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DYNAMIC BEHAVIOUR OF A CRACK AT MACRO- AND NANO- SCALE IN ANISOTROPIC PLANE BY BIEM*

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The two-dimensional in-plane time-harmonic elastodynamic problem for anisotropic infinite plane with a crack at macro and nano scale subjected to incident plane longitudinal P or shear SV wave is studied. The continuum mechanics model of surface elasticity proposed by Gurtin and Murdoch [1] is applied to account for the effects of surface elasticity for a crack at nano-level. The non-hypersingular traction boundary integral equation method (BIEM) is used in conjunction with closed form frequency dependent fundamental solution, obtained by Radon transform. In addition a parametric study for the dynamic stress intensity factor (SIF) and stress concentration field (SCF) sensitivity to the frequency, crack-size, surface effects and material anisotropy is presented.

1. State of the art

When the characteristic sizes of heterogeneities in materials shrink to nanometers, surface effects often play a crucial role in their mechanical behaviour. Therefore, the mechanical properties of the material in a thin surface layer may distinctly deviate from its bulk counterpart. A continuum mechanical model of surface elasticity was proposed by Gurtin and Murdoch [1] aiming to describe the surface effects. In this model a surface/interface is regarded as an elastic but negligibly thin membrane with the properties that is adhered to the underlying bulk material

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without slipping and with elastic constants different from the bulk. The equilibrium and constitutive equations in the bulk are the same as those in the classical elasticity theory, but the presence of surface stresses gives rise to non-classical boundary conditions taking into consideration the residual surface tension and surface deformation.

There exist papers demonstrating that Gurtin and Murdoch [1] model can well explain various size–dependent elastic properties when solve elastostatic and elastodynamic problems at nanoscale. We will discuss here only results for dynamic problems. Anti–plane wave motion problems for diffraction of SH waves by nano–inclusions were studied analytically in Fu et al. [2], Yang et al. [3], Fang et al. [4]. In–plane wave motion problems for diffraction of time–harmonic P- and SV- waves by nano–holes and nano–inclusions were solved analytically in Wang [5], Zhang et al. [6], Wang et al. [7], Ru et al. [8]. The most of the used computational techniques are analytical, there are few papers presenting BIEM as an efficient tool for solution of 2D problems and they all concern static solutions, except the results in Parvanova et al. [9].

The conclusions from the short review are as follows: a) There are a limited number of papers considering surface elasticity effects on the stresses near the crack-tip and all they are for static loads; b) To the authors knowledge there are no results for nano-cracks in the field of dynamic fracture mechanics even for elastic isotropic solids. For anisotropic cracked solids there is no results even for static problems.

The aim of this study is to consider the dynamic stress field near mode-I crack tip under incident time-harmonic plane P or SV wave in anisotropic plane by BIEM taking into account the surface elasticity effect in the frame of Gurtin and Murdoch [1] model. At macro-scale the crack is finite line crack (Figure 1 a)), while at nano-scale a blunt crack (Figure 1 b)) with crack root presenting by a semicircular shape of radius b is considered.

The paper is organized as follows: the formulation of the considered in-plane elastodynamic problem for a crack at macro- and nano-scale in an anisotropic plane subjected to incident time-harmonic wave is done in Section 2, while its reformulation via boundary integral equations along existing boundaries based on the analytically derived by Radon transform fundamental solution is given in Section 3. Numerical results are shown in Section 4, and finally conclusions are in Section 5.

2. Problem formulation

In a Cartesian coordinate system $Ox_1x_2x_3$ consider an anisotropic plane $x_3 = 0$ containing a line macro–crack $\Gamma = \Gamma^+ \cup \Gamma^-$ with length $|\Gamma| = 2a$ (Figure 1 a)) or nano-crack, a domain G with boundary S (Figure 1 b)) under incident time-harmonic with frequency ω plane longitudinal P or shear SV wave with incident angle φ in respect to the Ox_1 coordinate axis. Considered nano-crack is a blunt crack with crack root presenting by a semicircular shape of radius b with dimension falling in the interval 10^{-7} m $\div$ 10^{-10} m. The crack length is $|S| = 2[2(a-b) + \pi b]$.

Plane strain state, i.e. in-plane wave motion in respect to plane $x_3 = 0$ is considered. In this case the only non-zero field quantities are displacements u_1, u_2, stresses $\sigma_{11}, \sigma_{12}, \sigma_{22}$ all dependent on coordinates $x = (x_1, x_2)$ and frequency ω. What follows is to define the boundary-value problem (BVP) in frequency domain by the governing elliptic partial differential equations of second order and corresponding boundary conditions.

2.1. The governing equations

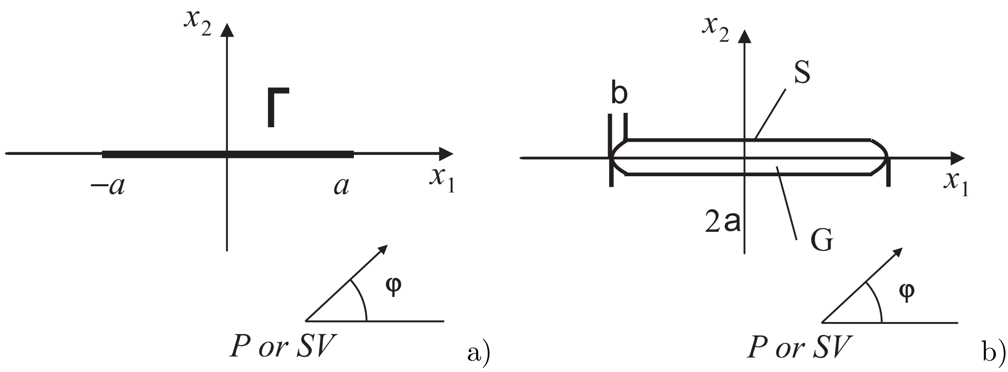

Figure 1: The problem geometry. Incident P or SV wave propagating in an anisotropic plane with : a) a line macro crack; b) a blunt nano-crack

2.1.1. Constitutive equation

In the case of general anisotropy 6 parameters $c_{11}, c_{12}, c_{16}, c_{22}, c_{26}, c_{66}$ characterize the stiffness matrix. Note that the contracted Voigt notation is applied to the fourth order stiffness matrix C_{ijkl}, where $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$, $C_{ijkl}g_{ij}g_{kl} > 0$ for any non-zero real symmetric tensor g_{ij}. The constitutive equations – Hookes law, is presented in the following form:

$$\sigma_{ij} = C_{ijkl} s_{kl}, \tag{1}$$

where s_{ij} is the strain tensor. The orthotropic material is characterized with the following four independent material constants $c_{11}, c_{12}, c_{22}, c_{66}$, because $c_{16} = c_{26} = 0$. For isotropic material the following relations are truth $c_{11} = c_{22} = \lambda + 2\mu$ and $c_{12} = \lambda$; $c_{66} = \mu$, where λ and μ are Lamé constants.

2.1.2. Kinematics relations

Under assumption of small displacements the kinematic relations are

$$s_{ij}(x,\omega) = 0.5[u_{i,j}(x,\omega) + u_{j,i}(x,\omega)], \quad i,j = 1,2,$$

where the symbol $()_{,i}$ represents the partial derivative with respect to x_i.

2.1.3. Equation of motion

In the plane $x_3 = 0$ the equation of motion is

$$\sigma_{ij,j}(x,\omega) + \rho\omega^2 u_i(x,\omega) = 0, \quad x \in R^2 \backslash \Gamma, \text{ or } x \in R^2 \backslash G, \tag{2}$$

in the case of zero body force, where ρ is the mass density.

2.2. Boundary conditions

2.2.1. Boundary conditions at macro-scale

Along the crack-faces traction free boundary conditions are satisfied, i.e.

$$t_j(x,\omega) = 0, \quad x \in \Gamma, \text{ or } x \in S, \tag{3}$$

where t_j is the total traction, which in the case of incident plane wave is presented as a superposition of incident and scattered wave. The same is truth for the total displacement and traction fields, i.e.

$$\begin{aligned} u_i(x,\omega) &= u_i^{in}(x,\omega) + u_i^{sc}(x,\omega), \quad x \in R^2 \backslash \Gamma, \text{ or } x \in R^2 \backslash G, \\ t_i(x,\omega) &= t_i^{in}(x,\omega) + t_i^{sc}(x,\omega), \quad x \in \Gamma, \text{ or } x \in S. \end{aligned} \tag{4}$$

At infinite the Sommerfeld's radiation condition for the scattered wave is satisfied.

The BVP for the scattered wave field u_i^{sc} consists of governing equation (2), the Sommerfeld's radiation condition at infinite and along the cracks boundary

$t_i^{sc} = -t_i^{in}$, $x \in \Gamma$, or $x \in S$ having in mind relation (4). Thus, to solve this BVP we must know the incident wave field for any direction of propagation in the elastic anisotropic plane. Following Dineva et al. [10], the incident wave field of plane wave propagating with frequency ω in the direction $\eta = (\eta_1, \eta_2)$ with angle φ with Ox_1 axis, i.e., $\eta_1 = \cos\varphi$, $\eta_2 = \sin\varphi$ at point x is as follows:

$$u_i^{in}(x, \omega) = p_i^j(\eta) \exp[-ik_j(\eta)\langle x, \eta\rangle], \tag{5}$$

where super-index $j = 1$ stands for P wave, while $j = 2$ is for SV wave, and $\langle ., . \rangle$ denotes scalar product in R^2. Here wave numbers are $k_i(\eta) = \sqrt{\dfrac{\rho}{a_i(\eta)}}\omega$ and $a_i(\eta) > 0, a_1 > a_2$, $p^i = (p_1^i, p_2^i)$ are eigenvalues and eigenvectors of the matrix

$$C(\eta) = \begin{pmatrix} c_{11}\eta_1^2 + c_{66}\eta_2^2 + 2c_{16}\eta_1\eta_2 & c_{26}\eta_2^2 + (c_{12} + c_{66})\eta_1\eta_2 \\ c_{16}\eta_1^2 + c_{26}\eta_2^2 + (c_{12} + c_{66})\eta_1\eta_2 & c_{22}\eta_2^2 + 2c_{26}\eta_1\eta_2 \end{pmatrix}.$$

Note that in the isotropic case $a_1 = \lambda + 2\mu$, $a_2 = \mu$ and only in this case the eigenvalues, eigenvectors and wave numbers do not depend on η.

Once having incident displacement field, the corresponding stresses are computed by Eq. (1) and the corresponding tractions on the boundary are $t_j^{in}(x, \omega) = \sigma_{ij}^{in}(x, \omega)n_i(x)$.

The knowledge of SIFs gives information for the strength and life time prediction of studied solid and structures. The computation of SIFs is possible by the usage of well-known traction formulae. If consider in-plane crack along the segment AB with local coordinates of points $A = (-a, 0)$, $B = (+a, 0)$ subjected to time-harmonic load, the traction formulae give:

$$K_I = \lim_{x_1 \to \pm a} \sqrt{2\pi(x_1 \mp a)}t_2, \quad K_{II} = \lim_{x_1 \to \pm a} \sqrt{2\pi(x_1 \mp a)}t_1, \tag{6}$$

where $t_i, i = 1, 2$ is the traction at a point $(x_1, 0)$ close to the crack-tips.

2.2.2. Boundary conditions at nano-scale

Non-classical boundary condition for the blunt nano-crack is applied along its surface S. Note that for blunt crack the surface is (see Fig. 1b)): $S = S^+ \cup S^- \cup S^l \cup S^r$, where S^+, S^- are upper and lower flat part of the crack and crack-tip is presented by a semicircles S^l, S^r with radius b with dimension falling in the interval 10^{-7} m $\div$ 10^{-10} m.

In the frame of the Gurtin and Murdoch [1] model the surface stress is expressed by

$$\sigma_{ij}^S = \tau_0\delta_{ij} + \frac{\partial E}{\partial \varepsilon_{ij}^S}. \tag{7}$$

The surface stress σ_{ij}^S is related to the deformation dependent surface energy density E, where ε_{ij}^S is the 2x2 strain tensor for the surface S, δ_{ij} is the Kronecker delta symbol, τ_0 is the residual surface tension under unstrained condition (along undeformed surface) which is independent on deformation and induce an additional static deformation. The residual surface tension τ_0 is often ignored in the dynamic analysis. Following Gurtin and Murdoch [1], it is assumed that the surface layer S with zero thickness has different elastic properties from that of the plane. Although the plane is elastic anisotropic, we assume the surface layer S is elastic and isotropic with surface Lamé constants λ^S and μ^S. The following conditions are satisfied in tangential and in normal direction along S, i.e. in local coordinate system of normal n and tangential l to S vectors

$$(8)\qquad \left| \begin{array}{l} \sigma_{nl}^M = \dfrac{\partial \sigma_{ll}^S}{\partial l} + \tau_0 \dfrac{\psi_{ll}^S}{\rho} \\ \sigma_{nn}^M = -\dfrac{\sigma_{ll}^S}{b} + \tau_0 \dfrac{\partial \psi_{ll}^S}{\partial l}. \end{array} \right.$$

Here the first equilibrium equation is written in the tangential plane and the second equation is in the normal direction to the surface boundary S, $\psi_{ll}^S = -\dfrac{u_l}{b} + \dfrac{\partial u_n}{\partial l}$, $\varepsilon_{ll}^S = \dfrac{u_n}{b} + \dfrac{\partial u_l}{\partial l}$, the super suffix M means the matrix material in the plane, $\sigma_{ll}^S = \tau_0 + \alpha^S \varepsilon_{ll}^S$, $\alpha^S = \lambda^S + 2\mu^S$ is the normal stress in tangential direction expressed by the constitutive equation for elastic isotropic behavior of the surface layer, b is the curvature radius of the boundary S, (u_n, u_l) are the displacement components in the local coordinate system (n, l). The discussed above boundary conditions (8) can be reformulated in the more compact form in respect to the tractions $t_k^M = \sigma_{kj}^M n_j$ developed along the interface boundary, see Parvanova et al. [9]. Note that when $\tau_0 = 0$ and $\lambda^S = \mu^S = 0$ the boundary condition (8) transforms into classical boundary condition (3) describing the traction free surface of the macro-crack.

3. BIEM formulation

3.1. Macro-crack

The formulated in Section 1. problem can be described by the following frequency-dependent non-hypersingular traction based BIE along the crack boundary Γ in respect to the unknown crack opening displacement (COD) of the scattered wave field defined as $\Delta u_i^{sc} = u_i^{sc}|_{\Gamma^+} - u_i^{sc}|_{\Gamma^-}$ along the crack Γ. Following Dineva et al.

[10] we obtain the following integro–differential equation for Δu_i^{sc}:
(9)

$$t_i^{in}(x,\omega) = C_{ijkl} n_j(x) \int_{\Gamma+} \left[\left(\sigma^*_{p\eta k}(x,\xi,\omega) \Delta u^{sc}_{p,\eta}(\xi,\omega) \right.\right.$$

$$- \left.\left. \rho\omega^2 u^*_{pk}(x,\xi,\omega) \Delta u^{sc}_p \right) \delta_{\lambda l} - \sigma^*_{m\lambda k}(x,\xi,\omega) \Delta u^{sc}_{m,l}(\xi,\omega) \right] n_\lambda(\xi) d\xi, \quad x \in \Gamma^+.$$

Here the couple x,ξ presents the position vectors of the source and receiver points, $\sigma^*_{ijq} = C_{ijkl} u^*_{kq,l}$, u^*_{kq} is the displacement fundamental solution of the governing equation (2), δ_{ij} is the Kronecker symbol, all indices takes values 1 and 2, the summation convention over repeated indexes is implied, while subscript commas denote partial differentiation with respect to ξ. In the BIE (9) all singular integrals are of CPV sense. Once having solution for the COD, the solution for displacement and stress at any point in $R^2 \backslash \Gamma$ can be obtained by the usage of the integral representation formulae. The total wave field is obtained by the usage of the superposition formulae (4).

3.2. Nano-crack

In here we use integro-differential equation for u_i^{sc} on S obtained analogous as (9), see Dineva et al. [10], but accounting for the surface effect:

$$c_{ij}(t_j^{in}(x,\omega) - t_j^M(x,\omega)) = C_{ijkl} n_j(x) \int_S \left[\left(\sigma^*_{p\eta k}(x,\xi,\omega) u^{sc}_{p,\eta}(\xi,\omega) \right.\right.$$

$$- \left.\left. \rho\omega^2 u^*_{pk}(x,\xi,\omega) u^{sc}_p \right) \delta_{\lambda l} - \sigma^*_{m\lambda k}(x,\xi,\omega) u^{sc}_{m,l}(\xi,\omega) \right] n_\lambda(\xi) d\xi, \quad x \in S.$$

where $u_i^M = u_i^{sc} + u^{in}$, $t_i^{in} + t_i^{sc} = t_i^M$, u_i^{in}, t_i^{in} are defined with (5), c_{ij} is the jump term depending on the local geometry at the source point x. Once having displacement along the surface S, the displacement and stresses at any observer point $x \notin S$ can be obtained by integral representation formulae.

4. Numerical results

The BIEM model presented in Section 3. describes the general anisotropic cracked solid and in this section we will consider two model cases of anisotropic solids:

(a) isotropic material with $\lambda = 3.315 \times 10^{10}$ N/m^2, $\mu = 2.21 \times 10^{10}$ N/m^2, i.e. Poisson's ratio $\nu = 0.3$ and correspondingly $c_{11} = c_{22} = \lambda + 2\mu$, $c_{12} = \lambda$, $c_{66} = \mu$;

(b) orthotropic material with $\lambda = \mu = 2.21 \times 10^{10}$ N/m^2, $c_{11} = c_{22} = \lambda + 2\mu$, $c_{12} = 1/30\mu$, $c_{66} = \mu$.

Density in both cases is $\rho = 2.7 \times 10^3$ kg. The following expression for the normalized frequency is used $\Omega = a\omega\sqrt{\rho/c_{22}}$, ω is the frequency of the incident

P wave with incident angle $\varphi = \pi/2$. We use dimensionless surface parameter $m = \dfrac{\alpha^S}{2\mu b b_m}$ wth $\alpha^S = 8.31$ N/m and in the numerical results we get $b_m = 1, 5, 25, 125, 625, \infty$. Note that $m = 0$ corresponds to the blunt crack without surface effect.

We use 5 boundary elements (BE) on the crack Γ, with a length $|\Gamma| = 2a$, $a = 5 \times 10^{-9}$ m, where $1^{-\mathrm{st}}$ and $5^{-\mathrm{th}}$ are quarter-point crack-tip BE with the length $0.15a$, on the blunt crack S, with a length $|S| = 2a[2(1-0.0375)+0.0375\pi] \approx 2.04281 \times 10^{-8}$ m we use 10 ordinary BE: 8 on $S^- \cup S^+$ and 2 on semi-circle S^l, S^r with radius $b = 0.0375a$.

In the numerical examples for the normal incident load (note that in this case SCF $K_{II} = 0$) the normalized SCF near the right crack-tip of Γ and close to S at the point $(x_1, 0)$ is evaluated using the formulae, see (6),

$$K_I((x_1,0),\omega) = \frac{\sigma_{22}((x_1,0),\omega)}{\sigma_{22}^{in}((x_1,0),\omega)}\sqrt{2\pi(x_1-a)}, \quad x_1 > a,$$

normalized SIF on the right crack-tip of Γ is

$$K_I(\omega) = \lim_{x_1 \to a} K_I((x_1,0),\omega).$$

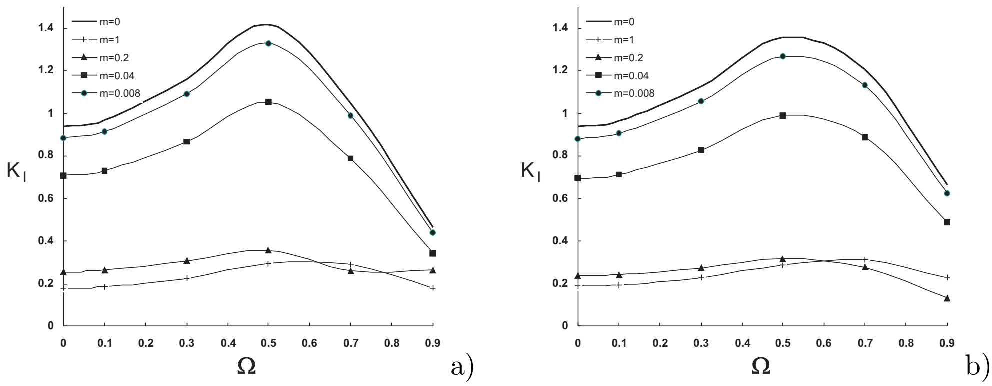

Figure 2: Normalized SIF K_1 versus normalized frequency Ω for different values of surface parameter $m = \dfrac{\alpha^S}{2\mu b b_m}$: a) isotropic plane; b) anisotropic plane

The results of the BIEM computations for normalized SIF produced by normal to the crack surface incident P-wave, which captures interface effects at the nanoscale, are plotted in Figure 2a) for the case (a) and in Figure 2b) for the case

(b) versus normalized frequency Ω for different values of dimensionless parameter m.

The obtained results in Figure 2 demonstrate the surface elasticity effect. With increasing of m (respectively decreasing of bb_m), the influence of the surface effect increases, and as a result the stress concentration field is reduced strongly in the whole considered frequency interval. In the case of $m = 0$, the obtained solution recovers those for the macro-crack. To the authors knowledge there is no available result for wave scattering by nano-crack in elastic isotropic/anisotropic plane. Figure 2 reveals the sensitivity of the SIF to the frequency of the dynamic load, to specific surface properties, to material anisotropy and to the size of the blunt crack root curvature. The above analyses demonstrate that surface elasticity has a considerable impact on the near-tip fields of a mode-I crack.

5. Conclusion

It is solved elastodynamic problem for anisotropic infinite plane with an in-plane crack at macro and nano scale under plane longitudinal P or shear SV wave. The novelty of the present numerical approach lies in the effective combination of four stages: (a) 2D mechanical modeling based on classical elastodynamic theory for the bulk anisotropic solid with macro- and nano- cracks combining with the linear fracture mechanics principles; (b) introduction of non-classical boundary conditions stemming from localized constitutive equations for the material interfaces; (c) efficient numerical implementation of the non-hypersingular traction boundary integral equations based on the closed form frequency dependent fundamental solution obtained by Radon transform, and (d) low computational cost as numerical modeling is restricted to the surface of the problem in question.

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