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ON SOLITON EQUATIONS IN CLASSICAL DIFFERENTIAL GEOMETRY

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In this survey report we shall briefly sketch certain problems from the classical differential geometry of curves and surfaces that lead to nonlinear partial differential equations known from soliton theory. Thus an alternative viewpoint on these completely integrable equations will be presented.

1. Introduction

It is rather well-known that the modern theory of integrable systems started with the discovery of inverse scattering method for KdV equation in the late 1960's [1]. It is less known, however, that many basic concepts and results in the theory of solitons have their true origins in the classical differential geometry of curves and surfaces. This remarkable fact motivates some to speak of "pre-history of soliton" [10].

It all began with French engineer and differential geometer Edmond Bour [11]. In his studies of surfaces of constant negative Gauss curvature (pseudo-spheres) Bour showed that the Gauss-Mainardi-Codazzi equations for a pseudo-spherical surface can be reduced to sine-Gordon equation

$$\omega_{uv} = \frac{\sin\omega}{\rho^2} \qquad \rho > 0$$

for the angle ω between the asymptotic lines of the surface. Further significant contributions to the field that shaped its classical period were made by Bäcklund, Bianchi and Darboux [9, 11].

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This report is aimed at providing a short survey on ideas and results illustrating how certain 2-dimensional nonlinear partial differential equations (PDEs) integrable through the inverse scattering method naturally arise in geometric context. This way we will give an alternative viewpoint on these completely integrable PDEs.

Report is organized as follows. Next section contains our main considerations and it is divided into two subsections demonstrating two different geometrical settings that give rise to nonlinear PDEs. In the first subsection we shall show how the analytical description of curves moving in real Euclidean space naturally produces evolution PDEs. We shall consider in detail the motion of inextensible curves of constant torsion which turns out to be connected to Dym equation. Other important examples are mKdV and NLS equations. The latter describes the binormal motion of inextensible curves.

Subsection 2.2. is dedicated to PDEs arising in the theory of surfaces in Euclidean space. More specifically, we will focus on a special but interesting class of surfaces called isothermic. As it will be discussed integrable PDEs like zoomeron and Davey-Stewartson II equations are deeply related to isothermic surfaces.

Finally, Section 3. contains several concluding remarks.

2. Integrable PDEs & Classical Differential Geometry

2.1. Motion of Curves

Many soliton PDEs naturally appear in the study of curves in real Euclidean space $\mathbb{E}^3$, see [8, 11]. We shall demonstrate here the relation existing between several classical scalar soliton equations and the motion of inextensible curves in $\mathbb{E}^3$. This is why we shall remind some basic facts from the differential geometry of curves in $\mathbb{E}^3$. For a more comprehensive exposition of this subject we refer to [4, 12].

Let $\gamma \subset \mathbb{E}^3$ be a smooth regular curve from the class C^k, $k \geq 3$ parametrized by its arc-length parameter s, i.e. $\gamma : \mathbf{r} = \mathbf{r}(s)$ for $\mathbf{r}$ being position vector taken with respect to some coordinate frame. We denote by $\mathbf{t}$ the unit tangent vector of γ, while $\mathbf{n}$ stands for its (principal) normal vector and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is its binormal vector. Then the curve is characterized by its Frenet-Serret moving frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ satisfying linear system:

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}_s = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} \tag{1}$$

where κ is curvature, τ is torsion and subscript means differentiation.

Suppose now γ moves in Euclidean 3-space sweeping a surface with position vector $\mathbf{r}(s,t)$ where $t \in (\alpha, \beta)$ is evolution parameter. **Further on, we shall restrict ourselves with the important particular case of inextensible curves.**

Definition 1. *A curve is said to be inextensible if* $s_t = 0$ *during motion.*

It is clear that for inextensible curves s and t are two independent variables.

Yet another essential assumption: **the Frenet-Serret frame of γ remains orthonormal during motion**. This implies that the evolution of $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is driven by linear equations of the form:

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}_t = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} \tag{2}$$

where a, b and c are some smooth functions.

The formal integrability condition of (1) and (2) leads to the following relations

$$a_s = \kappa_t + \tau b \tag{3}$$

$$b_s = \kappa c - \tau a \tag{4}$$

$$c_s = \tau_t - \kappa b\,. \tag{5}$$

between the coefficients a, b and c and the characteristics of curve.

On the other hand, a, b and c can be expressed through the components of velocity field $\mathbf{v}(s,t) := \mathbf{r}_t(s,t)$ in a very convenient way. Indeed, from the compatibility condition $\mathbf{r}_{st} = \mathbf{r}_{ts}$ one easily derives

$$u_s - \kappa v = 0 \tag{6}$$

$$v_s + \kappa u - \tau w = a \tag{7}$$

$$w_s + \tau v = b. \tag{8}$$

Above we have used the representation $\mathbf{v} = u\mathbf{t} + v\mathbf{n} + w\mathbf{b}$ of the velocity field.

An important special case of motion is when $\mathbf{v} = w\mathbf{b}$. The requirement of inextensibility of γ is crucial due to the following statement:

Theorem 1. *A pure binormal motion is possible for inextensible curves only.*

In the case of binormal motion of curves (7) and (8) immediately show that

$$a = -\tau w \qquad b = w_s \tag{9}$$

while system (3)–(5) is now reduced to

$$\kappa_t + 2w_s\tau + w\tau_s = 0 \tag{10}$$

$$\tau_t = \left(\frac{w_{ss} - \tau^2 w}{\kappa}\right)_s + \kappa w_s \tag{11}$$

For curves of constant curvature equation (10) immediately gives $w = \tau^{-1/2}$ and equation (11) turns into

$$\tau_t = \frac{1}{\kappa}\left[\left(\tau^{-1/2}\right)_{ss} - \tau^{3/2} + \kappa^2\tau^{-1/2}\right]_s \tag{12}$$

known as extended Dym equation.

From the extended Dym equation one can obtain usual Dym equation [11]

$$\tau_t = \left(\tau^{-1/2}\right)_{sss}. \tag{13}$$

Equation (13) is remarkable for it is completely integrable but does not have the Painlevé property.

Apart of the Frenet-Serret moving frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ one can employ so-called natural Frenet frame $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$ for $\mathbf{n}_1$ and $\mathbf{n}_2$ being relatively parallel normal vectors of unit length [2]. For a regular inextensible curve of the class C^3 parametrized by its arc-length parameter the natural Frenet frame is governed by equations:

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n}_1 \\ \mathbf{n}_2 \end{pmatrix}_s = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n}_1 \\ \mathbf{n}_2 \end{pmatrix}. \tag{14}$$

Above k_1 and k_2 play the role of curvatures connected to curvature κ and torsion τ through relations

$$\begin{aligned} k_1 &= \kappa\cos\theta \qquad k_2 = \kappa\sin\theta \qquad \Leftrightarrow \\ \kappa &= \sqrt{k_1^2 + k_2^2} \qquad \tau = \theta_s \quad \theta = \sphericalangle(\mathbf{n}_1, \mathbf{n}). \end{aligned}$$

Following ideas from [6] we shall show how one can easily obtain mKdV and NLS equations using the natural Frenet basis. Consider a curve γ moving into $\mathbb{E}^3$

in such a way that the natural Frenet frame remains orthonormal during motion, i.e. we have:

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n}_1 \\ \mathbf{n}_2 \end{pmatrix}_t = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n}_1 \\ \mathbf{n}_2 \end{pmatrix} \tag{15}$$

for some coefficients a_1, a_2 and a_3 being smooth functions of s and t. Then the integrability condition of (14) and (15) leads to equations

$$a_{1,s} = k_{1,t} - k_2 a_3, \tag{16}$$

$$a_{2,s} = k_{2,t} + k_1 a_3, \tag{17}$$

$$a_{3,s} = a_1 k_2 - a_2 k_1 \tag{18}$$

Like before there exists a simple connection between the components of the velocity $\mathbf{v} = \mathbf{r}_t = u\mathbf{t} + v_1\mathbf{n}_1 + v_2\mathbf{n}_2$ and the coefficients a_1 and a_2 as given below:

$$u_s = v_1 k_1 + v_2 k_2, \tag{19}$$

$$a_\sigma = v_{\sigma,s} + u k_\sigma \quad \sigma = 1, 2 \tag{20}$$

Assume now $v_1 = k_{1,s}$ and $v_2 = k_{2,s}$. From (19) and (20) we immediately get

$$u = \frac{k_1^2 + k_2^2}{2} \qquad a_\sigma = k_{\sigma,ss} + \frac{k_1^2 + k_2^2}{2} k_\sigma$$

and equations (16), (17) and (18) give rise to the following coupled mKdV equations

$$k_{\sigma,t} = k_{\sigma,sss} + \frac{3(k_1^2 + k_2^2)}{2} k_{\sigma,s} \tag{21}$$

or equivalently to a single complex mKdV equation

$$U_t = U_{sss} + \frac{3|U|^2}{2} U_s \tag{22}$$

for complex curvature function $U := k_1 + \mathrm{i}k_2$.

Consider now the case of pure binormal motion, i.e. $u = 0$. From (19) we determine

$$v_1 = -k_2 \qquad v_2 = k_1 \tag{23}$$

hence (20) and (18) give

$$a_3 = -\frac{k_1^2 + k_2^2}{2}.$$

Finally (16) and (17) are reduced to focusing NLS equation

$$\mathrm{i}U_t + U_{ss} + \frac{|U|^2}{2} U = 0 \tag{24}$$

for the complex curvature function introduced before.

2.2. Isothermic Surfaces

We have already discussed in the introduction of paper how soliton theory originated in the study of surfaces of constant negative Gauss curvature. Thus the sine-Gordon equation was the first soliton equation appearing in differential geometry. A few other interesting integrable PDEs appear in the theory of isothermic surfaces. In this subsection we shall follow the book by Rogers and Schief [11] which we recommend for further reading.

Let $S : \mathbf{r} = \mathbf{r}(u, v)$, $(u, v) \in D \subset \mathbb{R}^2$ be a parametric regular smooth surface from the class C^k, $k \geq 2$ in Euclidean 3-space $\mathbb{E}^3$. Assume (u, v) provide a conformal parametrization of S so that the first fundamental form reads:

$$I(u, v) = \mathrm{e}^{2\varphi(u,v)} \left(du^2 + dv^2\right) \tag{25}$$

for φ being some smooth function. Sometimes u and v are called isothermic parameters because these are harmonic functions, i.e.

$$\Delta_g u = \Delta_g v = 0.$$

Above Δ_g is the Laplace-Beltrami operator associated with the metric g induced on the surface. Harmonic functions, in turn, can be viewed as steady solutions to heat equation whose level sets are known as isotherms in physics.

If conformal parameters form conjugate lines, i.e. the second fundamental form of S is diagonal, then we shall call such parameters isothermic. **The conjugacy requirement above is nontrivial!**

Definition 2. *A surface is called isothermic if it enjoys an isothermic parametrization.*

In accordance with our discussion above, for an isothermic surface the second fundamental form can be written down as:

$$II(u, v) = \mathrm{e}^{2\varphi(u,v)} \left(\kappa_1 du^2 + \kappa_2 dv^2\right) \tag{26}$$

where u and v are isothermic parameters and κ_1 and κ_2 are principal curvatures.

Isothermic surfaces are a large class of surfaces including surfaces of revolution, constant mean curvature surfaces and quadrics. An example of isothermic surface is Enneper surface, see Fig. 1.

Let $\{\mathbf{r}_u, \mathbf{r}_v, \mathbf{N}\}$ be the Gauss moving frame of S. Taking into account (25) and (26) the Gauss-Weingarten equations for an isothermic surface of the class

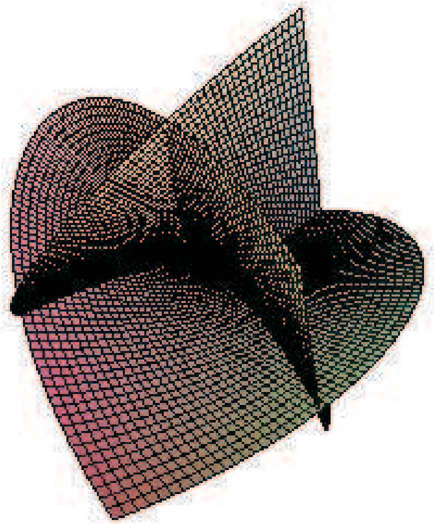

Figure 1: Enneper surface.

C^k, $k \geq 3$ can be written down in the following matrix form:

$$\left(\begin{array}{c}\mathbf{r}_u\\ \mathbf{r}_v\\ \mathbf{N}\end{array}\right)_u = \left(\begin{array}{ccc}\varphi_u & -\varphi_v & \kappa_1 \mathrm{e}^{2\varphi}\\ \varphi_v & \varphi_u & 0\\ -\kappa_1 & 0 & 0\end{array}\right)\left(\begin{array}{c}\mathbf{r}_u\\ \mathbf{r}_v\\ \mathbf{N}\end{array}\right), \tag{27}$$

$$\left(\begin{array}{c}\mathbf{r}_u\\ \mathbf{r}_v\\ \mathbf{N}\end{array}\right)_v = \left(\begin{array}{ccc}\varphi_v & \varphi_u & 0\\ -\varphi_u & \varphi_v & \kappa_2 \mathrm{e}^{2\varphi}\\ 0 & -\kappa_2 & 0\end{array}\right)\left(\begin{array}{c}\mathbf{r}_u\\ \mathbf{r}_v\\ \mathbf{N}\end{array}\right). \tag{28}$$

The integrability condition of overdetermined system (27), (28) gives rise to the Gauss-Mainardi-Codazzi equations

$$\varphi_{uu} + \varphi_{vv} + \kappa_1\kappa_2 \mathrm{e}^{2\varphi} = 0 \tag{29}$$

$$\kappa_{1,v} + (\kappa_1 - \kappa_2)\varphi_v = 0 \tag{30}$$

$$\kappa_{2,u} + (\kappa_2 - \kappa_1)\varphi_u = 0 \tag{31}$$

After introducing the new variable

$$z = -\frac{(\kappa_1 + \kappa_2)\mathrm{e}^{\varphi}}{\sqrt{2}}$$

in (29)–(31) we get classical Calapso equation

$$\Delta\left(\frac{z_{uv}}{z}\right)+(z^2)_{uv}=0 \qquad \Delta:=\partial_u^2+\partial_v^2. \tag{32}$$

The Calapso equation is completely integrable and upon formal change of variable $v\to \mathrm{i}v$ it turns into

$$\Box\left(\frac{z_{uv}}{z}\right)+(z^2)_{uv}=0, \qquad \Box:=\partial_u^2-\partial_v^2 \tag{33}$$

Equation (33) is completely integrable and originally it was obtained by Calogero and Degasperis [3]. What makes (33) remarkable is that it enjoys soliton solutions with varying velocity (boomerons and trappons) as well as with varying amplitude (zoomerons). Thus there exists a whole “zoo” of various species of solutions hence the name zoomeron equation was derived.

Let us consider a surface S immersed into $m+2$ dimensional real Euclidean space. Now the immersion of S in $\mathbb{E}^{m+2}$ is characterized by a field of m second fundamental forms $\underline{II}(u,v)$.

Definition 3. *A surface in $\mathbb{E}^{m+2}$ is isothermic if there exist parameters u and v such that its first and second fundamental forms read:*

$$\begin{aligned} I(u,v) &:= \mathrm{e}^{2\varphi(u,v)}\left(du^2+dv^2\right)\\ \underline{II}(u,v) &:= \mathrm{e}^{2\varphi(u,v)}\left(\underline{\kappa_1}du^2+\underline{\kappa_2}dv^2\right)\end{aligned}$$

where φ is a smooth function while $\underline{\kappa}_1$ and $\underline{\kappa}_2$ are principal curvature fields of S.

Like in the case of isothermic surfaces in 3-space one can derive vector Calapso equation

$$\underline{z}_{uv}=\sigma\underline{z} \qquad \Delta\sigma+\left(\underline{z}^2\right)_{uv}=0 \tag{34}$$

upon introduction of new variables

$$\underline{z}=-\frac{(\underline{\kappa_1}+\underline{\kappa_2})\mathrm{e}^{\varphi}}{\sqrt{2}} \qquad \sigma=\mathrm{e}^{-\varphi}\left(\mathrm{e}^{\varphi}\right)_{uv}.$$

For an isothermic surface in $\mathbb{E}^4$ $\underline{z}$ is a 2-vector. In terms of complex-valued function Φ such that $\underline{z}=(\mathrm{Re}\,(\Phi),\mathrm{Im}\,(\Phi))$ equation (34) is transformed into:

$$\Phi_{uv}=\sigma\Phi \qquad \Delta\sigma+\left(|\Phi|^2\right)_{uv}=0. \tag{35}$$

This equation is a stationary reduction of classical Davey-Stewartson II equation

$$\mathrm{i}\Phi_t=\Phi_{uv}-\sigma\Phi \qquad \Delta\sigma+\left(|\Phi|^2\right)_{uv}=0.$$

Thus we have just seen that Davey-Stewartson II equation is intimately connected to isothermic surfaces in $\mathbb{E}^4$.

3. Conclusions

In this short survey report we have considered two typical situations in classical differential geometry that lead to 2-dimensional integrable PDEs: the motion of inextensible curves in Euclidean space and the theory of surfaces in Euclidean space. As a matter of fact, those can be viewed two different aspects of the same problems — while the theory of surfaces provides a stationary setting, inextensible curve motion gives a dynamical one. Indeed, the integrability condition of Frenet-Serret equations (1) (resp. (14)) and (2) (resp. (15)) is an analogue of Gauss-Mainardi-Codazzi equations being the integrability condition of classical Gauss-Weingarten equations for a surface.

We have mainly been focused in this report on the simplest case of curves and surfaces in real Euclidean 3-space. As we have seen, this setting gives rise to 2-dimensional scalar PDEs. A natural question is to find similar geometric setting that produces the multicomponent counterparts of the afore-discussed scalar equations. This problem has partially been solved in the case of mKdV and NLS equations, see [6, 7]. For example, vector mKdV equation

$$\vec{k}_t = \vec{k}_{sss} + \frac{3|\vec{k}|^2}{2}\vec{k}_s$$

naturally arises when a curve moves in $\mathbb{E}^{m+1}$. Above $\vec{k} = (k_1, k_2, \ldots, k_m)$ stands for the curvature vector of the curve taken with respect to natural Frenet frame and $|\vec{k}|^2 = k_1^2 + k_2^2 + \cdots + k_m^2$.

Another appealing direction for further study is to consider motion of curves in a pseudo-Rimannian manifold with pseudo-Euclidean space being an important special case. What makes pseudo-Rimannian case harder and more interesting at the same time is the existence of isotropic vectors. This makes the standard moving frames inconvenient and requires picking up different basis to provide a more appropriate description.

At the end, it is worth mentioning that completely integrable systems may also arise in a differential geometric context rather different than we have discussed here. For instance, certain N-wave systems appear in the study of isoparametric hypersurfaces in spheres [5].

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