \times 10^6$ N/m^2 and electrical displacement with amplitude $U_0 = k\dfrac{\varepsilon_{11}^M}{e_{15}^M}\sigma_0$, where $k = 0.1$ for electromechanical load and $k = 10^{-4}$ for "pure" mechanical load.

This means that $G_t = \partial G$, $G_u = \emptyset$ and the boundary conditions (4) are:

- on G_1G_2: $\tau_3^{M0} = -\sigma_0, \tau_4^{M0} = -D_0$;
- on G_2G_3: $\tau_3^{M0} = \tau_4^{M0} = 0$;
- on G_3G_4: $\tau_3^{M0} = \sigma_0, \tau_4^{M0} = D_0$;
- on G_4G_1: $\tau_3^{M0} = \tau_4^{M0} = 0$.

For heterogeneities at nano–scale as nano-hole in our case we have: the side of the square is $a = 10^{-7}m$; material parameters inside I for hole are 0; material parameters on $S = \partial I$ for hole and for an inclusion are: $c_{44}^S = 0.1c_{44}^M$, $e_{15}^S = 0.1e_{15}^M$, $\varepsilon_{11}^S = 0.1\varepsilon_{11}^M$, $\rho^S = \rho^M$.

Then simulating our CNN IDE model (11) we obtain the following solutions (Figure 4):

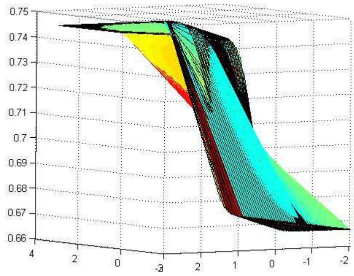

Figure 4: Validation of the obtained results

5. Stabilizing feedback control for IDE CNN model

Let us extend the model (11) by adding to each cell the local linear feedback:

$$\frac{du_{ij}}{dt} = D(u_{i-1j} - 2u_{ij} + u_{i+1j}) - C_1 \int_S G(u_{ij}))dt - ku_{ij}, \tag{14}$$

where k is the feedback controls coefficient, which is assumed to be equal for all cells.

The problem is to prove that this simple and available for the implementation feedback can stabilize the IDE CNN model (11). In the following we present a proof of this statement and give sufficient condition on the feedback coefficient values which provide stability of the CNN nonlinear model (14).

As a first step, we examine the the stability conditions of the system (14), linearized in the neighborhood of the zero equilibrium point E_0. This system in a vector-matrix form is given by

$$\frac{dz}{dt} = J(k)z$$

$J(k)$ is the Jacobian matrix of the controlled IDE CNN in E_0.

Theorem 2. *Let the parameters of IDE CNN system and feedback coefficient k (14) have positive values. Then its linearized in E_0 model is assymptotically stable for all $k > 0$.*

P r o o f. Define the quadratic Lyapunov function candidate $L(z) = \frac{1}{2}z^T z$. Then its derivative along the linearized control IDE CNN is $\frac{dL(z)}{dt} = \frac{1}{2}z^T(J^T(k)+$

$J(k))z = -z^T Q(k)z$. Therefore $\dfrac{dL(z)}{dt} < 0$ implies a positive definiteness of $Q(k)$. It can be shown that $Q(k)$ positive definiteness implies $k > 0$.

For verification of the above statement the eigenvalues of $J(k)$ were calculated related on the values of feedback coefficient k. Stability of the linear system requires that the eigenvalues $\lambda_j^i, i = 1, \ldots, 4$ satisfy the inequality $\max_i Re\lambda_j^i < 0$. Dependence of the $\max_i Re\lambda_j^i$ on k for the parameter set, which is defined in the Theorem 1, is represented in Figure 5. $\square$

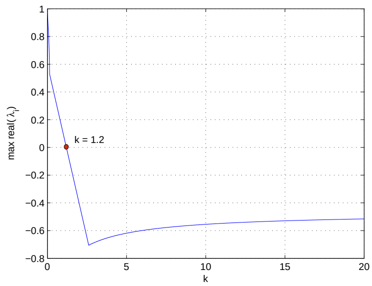

Figure 5: Dependence of real part value of dominant eigenvalue on the feedback coefficient IDE CNN model (14)

The critical value of $k = 1.2$, for which the $\max_i Re\lambda_j^i = 0$, is marked in the figure. For this parameter set the system gives the critical value $k = 1.67$.

6. Conclusion and discussions

In this paper we study integro-differential equation which arise in piesoelectrical material with nano-holes or nano-inclusions. We derive the algorithm for determination of edge of chaos regime in the IDE CNN model (11). Then we apply feedback control in order to stabilize the model under consideration. Computer simulations and validation are provided.

The characteristic that is of interest in mechanics is normalized dynamic Stress Concentration Field (SCF) $|\sigma_{\varphi 3}/\sigma_0|$ and the normalized dynamic Electric Field Concentration Field (EFCF) $|e_{15}^M E_\varphi/\sigma_0|$ along the perimeter of the inhomogeneity, φ is the polar angle of the observer point.

Numerical simulations show [3, 4, 6] that the stress concentration field near defects is strongly influenced by the type and the size of the defect(crack, hole or inclusion), the material anisotropy, the defect location and geometry, the dynamic load characteristics and the mutual interactions between defects and between them and the solid's boundary.

Computational nanomechanics has a high priority in Europe, because it concerns the development and creation of new smart materials and devices based on them. The present paper addresses the vital component of accurate description and computation of the wave motions and stress concentrations that are developed in the multifunctional materials with nano-structures.

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