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A GENERALIZED MOUNTAIN PASS THEOREM

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We present a *new* variational characterization of multiple critical points for *even* energy functionals functionals corresponding to non-linear Schrödinger equations of the following type:

$$\begin{cases} -\Delta u + V(x)u - q(x)|u|^\sigma u = \lambda u, & (x \in \mathbf{R}^N) \\ u \in H^1(\mathbf{R}^N) \setminus \{0\}. \end{cases}$$

We assume $N \geq 3$, $q(x) \in L^\infty(\mathbf{R}^N)$, $q(x) > 0$ a.e. with $\lim_{|x|\to\infty} q(x) = 0$ and $0 < \sigma < \frac{4}{N-2}$. Our results cover the following three cases in a *uniform* way:

1. $V(x) \equiv 0$;
2. $V(x)$ is a Coulomb potential and
3. $V(x) \in L^\infty(\mathbf{R}^N)$ with $V(x+k) \equiv V(x)$ for all $k \in \mathbf{Z}^N$.

The eigenvalue $\lambda \in \mathbf{R} \setminus \sigma(-\Delta + V)$ thereby *may or may not lie inside a spectral gap.*

Our variational characterization is "simple" and well suited for discussing *multiple bifurcation* of solutions.

The detailed presentation of all the results can be found in [3], [4] and [5].

1. The concrete setting

We consider the non-linear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u - q(x)|u|^\sigma u = \lambda u, & (x \in \mathbf{R}^N) \\ u \in H^1(\mathbf{R}^N) \setminus \{0\} \end{cases}$$

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and we assume $N \geq 3$, $q(x) \in L^\infty(\mathbf{R}^N)$, $q(x) > 0$ a.e. with $\lim_{|x|\to\infty} q(x) = 0$ and $0 < \sigma < \frac{4}{N-2}$.

We thereby consider 3 cases:

1. $V(x) \equiv 0$;
2. $V(x)$ is a Coulomb potential and
3. V is periodic: $V(x) \in L^\infty(\mathbf{R}^N)$ with $V(x+k) \equiv V(x)$ for all $k \in \mathbf{Z}^N$.

In order to formulate this problem in a functional analysis setting, we consider two self-adjoint operators A and $L : H^1(\mathbf{R}^N) \to H^1(\mathbf{R}^N)$ defined by

$$(Au|v)_{H^1(\mathbf{R}^N)} = \int_{\mathbf{R}^N} (\nabla u \cdot \nabla v + V\ u\ v)\ dx \qquad (u, v \in H^1(\mathbf{R}^N)\,)$$

and

$$(Lu|v)_{H^1(\mathbf{R}^N)} = \int_{\mathbf{R}^N} u\ v\ dx \qquad (u, v \in H^1(\mathbf{R}^N)\,)$$

In order to formulate the non-linear part of the considered Schrödinger equation, we introduce the non-linear operator

$$\Phi(u) := \frac{1}{2+\sigma} \int_{\mathbf{R}^N} q(x)|u(x)|^{2+\sigma}\ dx \qquad (u \in H^1(\mathbf{R}^N) \subset L^{2+\sigma}(\mathbf{R}^N))$$

and we remark that (among others)

- $\nabla\Phi(u) = q(x)|u|^\sigma u$,
- Φ is weakly continuous: $u_n \rightharpoonup u \Longrightarrow \Phi(u_n) \to \Phi(u)$ and
- $\nabla\Phi$ is compact: $u_n \rightharpoonup u \Longrightarrow \nabla\Phi(u_n) \to \nabla\Phi(u)$.

Remark. We can now write our Schrödinger equation as

$$\underbrace{-\Delta u + V(x)u}_{=Au} - \underbrace{q(x)|u|^\sigma u}_{=\nabla\Phi(u)} = \underbrace{\lambda u}_{=\lambda L u}\ .$$

We consider an interval $]\lambda^-, \lambda^+[$ in the resolvent set of A (with respect to L):

1. If $V \equiv 0$, we put $\lambda^- := -\infty$ and $\lambda^+ := 0$.
2. If V is a Coulomb potential, λ^- and λ^+ are two consecutive eigenvalues, but we do not exclude the case where λ^- is the largest eigenvalue and $\lambda^+ = 0$ if this potential has only a finite number of eigenvalues.
3. If V is k-periodic, we assume that $]\lambda^-, \lambda^+[$ is a spectral gap with λ^- and λ^+ belonging to the spectrum

We consider the following problem:

For every *fixed* value of $\lambda \in \left]\lambda^-, \lambda^+\right[$, find functions $u \in H^1(\mathbf{R}^N) \setminus \{0\}$ such that
$$(A - \lambda L)u - \nabla \Phi(u) = \lambda L u.$$

Replacing

$$-\Delta u + V(x)u - q(x)|u|^\sigma u = \lambda u, \quad (x \in \mathbf{R}^N)$$

by

$$-\Delta u + (V(x) - \lambda_0)u - q(x)|u|^\sigma u = (\lambda - \lambda_0)u, \quad (x \in \mathbf{R}^N),$$

we may assume that
$$0 \in \left]\lambda^-, \lambda^+\right[.$$

Remark that, if $V \equiv 0$, this shifting is of no interest! However, in what follows we will focus our notations on the case where V is a periodic potential; in the other cases, the main ideas remain the same, but formally, there are obvious changes.

2. The corresponding abstract setting

The above considered problem can be viewed as a special case of a problem in a real, separable Hilbert space $(H, \langle\cdot|\cdot\rangle)$ where we consider

- two linear, self-adjoint operators $A, L : H \to H$ and
- a non-linear operator $\Phi : H \to \mathbf{R}$ with the appropriate properties corresponding to the concrete problem.

Now let us introduce the *energy functional*

$$\mathscr{E}_\lambda(u) := \frac{1}{2}\langle (A - \lambda L)u|u\rangle - \Phi(u), \qquad (u \in H).$$

Then our problem reduces to:

For every *fixed* value of $\lambda \in \left]\lambda^-, \lambda^+\right[$, find elements $u \in H \setminus \{0\}$ such that

$$\nabla \mathscr{E}_\lambda(u) = 0,$$

i.e. find *multiple, non-trivial critical point* of the energy functional $\mathscr{E}$.

3. The behaviour of the energy functional $\mathscr{E}_\lambda$

The behaviour of the quadratic part

$$u \mapsto \frac{1}{2}\langle (A - \lambda L u)|u\rangle, \qquad (u \in H)$$

induces a decomposition of H in an orthogonal sum $Y \oplus Z$: Let us denote by $\{P_\lambda\}_{\lambda\in\mathbf{R}}$ the decomposition of unity corresponding to the self-adjoint operator A. Then we put

$$Y := \underbrace{P_0}_{=:P} X \qquad \text{an} \qquad Z := \underbrace{(\mathbb{1} - P_0)}_{=:Q} X$$

We have (see Stuart [6])

- $\langle (A-\lambda L)z|z\rangle \geq n(\lambda)\|z\|^2$ for all $z \in Z$ and for all $\lambda \in]\lambda^-, \lambda^+[$;
- $\langle (A-\lambda L)y|y\rangle \leq -m(\lambda)\|y\|^2$ for all $y \in Y$ and for all $\lambda \in]\lambda^-, \lambda^+[$.

Thereby, $n(\lambda)$ and $m(\lambda)$ are both positive constants that depend only on λ (see Figure 1).

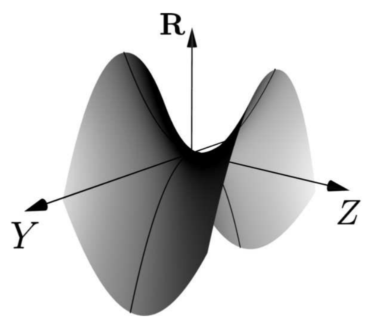

Figure 1: The behaviour of the quadratic part of the energy functional $\mathscr{E}_\lambda$

Remark.

1. If $V \equiv 0$, $\dim Y = 0$;
2. If V is a Coulomb potential, $\dim Y$ is finite;
3. If V is k-periodic, $\dim Y = \infty$.

Assuming that $PL = LP$, we can introduce the scalar product $\langle (A-\lambda L)(Q-P)u|v\rangle$ on X; remark that this scalar product induces a norm $\|u\|_\lambda$ that is equivalent to the usual norm on X. Moreover, the energy function can now be written as

$$\mathscr{E}_\lambda(u) = \frac{1}{2}\langle (A-\lambda L)(Q-P)u|v\rangle - \Phi(u) = \frac{1}{2}\left[\|Qu\|_\lambda^2 - \|Pu\|_\lambda^2\right] - \Phi(u).$$

Concerning the behaviour of the non-linear term Φ, let us mention that all functions

$$[0,+\infty[\to \mathbf{R}, \qquad t\mapsto \Phi(tu)=t^{2+\sigma}\Phi(u/\|u\|_\lambda)$$

(for a fixed $u\in H\setminus\{0\}$) are strictly and strongly non-decreasing. Overlaying this behaviour with the behaviour of the quadratic part leads us to the behaviour of the energy functional as shown in Figure 2.

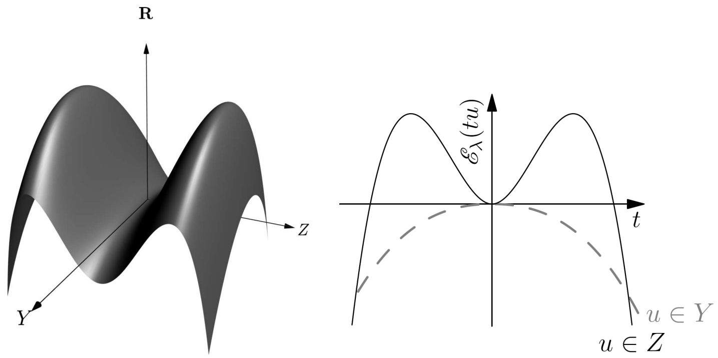

Figure 2: The behaviour of the energy functional $\mathscr{E}_\lambda$ on $Y\oplus Z$

The behaviour of this energy functional $\mathscr{E}_\lambda$ on the subspace Z

$$\mathscr{E}_\lambda(z)=\frac{1}{2}\|z\|_\lambda^2-\Phi(z) \qquad (z\in Z)$$

can be studied on different rays emerging from the origin. It turns out that there is a minimal ramp $\alpha_\lambda>0$ that this energy levels must cross on each such ray:

Proposition 1. *For each fixed* $\lambda\in\,]-\infty,0]$, *there exists* $\rho_\lambda>0$ *and* $\alpha_\lambda>0$ *such that, for all* $z\in Z$ *with* $\|z\|=\rho_\lambda$,

$$\mathscr{E}_\lambda(z)\geq\alpha_\lambda>0$$

(see Figure 3*).*

We even get a kind of localization for critical levels:

Proposition 2. *If* $u\in H$ *is a critical point of the energy functional*

$$\mathscr{E}_\lambda(u)=\frac{1}{2}\left[\|Qu\|_\lambda^2-\|Pu\|_\lambda^2\right]-\Phi(u),$$

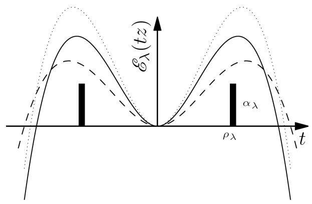

Figure 3: The positive ramp α_λ for the energy functional $\mathscr{E}_\lambda$ on Z

then

$$\|Qu\|_\lambda^2 > \|Pu\|_\lambda^2$$

and the corresponding critical level is strictly positive: $\mathscr{E}_\lambda(u) > 0$.

Thus critical values must belong to the "cone" $\{u \in H \ : \ \|Qu\|_\lambda^2 > \|Pu\|_\lambda^2\}$.

We will use variational techniques for proving the existence of critical levels. Typically, Palais-Smale conditions are used in this context in order to get some kind of compactness. Since all critical levels are strictly positive, we only need thus a property for positive levels:

Proposition 3. *For all fixed values of* $\lambda \in]\lambda^-, \lambda^+[$, *the energy functional* $\mathscr{E}_\lambda$ *satisfies the following Palais-Smale condition:*

Every sequence $\{u_n\}_{n=1}^\infty$ *in* H *with*

$$\lim_{n\to\infty} E_\lambda(u_n) = c > 0 \quad \textit{and} \quad \lim_{n\to\infty} \nabla E_\lambda(u_n) = 0$$

has a convergent subsequence.

Let us remark that this proposition heavily depends on the fact that Φ is weakly sequentially continuous.

4. How to find critical levels when the potential is zero

Let us first assume that $V \equiv 0$, so that $X = Z$. We will extend the so obtained results in the next section where we will assume that $\dim Y > 0$.

4.1. The sphere $\mathscr{B}_\lambda$

Il $\dim Y = 0$, the energy function reduces to $\mathscr{E}_\lambda(u) := \frac{1}{2}\|u\|_\lambda^2 - \Phi(u)$. In this case, all rays emerging from the origin will have to cross a ramp of strictly positive height (see Figure 4). Thus, we consider a sphere $\mathscr{B}_\lambda$ on which every such ray will assume an energy of at least $\alpha_\lambda > 0$:

$$\mathscr{B}_\lambda := \{z \in Z \ : \ \|z\|_\lambda = \rho_\lambda\}.$$

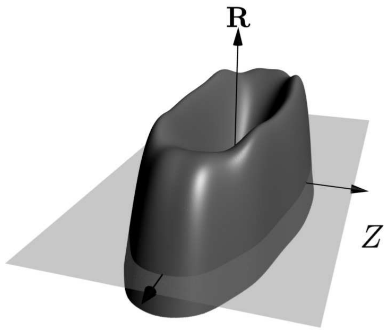

Figure 4: The behaviour of the energy functional when $\dim Y = 0$

4.2. The set $\mathscr{A}_{\lambda,m}$

Let E_m be a subspace of $X = Z$ of dimension $m \geq 1$.Then

Proposition 4. *There exists $R_{\lambda,m} > \rho_\lambda > 0$ such that*

$$z \in E_m, \|z\|_\lambda = R_{\lambda,m} \qquad \Longrightarrow \qquad \mathscr{E}_\lambda(z) \leq 0.$$

Remark that such a threshold value can be established due to the fact that we restrict $\mathscr{E}$ to a finite dimensional subspace.

We put (see Figure 5)

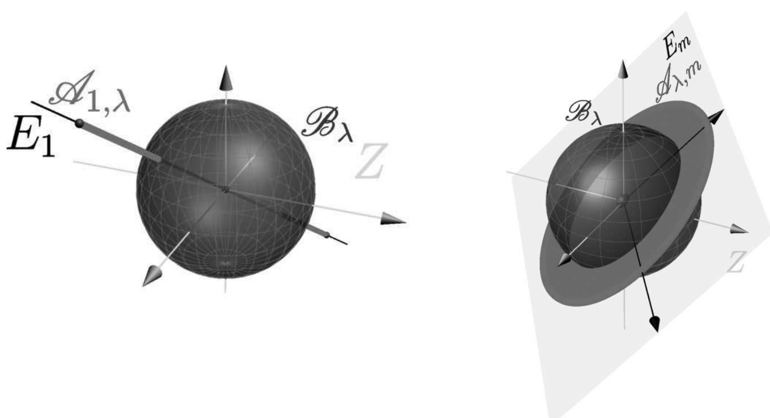

Figure 5: The set $\mathscr{A}_{\lambda,m}$ for $m = 1$ and for $m > 1$ when $\dim Y = 0$

$$\begin{aligned} \partial\mathscr{A}_{\lambda,m} &:= \{0\} \cup \{z \in E_m \ : \ \|z\|_\lambda = R_{\lambda,m}\}, \\ \mathscr{A}_{\lambda,m} &:= \{z \in E_m \ : \ 0 < \|z\|_\lambda < R_{\lambda,m}\} \quad \text{and} \\ \overline{\mathscr{A}_{\lambda,m}} &:= \{z \in E_m \ : \ \|z\|_\lambda \le R_{\lambda,m}\} \end{aligned}$$

4.3. The set $\Gamma_{\lambda,m}$

We collect in a set $\Gamma_{\lambda,m}$ all odd homeomorphisms $\gamma : \overline{\mathscr{A}_{\lambda,m}} \to \gamma(\overline{\mathscr{A}_{\lambda,m}}) \subset X$ that keep the border $\partial\mathscr{A}_{\lambda,m}$ fixed: $\forall z \in \partial\mathscr{A}_{\lambda,m}, \quad \gamma(z) = z$. We can then formulate a central linking property (see Figure 6):

Proposition 5. *We have, for all* $\gamma \in \Gamma_{\lambda,m}$,

$$\gamma(\mathscr{A}_{\lambda,m}) \cap \mathscr{B}_\lambda \neq \emptyset \qquad \textit{and thus} \quad \text{genus}(\gamma(\mathscr{A}_{\lambda,1}) \cap \mathscr{B}_\lambda) \ge m.$$

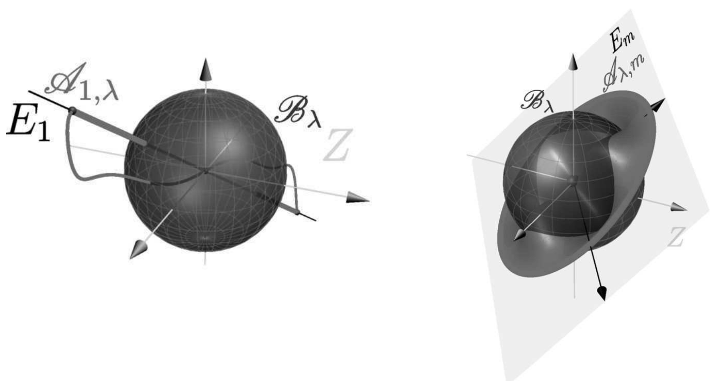

Figure 6: The linking of $\mathscr{A}_{\lambda,m}$ and $\mathscr{B}_\lambda$ when $\dim Y = 0$

4.4. The culmination point on $\gamma(\mathscr{A}_{\lambda,m})$

Clearly, for each $\gamma \in \Gamma_{\lambda,m}$,

$$\sup_{z\in\overline{\mathscr{A}_{\lambda,m}}} \mathscr{E}_\lambda(\gamma(z)) = \max_{z\in\mathscr{A}_{\lambda,m}} \mathscr{E}_\lambda(\gamma(z)) \ge \alpha_\lambda > 0.$$

Thus we may consider the value

$$d_{m,0}(\lambda) := \inf_{\gamma\in\Gamma_{\lambda,m}} \sup_{z\in\overline{\mathscr{A}_{\lambda,m}}} \mathscr{E}_\lambda(\gamma(z)) = \inf_{\gamma\in\Gamma_{\lambda,m}} \max_{z\in\mathscr{A}_{\lambda,m}} \mathscr{E}_\lambda(\gamma(z)) \ge \alpha_\lambda > 0$$

as a candidate for a critical level of $\mathscr{E}_\lambda$. That this value is in fact a critical level can be shown with the help of the classical deformation theorem. Indeed, if $d_{m,0}(\lambda)$ would not be a critical value of $\mathscr{E}_\lambda$, we can get a contradiction in the following way:

- choose some $\gamma_0 \in \Gamma_{\lambda,m}$ such that $\max_{z\in\mathscr{A}_{\lambda,m}} \mathscr{E}_\lambda(\gamma_0(z))$ is close enough to $d_{m,0}(\lambda)$.
- the classical deformation theorem gives a new element $\gamma_1 \in \Gamma_{\lambda,m}$ with

$$\max_{z\in\mathscr{A}_{\lambda,m}} \mathscr{E}_\lambda(\gamma_1(z)) < d_{m,0}(\lambda),$$

  contradicting in this way the definition of $d_{m,0}(\lambda)$.

Proposition 6. *For $m = 1, 2, 3, \ldots$, the values*

$$d_{m,0}(\lambda) := \inf_{\gamma\in\Gamma_{\lambda,m}} \sup_{z\in\overline{\mathscr{A}_{\lambda,m}}} \mathscr{E}_\lambda(\gamma(z)) = \inf_{\gamma\in\Gamma_{\lambda,m}} \max_{z\in\mathscr{A}_{\lambda,m}} \mathscr{E}_\lambda(\gamma(z)) \geq \alpha_\lambda > 0$$

are all strictly positive critical level of the energy functional $\mathscr{E}_\lambda$. Thus, for every fixed value $\lambda \in]\lambda^-, \lambda^+[$, there exists (at least) a pair of critical points $\pm u_{\lambda,m,0}$ of $\mathscr{E}_\lambda$ corresponding to the level $d_{m,0}(\lambda)$.

Remark that $d_{1,0}(\lambda) = d_{2,0}(\lambda) = d_{3,0}(\lambda) = \cdots$ is possible; hence we can say nothing about the existence of multiple critical levels. There are several way-outs:

1. One may try to show that $\lim_{m\to\infty} d_{m,0}(\lambda) = \infty$. Let us, however, remark that this approach is not suitable for studying multiple bifurcation; indeed, the usual bifurcation relies on

$$\lim_{\lambda\nearrow\lambda^+} d_{m,0}(\lambda) = 0$$

   for some m.
2. One can use the approach given by Amborsetti-Rabinowitz (see [1]). Remark, however, that this approach will no longer be possible in the case where $\dim Y = \infty$ (at least not without strong modifications).

Le us formulate our way-out.

Recall that for any odd homeomorphism $\gamma \in \Gamma_{\lambda,m}$, we have

$$\text{genus}(\gamma(\mathscr{A}_{\lambda,m}) \cap \mathscr{B}_\lambda) \geq m.$$

Thus, if $U \subset Z$ is an odd, open set with $\text{genus}(U) \leq j$, we have

$$\gamma(\mathscr{A}_{\lambda,m} \setminus U) \cap \mathscr{B}_\lambda \neq \emptyset, \qquad \text{for } j = 1, 2, \dots, m-1.$$

Thus

$$\sup_{z \in \mathscr{A}_{\lambda,m} \setminus U} \mathscr{E}_\lambda(z) = \max_{z \in \mathscr{A}_{\lambda,m} \setminus U} \geq \alpha_\lambda > 0$$

and we may consider

$$d_{m,j}(\lambda) = \inf_{\substack{\gamma \in \Gamma_{\lambda,m} \\ \text{genus}(U) \leq j}} \sup_{z \in \mathscr{A}_{\lambda,m} \setminus U} \mathscr{E}_\lambda(z) = \inf_{\substack{\gamma \in \Gamma_{\lambda,m} \\ \text{genus}(U) \leq j}} \max_{z \in \mathscr{A}_{\lambda,m} \setminus U} \mathscr{E}_\lambda(z)$$

as a candidate for a strictly positive critical level of the energy functional $\mathscr{E}_\lambda$. It turns out that this value is in fact a critical level for $\mathscr{E}_\lambda$.

Indeed, if $d_{m,j}(\lambda)$ (with $m \geq 1$ and $j \in \{0, 1, 2, \dots, m-1\}$) would not be a critical value of $\mathscr{E}_\lambda$, we get a contradiction in the following way:

- choose some $\gamma_0 \in \Gamma_{\lambda,m}$ and some open set $U \subset Z$ of genus $\leq j$ such that $\max_{z \in \mathscr{A}_{\lambda,m} \setminus U} \mathscr{E}_\lambda(\gamma_0(z))$ is close enough to $d_{m,j}(\lambda)$.
- the classical deformation theorem gives a new element $\gamma_1 \in \Gamma_{\lambda,1}$ with

$$\max_{z \in \mathscr{A}_{\lambda,m} \setminus U} \mathscr{E}_\lambda(\gamma_1(z)) < d_{m,j}(\lambda),$$

  contradicting the definition of $d_{m,j}(\lambda)$.

In order to formalize the above described process, we put

- $G_0 := \{\emptyset\}$
- $G_j = \{U \in Z \mid U = -U,\ U \text{ open},\ 0 \notin \bar{U},\ U \text{ of genus } \leq j\}$, for $j = 1, 2, 3, \dots, m-1$.

The we get the following result:

Proposition 7. *For $j = 0, 1, \dots, m-1$, the values*

$$d_{m,j}(\lambda) := \inf_{\substack{\gamma \in \Gamma_{\lambda,m} \\ U \in G_j}} \max_{u \in \mathscr{A}_{\lambda,m} \setminus U} \mathscr{E}_\lambda(\gamma(u))$$

are critical values of the energy $\mathscr{E}_\lambda$ with

$$0 < \alpha_\lambda \leq d_{m,m-1}(\lambda) \leq d_{m,m-2}(\lambda) \leq \cdots \leq d_{m,0}(\lambda).$$

Remark that we still cannot exclude the possibility that

$$d_{m,m-1}(\lambda) = d_{m,m-2}(\lambda) = \cdots = d_{m,0}(\lambda)$$

However, if for example $d_{m,1}(\lambda) = d_{m,0}(\lambda)$, there are two possible cases:

1. If $d_{m,1}(\lambda) = d_{m,0}(\lambda)$ is an accumulated point of critical levels, then there exists an infinite number of critical points with energies $\leq d_{m,0}(\lambda) + \varepsilon$, for all $\varepsilon > 0$.

2. Else, $d_{m,1}(\lambda) = d_{m,0}(\lambda)$ is an isolated critical level and we may argue as follows

   - Choose some $\gamma \in \Gamma_{\lambda.m}$ with
   $$\max_{u \in \mathscr{A}_{\lambda,m}} \mathscr{E}_\lambda(\gamma(u)) \leq d_{m,0}(\lambda) + \varepsilon.$$
   - Suppose one can choose an open set $U \in G_1$ that contains all critical points at level $d_{m,0}(\lambda)$. By a classical deformation, we get some $\gamma_1 \in \Gamma_{\lambda,m}$ with
   $$\max_{u \in \mathscr{A}_{\lambda,m} \setminus U} \mathscr{E}_\lambda(\gamma_1(u)) \leq d_{m,0}(\lambda) - \varepsilon = d_{m,1}(\lambda) - \varepsilon.$$
   Thus such a U cannot exist. This implies that there is an infinite number of critical points corresponding to this critical level.

Thus we get the following theorem:

Theorem 1. *For $j = 0, 1, \ldots, m-1$, the values*

$$d_{m,j}(\lambda) := \inf_{\substack{\gamma \in \Gamma_{\lambda,m} \\ U \in G_j}} \max_{u \in \mathscr{A}_{\lambda,m} \setminus U} \mathscr{E}_\lambda(\gamma(u))$$

are strictly positive critical values of the energy $\mathscr{E}_\lambda$

If any of these levels $d_{m,j}(\lambda)$ coincide, then there is an infinite number of critical points of $\mathscr{E}_\lambda$ that correspond to critical levels $\leq d_{m,j}(\lambda) + \varepsilon$, for all $\varepsilon > 0$.

Remark. In fact, we have the usual multiplicity: If 3 of these levels coincide without being an accumulation point of critical levels, the corresponding set of critical points is of genus ≥ 3, a.s.o.

Remark. This approach is well suited for the analysis of multiple bifurcations.

Remark. This approach can be used even when $\dim Y > 0$

5. How to find critical levels when the potential is a Coulomb potential

If V is a Coulomb potential, we have

$$0 < \dim Y < +\infty.$$

The energy functional $\mathscr{E}_\lambda$ has thus the form illustrated in Figure 2:

We adapt now the above given approach for $\dim Y = 0$ to the present case.

5.1. The set $\mathscr{B}_\lambda$

The definition of the set

$$\mathscr{B}_\lambda := \{z \in Z \ : \ \|z\|_\lambda = \rho_\lambda\}.$$

remains the same, but it is no longer a full sphere (see Figure 7).

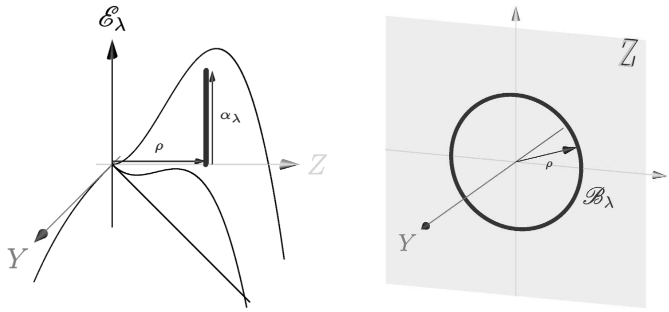

Figure 7: The set $\mathscr{B}_\lambda$ and the radial behaviour of the energy functional $\mathscr{E}_\lambda$ in the presence of a Coulomb potential

5.2. A first set of critical levels

We fix a m-dimensional subspace F_m in Z and we put $E_m := Y \oplus E_m$. Then

Proposition 8. *For each* $m \in \{1, 2, \ldots\}$ *(with* $m \leq \dim Z$*), there exists* $R_{\lambda,m} > \rho_\lambda > 0$ *such that*

$$u = y + z \in E_m = Y \oplus F_m, \|u\|_\lambda = R_{\lambda,m} \Longrightarrow \mathscr{E}_\lambda(u) \leq 0.$$

We put (see Figure 8)

$$\begin{aligned} \partial\mathscr{A}_{\lambda,m} &:= \{0\} \cup \{u \in E_m =: \ \|u\|_\lambda = R_{\lambda,m}\}, \\ \mathscr{A}_{\lambda,m} &:= \{u \in E_m =: \ 0 < \|u\|_\lambda < R_{\lambda,m}\} \quad \text{and} \\ \overline{\mathscr{A}_{\lambda,m}} &:= \{u \in E_m \ : \ \|u\|_\lambda \leq R_{\lambda,m}\} \end{aligned}$$

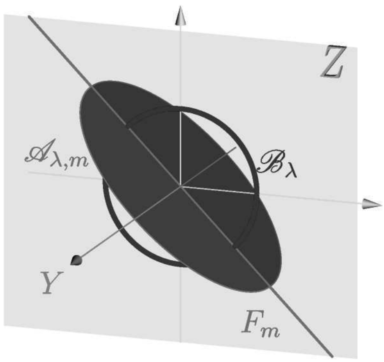

Figure 8: The linking of $\mathscr{A}_\lambda$ and $\mathscr{B}_\lambda$ in the presence of a Coulomb potential

We collect in a set $\Gamma_{\lambda,m}$ all *odd* homeomorphisms $\gamma : \overline{\mathscr{A}_{\lambda,m}} \to \gamma(\overline{\mathscr{A}_{\lambda,m}}) \subset X$ that keep the border $\partial\mathscr{A}_{\lambda,m}$ fixed: $\forall z \in \partial\mathscr{A}_{\lambda,m}, \quad \gamma(z) = z$.

We have again the following linking result:

Proposition 9. *We have, for all* $\gamma \in \Gamma_{\lambda,m}$ $(m = 1, 2, \dots, \leq \dim Z)$,

$$\text{genus}(\gamma(\mathscr{A}_{\lambda,m}) \cap \mathscr{B}_\lambda) \geq m$$

(see Figure 9).

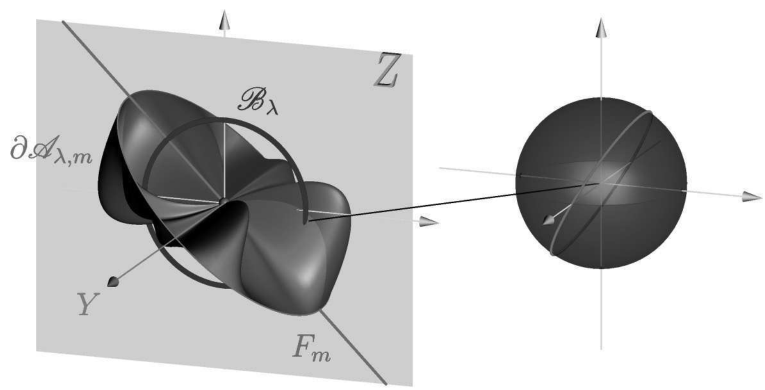

Figure 9: The preservation of the linking property in the presence of a Coulomb potential

Thus, if $U \subset H$ is an odd, open set with genus$(U) \leq j$ $(j = 1, 2, \dots, m-1)$, we have

$$\gamma(\mathscr{A}_{\lambda,m} \setminus U) \cap \mathscr{B}_\lambda \neq \emptyset, \qquad \text{for } j = 1, 2, \dots, m-1.$$

Hence we get

$$\sup_{z\in\mathscr{A}_{\lambda,m}\setminus U}\mathscr{E}_\lambda(z)=\max_{z\in\mathscr{A}_{\lambda,m}\setminus U}\geq\alpha_\lambda>0$$

and we may consider

$$d_{m,j}(\lambda)=\inf_{\substack{\gamma\in\Gamma_{\lambda,m}\\ \mathrm{genus}(U)\leq j}}\sup_{z\in\mathscr{A}_{\lambda,m}\setminus U}\mathscr{E}_\lambda(z)=\inf_{\substack{\gamma\in\Gamma_{\lambda,m}\\ \mathrm{genus}(U)\leq j}}\max_{z\in\mathscr{A}_{\lambda,m}\setminus U}\mathscr{E}_\lambda(z)$$

as a candidate for a critical level of the energy functional $\mathscr{E}_\lambda$.

As above, we put

- $G_0 := \{\emptyset\}$
- $G_j = \{U \in Z \mid U = -U,\ U \text{ open},\ 0 \notin \bar{U},\ U \text{ of genus } \leq j\}$ for $j = 1, 2, 3, \ldots, m-1$

and we get the following result:

Theorem 2. *For $j = 0, 1, \ldots, m-1$, the values*

$$d_{m,j}(\lambda) := \inf_{\substack{\gamma\in\Gamma_{\lambda,m}\\ U\in G_j}}\max_{u\in\mathscr{A}_{\lambda,m}\setminus U}\mathscr{E}_\lambda(\gamma(u))$$

are critical values of the energy $\mathscr{E}_\lambda$ with

$$0<\alpha_\lambda\leq d_{m,m-1}(\lambda)\leq d_{m,m-2}(\lambda)\leq\cdots\leq d_{m,0}(\lambda).$$

Moreover, if any of these levels $d_{m,j}(\lambda)$ coincide, then there is an infinite number of critical points of I_λ that correspond to this critical level $\leq d_{m,j}(\lambda)+\varepsilon$, for all $\varepsilon > 0$.

Remark. In fact, we have the usual multiplicity: If 3 of these levels coincide without being an accumulation point of critical levels, the corresponding set of critical points is of genus ≥ 3, a.s.o.

6. How to find critical levels when the potential is periodic

If V is a periodic potential, we have $\dim Y = +\infty$. If one tries to extend the above presented approach (Coulomb potential) to the case where $\dim Y = +\infty$, new difficulties arise:

- The sets $\overline{\mathscr{A}_{\lambda,m}}$ are no longer compact and

- we lose the proof of the fact that $\gamma(\mathscr{A}_{\lambda,m}) \cap \mathscr{B}_\lambda \neq \emptyset$, since this proof uses the fact that $\dim Y < +\infty$.

In order to preserve the compacity of the sets

$$\overline{\mathscr{A}_{\lambda,m}} := \{u \in Z \oplus F_m =: \ 0 \leq \|u\| \leq R_{\lambda,m}\},$$

Wojciech Kryszewski and Andrzej Szulkin have introduced in [2] a so-called τ-topology on $H = Y \oplus Y$. On the set $\overline{\mathscr{A}_{\lambda,m}}$ this topology reduces to

- weak convergence on Y and
- usual (strong) convergence on Z.

It turns out that the energy functional $\mathscr{E}_\lambda$ is upper-continuous on the part where it is positive, so that

$$\sup_{u\in\overline{\mathscr{A}_{\lambda,m}}} \mathscr{E}_\lambda(u) = \max_{u\in\overline{\mathscr{A}_{\lambda,m}}} \mathscr{E}_\lambda(u).$$

In order to recover the linking result, we can adapt an idea that Wojciech Kryszewski and Andrzej Szulkin (see [2]) have developed for $m = 1$. The main idea is therein is to admit in the set $\Gamma_{\lambda,m}$ only transformations γ that are locally finite dimensional. For any such γ, one can choose a finite dimensional cross-section T of $Y \oplus E_m$ in such a way that $\operatorname{genus}(\gamma(\mathscr{A}_{\lambda,m} \cap T) \cap \mathscr{B}_\lambda) \geq m$; thus we get

$$\operatorname{genus}(\gamma(\mathscr{A}_{\lambda,m}) \cap \mathscr{B}_\lambda) \geq m.$$

Again, we put

- $G_0 := \{\emptyset\}$
- $G_j = \{U \in Z \mid U = -U,\ U \text{ open},\ 0 \notin \bar{U},\ U \text{ of genus } \leq j\}$, for $j = 1, 2, 3, \ldots, m-1$

and we get the following result:

Theorem 3. *For $j = 0, 1, \ldots, m-1$, the values*

$$d_{m,j}(\lambda) := \inf_{\substack{\gamma\in\Gamma_{\lambda,m} \\ U\in G_j}} \max_{u\in\mathscr{A}_{\lambda,m}\setminus U} \mathscr{E}_\lambda(\gamma(u))$$

are critical values of the energy $\mathscr{E}_\lambda$ with

$$0 < \alpha_\lambda \leq d_{m,m-1}(\lambda) \leq d_{m,m-2}(\lambda) \leq \cdots \leq d_{m,0}(\lambda).$$

If any of these levels $d_{m,j}(\lambda)$ coincide, then there is an infinite number of critical points of $\mathscr{E}_\lambda$ that correspond to this specific level $\leq d_{m,j}(\lambda) + \varepsilon$, for all $\varepsilon > 0$.

Remark. In fact, we have once more the usual multiplicity: If 3 of these levels coincide without being an accumulation point of critical levels, the corresponding set of critical points is of genus ≥ 3, a.s.o.

Final remark. As yet mentioned above, the complete presentation of all results can be found in [3], [4] and [5].

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