

## COMPUTING AND VISUALIZING SOLUTION SETS OF INTERVAL LINEAR SYSTEMS\*

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ABSTRACT. The computation of the exact solution set of an interval linear system is a nontrivial task [2, 13]. Even in two and three dimensions a lot of work has to be done. We demonstrate two different realizations. The first approach (see [16]) is based on Java, Java3D, and the BigRational package [21]. An applet allows modifications of the matrix coefficients and/or the coefficients of the right hand side with concurrent real time visualization of the corresponding solution sets. The second approach (see [5]) uses Maple and `intpakX` [22, 8, 12] to implement routines for the computation and visualization of two and three dimensional solution sets. The regularity of the interval matrix  $A$  is verified by showing that  $\rho(|I - \text{mid}^{-1}(A) * A|) < 1$  [14]. Here,  $I$  means the identity matrix,  $\text{mid}(A)$  denotes the midpoint matrix and  $\rho$  denotes the spectral radius of a real matrix.

**1. Introduction.** We will first give a brief review of methods for solving interval linear systems of equations exactly. We will first concentrate on the

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so called Sign-Accord Algorithm due to Jiří Rohn. This algorithm allows the computation of the interval hull of the solution set. To be able to compute and to visualize the solution set itself, we use a characterization of this set due to Alefeld/Mayer, which is based on a theorem by Rohn. The solution set of a system with regular interval matrix (in general, this set is not convex) can be described using half spaces computable from the bounds of the entries of the interval matrix and the interval right hand side. If all entries of the (regular) interval matrix are point intervals the true solution set is a parallelepiped (i.e. convex).

Two packages to compute and visualize the exact solution set of a regular linear interval system are described. Both use the Alefeld/Mayer/Rohn characterization of the solution set. The first package has been realized by Gregor Paw [16]. It is implemented as a Java applet. To be able to compute the corners of the solution set exactly, the implementation uses the multiple precision package `BigRational` [21] for rational computations. The 3D graphics implementation is based on the Java3D package. The graphical user interface of the applet makes it easy to explore solution sets. E.g., zooming, shifting, and rotations may be controlled with the mouse. Several examples can be chosen from a predefined set of linear interval equations.

The second package has been implemented by Sven Braun [5] in Maple using the `intpakX` package [7, 12] to compute enclosures for determinants to show regularity and/or to compute an upper bound of the spectral radius of  $|I - R * A|$ . Here,  $R$  denotes an approximate inverse to the midpoint matrix  $\text{mid}(A)$ . A bound smaller than 1 proves that the interval matrix  $A$  is regular. The corners of the solution set are computed exactly using Maple's rational arithmetic, and Maple's plot facilities are used to visualize the solution set. Other visualization tools may be found in [17, 18].

We denote by  $I\mathbb{R}$ ,  $I\mathbb{R}^n$ , and  $I\mathbb{R}^{m \times n}$  the sets of real intervals, of interval vectors with  $n$  components, and interval matrices with  $m$  rows and  $n$  columns, respectively.

**2. Characterizations of the solution set.** To solve the interval linear system of equations

$$Ax = b ,$$

with  $A = [\underline{A}, \overline{A}] \in I\mathbb{R}^{n \times n}$  being an interval matrix and  $b = [\underline{b}, \overline{b}] \in I\mathbb{R}^n$  being an interval vector, means to compute the solution set

$$\Sigma(A, b) := \{x \in \mathbb{R}^n \mid \dot{A}x = \dot{b} \text{ for some real } \dot{A} \in A, \dot{b} \in b\}.$$

For regular interval matrices  $A$  it holds

$$\Sigma(A, b) = \{\dot{A}^{-1}\dot{b} \in \mathbb{R}^n \mid \dot{A} \in A, \dot{b} \in b\}.$$

If  $A$  is not regular, the solution set is unbounded.

The computation of  $\Sigma(A, b)$  is very costly. Therefore, in the field of self-verifying numerical methods often an interval vector including the true solution set is computed. The best possible one is the interval hull  $\text{ihull}(\Sigma) := \bigcap_{X \in I\mathbb{R}^n, X \supseteq \Sigma} X$ .

But computing  $\text{ihull}(\Sigma)$  is computationally still very costly [13]. In an interval setting the typical task is to compute a more or less sharp interval vector  $z \in I\mathbb{R}^n$  containing  $\text{ihull}(\Sigma)$ . Frequently used methods are

- Interval Gaussian Elimination (using preconditioning),
- Rump/Krawczyk iteration  $\dot{R}b + (I - \dot{R}A)z^k \subset \text{int}(z^k) \Rightarrow z^k \supset \text{ihull}(\Sigma)$ ,
- Hansen-Sengupta algorithm (based on the Gauss-Seidel method).

Before we describe the computation of the solution set and/or its interval hull, we list some properties of  $\Sigma$ .

A first characterization of the solution set is due to Oettli and Prager

**Theorem 1** (Oettli and Prager [15], 1964).

$$\Sigma(A, b) = \{x \in \mathbb{R}^n \mid |m(A)x - m(b)| \leq r(A)|x| + r(b)\}.$$

$m(A) \in \mathbb{R}^{n \times n}$  denotes the midpoint matrix,  $m(b)$  is the midpoint of the interval vector  $b$ ,  $r(A)$  is the radius matrix, and  $|A| \in \mathbb{R}^{n \times n}$  is the nonnegative matrix of absolute values.

The following characterization of the solution set is due to Beack.

**Theorem 2** (Beack [4], 1972).

$$x \in \Sigma(A, b) \iff Ax \cap b \neq \emptyset.$$

$\Sigma(A, b)$  may be computed using the corner solutions  $\Sigma_{\text{cor}}(A, b)$  consisting of the vectors  $\dot{A}^{-1}\dot{b}$ , where all elements in the point matrix  $\dot{A}$  and point vector  $\dot{b}$  are chosen as one of the end points of the corresponding interval element in  $A$  and  $b$ , resp. The number of corner solutions is  $2^{n^2+n}$ .

Let  $\text{conv}()$  denote the convex hull of a bounded set. Then it holds:

**Theorem 3** (Beck [4], 1972).

$$\text{conv } \Sigma(A, b) = \text{conv } \Sigma_{\text{cor}}(A, b).$$

Thus,  $\text{conv } \Sigma(A, b)$  can be found via the computation of  $\Sigma_{\text{cor}}(A, b)$ . Each element of  $\Sigma_{\text{cor}}(A, b)$  is found by solving a real linear system of order  $n$ . The prohibitive complexity of this approach is  $O(n^3 2^{n^2+n})$  operations to compute  $\Sigma(A, b)$ . For  $n=5$  about  $1.34 \cdot 10^{11}$  operations are required!

The set of extreme solutions  $\Sigma_{\text{extr}}$  is defined as follows.

$$\Sigma_{\text{extr}}(A, b) := \{x \in \mathbb{R}^n \mid |m(A)x - m(b)| = r(A)|x| + r(b)\} \subseteq \Sigma_{\text{cor}}.$$

Using this definition Rohn showed in 1989 the following

**Theorem 4** (Rohn [19], 1989).

$$\text{conv } \Sigma(A, b) = \text{conv } \Sigma_{\text{cor}}(A, b) = \text{conv } \Sigma_{\text{extr}}(A, b).$$

It holds:  $\Sigma_{\text{extr}}(A, b)$  contains only  $2^n$  elements. But to compute an element of  $\Sigma_{\text{extr}}(A, b)$  now a **non**-linear system of equations has to be solved.

Rohn's Sign-Accord Algorithm [19] is probably the best method known to compute  $\Sigma_{\text{extr}}(A, b)$ . To be able to formulate this algorithm, let us introduce some notations. The set of sign vectors is denoted by

$$S^n := \{y \in \mathbb{R}^n \mid y_i \in \{-1, 1\}, i = 1(1)n\},$$

and the diagonal matrix corresponding to  $y \in S^n$  is denoted by

$$D_y := \text{diag}(y_1, \dots, y_n).$$

Using these notations we can formulate the following characterization:

**Theorem 5** (Rohn [19], 1989).

$$\Sigma_{\text{extr}}(A, b) = \{x \in \mathbb{R}^n \mid \exists y \in S^n \text{ s.t. } m(A)x - m(b) = D_y(r(A)|x| + r(b))\}.$$

To compute an element of  $\Sigma_{\text{extr}}(A, b)$ , a **non**-linear system of equations has to be solved.

For sign vectors  $y, z \in S^n$  we define the point matrix

$$A_{yz} := m(A) - D_y r(A) D_z, \text{ and the point vector}$$

$$b_y := m(b) + D_y r(b).$$

After performing the following computations

$X := \emptyset$

for each  $y \in S^n$  do

    find the corresponding extreme solution  $x_y$

$X := X \cup \{x_y\}$

it holds  $\text{conv}(X) = \text{conv} \Sigma(A, b)$  and therefore  $\text{ihull}(X) = \text{ihull} \Sigma(A, b)$ . Note that it does *not* hold that  $\text{conv}(X) = \Sigma(A, b)$ .

In this computation the extreme solution is found by the Sign-Accord Algorithm due to Rohn [19]. This algorithm works as follows (see also [6]):

    given the sign vector  $y \in S^n$ .

    choose initial  $z = \text{sign}(m^{-1}(A)b_y) \in S^n$ ; terminate:= false;

    repeat

        solve linear system  $A_{yz}x = b_y$

        if all  $z_i x_i \geq 0$  then  $x_y := x$ ; terminate:= true

        else

$k :=$  smallest  $j$  with  $z_j x_j < 0$

$z_k := -z_k$

        end

    until terminate

The stopping criterion is that the sign of  $x$  is in accordance with the sign vector  $z$ . Rohn showed in 1989 that the algorithm stops after at most  $2^n$  iteration steps. The complexity is between  $O(n^3 2^n)$  and  $O(n^3 2^{2n})$ .

Let us now discuss the Alefeld/Mayer/Rohn characterization of the solution set of an interval system of linear equations. Let  $O_k$  denote a closed orthant of  $\mathbb{R}^n$  (there are  $2^n$  orthants).

**Theorem 6** (Alefeld/Mayer [2], 1995).

- $\Sigma(A, b)$  is not convex.
- If  $\Sigma(A, b) \cap O_k \neq \emptyset$ , it is convex, compact, connected, and a polytope.
- $\Sigma(A, b) = \cup_{k=1}^{2^n} (\Sigma(A, b) \cap O_k)$ .

**2.1. How to compute  $\Sigma(A, b) \cap O_k$ ?** A fixed orthant  $O$  is given by the sign vector  $s = (s_i) \in S^n$  (i.e.  $s_i \in \{-1, +1\}$  for  $i = 1(1)n$ ) corresponding to the signs of the components of an interior point of  $O$ . Hence, if  $O$  denotes some orthant  $O$  fixed by the signs  $s_1, s_2, \dots, s_n$ , then  $x = (x_i) \in O$  fulfills

$$x_i \begin{cases} \geq 0 & \text{if } s_i = +1, \\ \leq 0 & \text{if } s_i = -1. \end{cases}$$

For  $i, j = 1(1)n$ , let

$$c_{ij} := \begin{cases} \underline{a}_{ij} & \text{if } s_j = +1, \\ \bar{a}_{ij} & \text{if } s_j = -1 \end{cases}, \quad d_{ij} := \begin{cases} \bar{a}_{ij} & \text{if } s_j = +1, \\ \underline{a}_{ij} & \text{if } s_j = -1. \end{cases}$$

Denote by  $\underline{H}_i, \bar{H}_i, i = 1(1)n$ , the half spaces

$$\underline{H}_i := \left\{ y \in \mathbb{R}^n \mid \sum_{j=1}^n c_{ij} y_j \leq \bar{b}_i \right\},$$

$$\bar{H}_i := \left\{ y \in \mathbb{R}^n \mid \sum_{j=1}^n d_{ij} y_j \geq \underline{b}_i \right\}.$$

Then

**Theorem 7** (Rohn, 1989).

$$\Sigma(A, b) \cap O = \bigcap_{i=1}^n (\underline{H}_i \cap \bar{H}_i) \cap O.$$

Note that  $\underline{H}_i, \bar{H}_i$  depend on the choice of the orthant  $O$  ( $2^n$  possibilities). Denote the half spaces  $\underline{H}_i, \bar{H}_i$  corresponding to orthant  $O_k$  by  $\underline{H}_i^k, \bar{H}_i^k, k = 1(1)2^n$ . Then we have the following representation of the exact solution set of the interval system of linear equations

**Theorem 8** (Alefeld/Mayer/Rohn, [2]).

$$\Sigma(A, b) = \bigcup_{k=1}^{2^n} \underbrace{\left( \bigcap_{i=1}^n (\underline{H}_i^k \cap \bar{H}_i^k) \cap O_k \right)}_{\emptyset \text{ or a convex polytope}}.$$

To find the corners of the sets  $\Sigma(A, b) \cap O_k$ , the linear inequalities are transformed into systems of linear equations augmented by  $n$  further linear equations describing the actual orthant  $O_k$ . So for each orthant we get  $\binom{3n}{n}$  equations. To find all corners of  $\Sigma(A, b)$  we have to solve  $2^n \binom{3n}{n}$  linear  $n \times n$  point systems. Thus, for  $n = 2$  we have to solve  $60 \quad 2 \times 2$  and for  $n = 3$  we have to solve  $672 \quad 3 \times 3$  systems of linear equations.

**Theorem 9.** *The solution  $x$  to a (regular) point system is a corner of the convex set  $\Sigma(A, b) \cap O_k$  if and only if  $x$  simultaneously fulfills all inequalities corresponding to the orthant  $O_k$ .*

**2.2. How to plot  $\Sigma(A, b)$ ?** We are now ready to outline the method used in the packages realized by Braun and Paw to compute the exact solution set of a linear interval system in 2 or 3 dimensions:

a) Check whether  $A \in I\mathbb{R}^n$  is regular:

- $0 \notin \det A$  or
- $\rho(|I - \dot{R}A|) < 1$  ( $\dot{R} \approx (\text{mid}(A))^{-1}$ )

Give a warning if this check fails (note that it is still possible that the interval matrix of the system is regular).

- b) Compute all vertices of the sets  $\Sigma(A, b) \cap O_k$ . This results in unordered sets of points in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , respectively.
- c) To plot the polytope in each orthant the vertices of each face of the polytope have to be found and sorted appropriately.

For details see the source codes of the packages [5, 16], which are available from the author.

**3. Examples.** In this section we show some screenshots made with the Java applet written by Gregor Paw [16]. The first screenshot (see Figure 1) shows the graphical user interface of the applet. In the left part of the figure the solution set of the classical Barth/Nuding [3] example  $Ax = b$  with

$$(1) \quad A := \begin{pmatrix} [2, 4] & [-2, 0] \\ [-1, 0] & [2, 4] \end{pmatrix}, \quad b := \begin{pmatrix} [1, 2] \\ [-2, 2] \end{pmatrix}$$

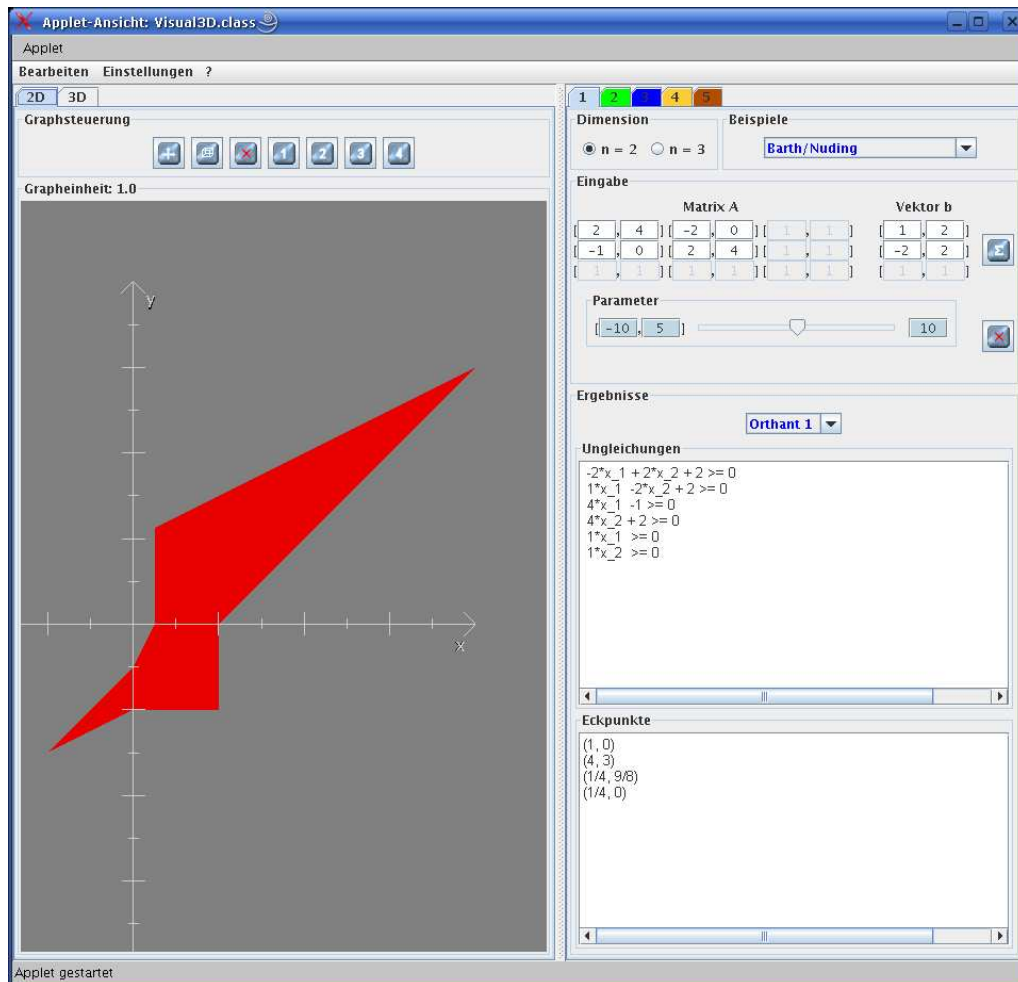


Fig. 1. Example of Barth/Nuding



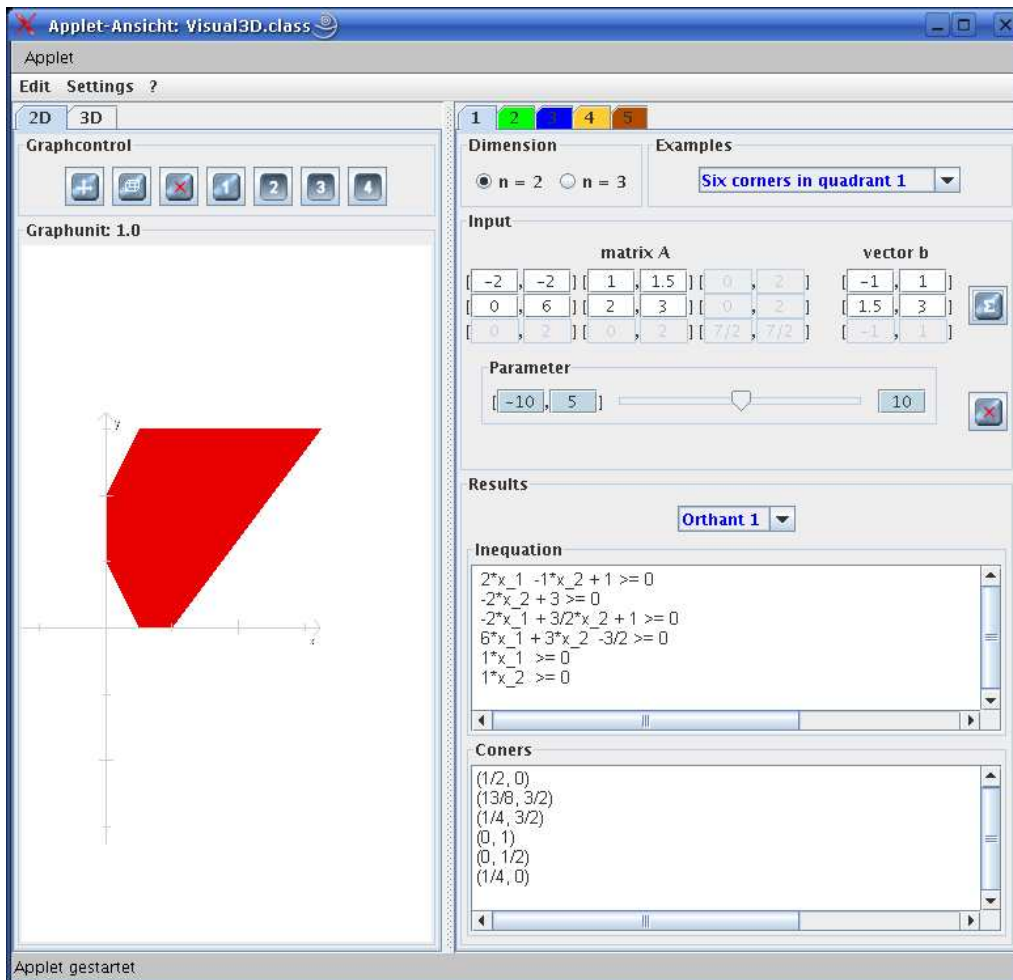


Fig. 2. General shape of the solution set in one quadrant

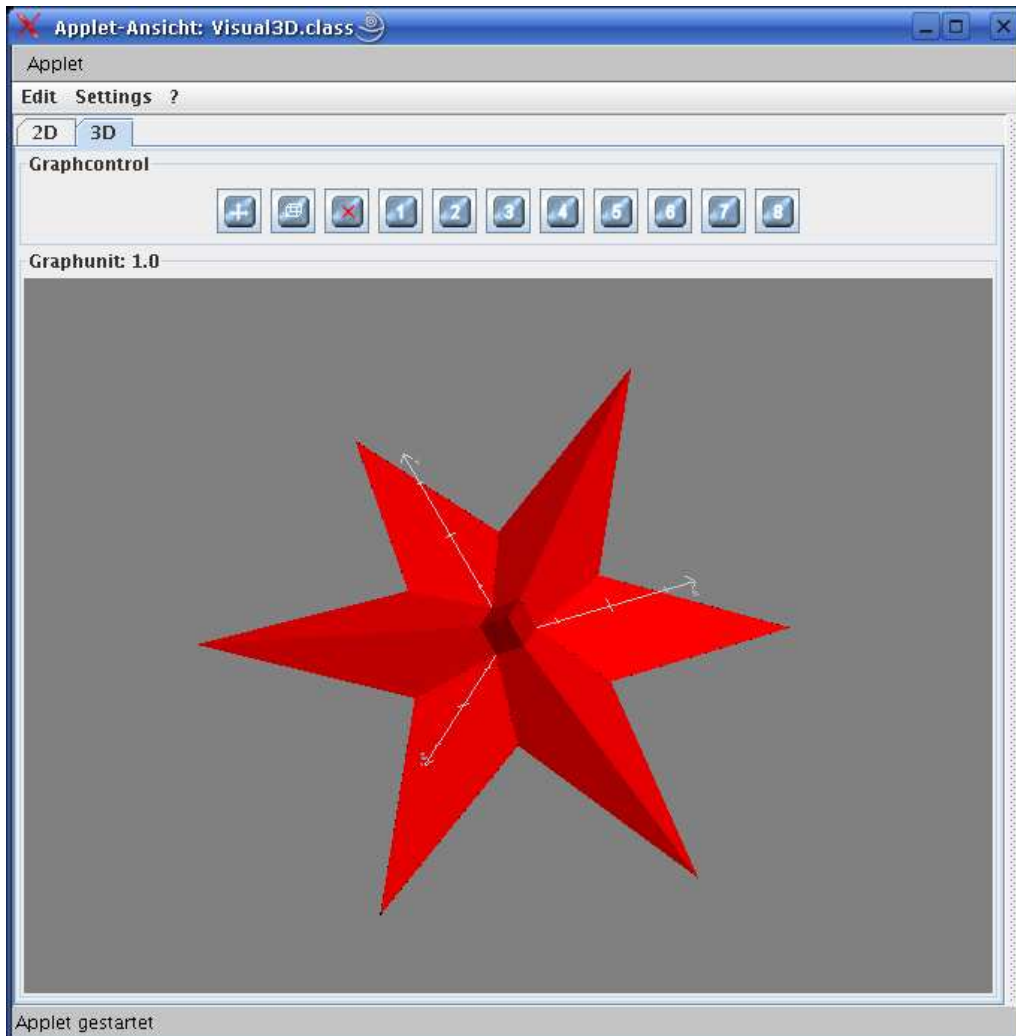


Fig. 3. Neumaier's example

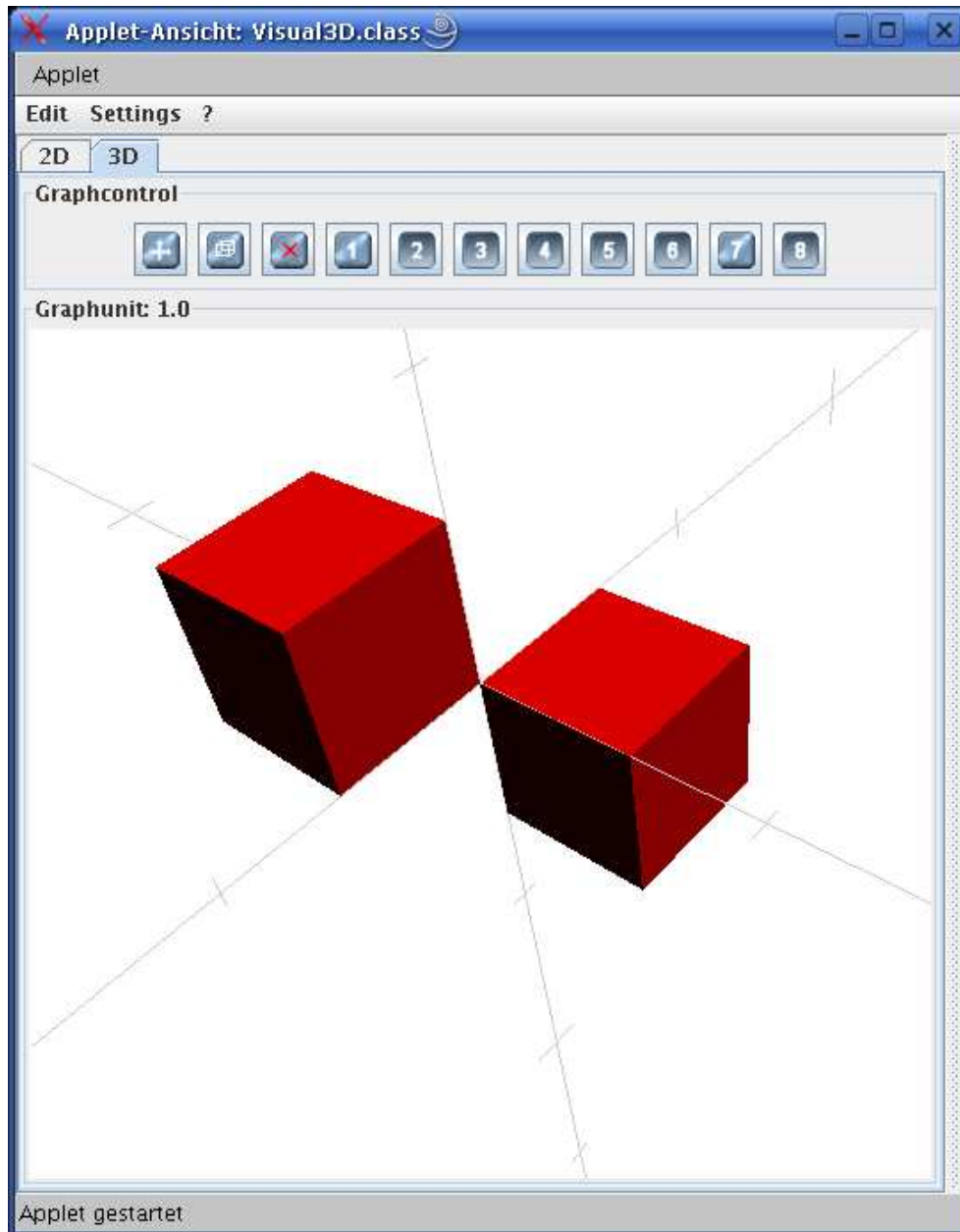


Fig. 4. Neumaier's example restricted to orthants 1 and 7

is displayed. The interval matrix and the right hand side are stored as a predefined example. They are displayed in the `matrix A` and `vector b` field, respectively. Under the heading `Inequation` we see the six active inequalities describing the boundaries of the half spaces defining that part of the solution set belonging to orthant 1. Under the heading `Corners` we see the exact coordinates of all corners of the solution set in orthant 1.

The shape of the intersection of the solution set with the first orthant shown in Figure 2 is the most general one in 2D. More than six different line segments (hexagon) are not possible. The interval matrix  $A$  and the interval right hand side  $b$  of the corresponding system are

$$(2) \quad A := \begin{pmatrix} [-2, -2] & [1, 3/2] \\ [0, 6] & [2, 3] \end{pmatrix}, \quad b := \begin{pmatrix} [-1, 1] \\ [3/2, 3] \end{pmatrix}.$$

If no points of the axes belong to such a 2D solution the boundary is given by at most four different straight line segments and by at most 9 different facets in 3D.

Figure 3 shows the 3D solution set of  $Ax = b$  with

$$(3) \quad A := \begin{pmatrix} 3.5 & [0, 2] & [0, 2] \\ [0, 2] & 3.5 & [0, 2] \\ [0, 2] & [0, 2] & 3.5 \end{pmatrix}, \quad b := \begin{pmatrix} [-1, -1] \\ [-1, -1] \\ [-1, -1] \end{pmatrix}.$$

This example is taken from the book by Neumaier [14]. A corresponding graphics is shown on the cover of the book cited.

In Figure 4 only parts of the solution set of Neumaier's example (3) are displayed. The `Graphcontrol` field of the applet has been used to activate only that parts of the solution set lying in orthant 1 and in orthant 7. The remaining parts of the solution set are hidden. The two cubes have only the point  $(0,0,0)$  in common.

**4. Conclusion.** Both packages are very well suited for displaying and manipulating solution sets of 2 and 3 dimensional interval systems of linear equations. The user may explore the variations of the shape of the solution set by varying e.g. the interval coefficients of the matrix and/or the right-hand side. Both packages also deliver the exact coordinates of all corner points of the exact solution set. To run the package written by Sven Braun, one needs a valid Maple licence. To get the source code of the packages, please send an email to [kraemer@math.uni-wuppertal.de](mailto:kraemer@math.uni-wuppertal.de)

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