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APPLICATION OF DISCRETE DIVIDENDS TO AMERICAN OPTION PRICING

Dessislava Koleva-Petkova, Mariyan Milev

Dividends are a detail of financial instruments pricing which is often being oversimplified. However, companies do declare (and pay out) flows which can be significant. In this paper we briefly review some known approaches to this topic. We analyse a few known drawbacks with application to American option pricing. Due to the fact that these options rely on numerical methods for their pricing, applying discrete dividends to the chosen approach may affect the solution quality. As we will show shortly, for some methods there are flaws affecting positivity and smoothness of the numerical solution while others are too computationally heavy. We find that applying discrete dividends to an exponentially fitted scheme (the Duffy scheme) overcomes these problems and we manage to obtain a smooth and sensible solution.

1. Introduction

American options are widely employed financial instruments. Due to the contracts specifics, we only rely on numerical methods for finding their fair value. As the continuity of the underlying stock process can be affected by the presence of dividends, when solving for the option price the latter should be taken into account. In Section 2 we will review different existing approaches for these real life equity attributes. In Section 3 we place an emphasis on the application to finite difference methods. We review certain problems with Crank-Nicolson scheme and then suggest a modification of the Duffy scheme which overcomes these flaws. Section 4 gives a brief conclusion.

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2. Background in dividend application

We consider a standard *geometric Brownian motion* diffusion process with constant coefficients r and σ for the evolution of the underlying asset price S

$$dS/S = rdt + \sigma dW_t,$$

where r and σ denote, respectively, interest rate and volatility in percentages and belong to the interval $[0, 1]$. If t is the time to expiry T of the contract, $0 \le t \le T$, the price $V(S, t)$ of the option satisfies the Black-Scholes PDE ([2])

$$-\frac{\partial V}{\partial t} + r\, S\, \frac{\partial V}{\partial S} + \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 V}{\partial S^2} - r\, V = 0,$$

endowed with its initial and boundary conditions. The solution $V(S, t)$ depends on the two independent variables S and t. It should be noted that the option price can move on the positive real axis interval $[0, +\infty)$.

2.1. Applying a continuous dividend yield

Let q denote the dividend yield expressed as a continuously compounded annual percentage. Then the modified BS equation will take the form

$$-\frac{\partial V}{\partial t} + (r - q)\, S\, \frac{\partial V}{\partial S} + \frac{1}{2}\, \sigma^2 S^2\, \frac{\partial^2 V}{\partial S^2} - r\, V = 0.$$

The call and put European option prices as defined by [12] will be

$$Call = Se^{-q(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2)$$

$$Put = Ke^{-r(T-t)} N(-d_2) - Se^{-q(T-t)} N(-d_1)$$

with $d_1 = \dfrac{ln(S/K) + (r - q + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$ and $d_2 = d_1 - \sigma\sqrt{T - t}$.

Despite the simplicity of this approach the continuous dividend assumption is oversimplifying and lacks accuracy.

2.2. Applying an approximation to the spot or the strike value

Assuming a single dividend of size D paid out at time t_D, the spot value of the underlying asset price in the BS formula can be shifted with the amount of the discounted dividend, i.e. taking $\tilde{S} = S - De^{-r(T-t_D)}$. For multiple dividends one takes $\tilde{S} = S - \sum_{0<t_i\leq T} D_i e^{-r(t_i-t)}$ for constant interest rate r and with D_i, $i = 0, 1, .., T$ denoting the dividend paid out at time t_i. The problem with this approach is that generally call options are undervalued. [3] suggests a similar approach for American options which suffers from the same drawback.

As with the spot case, we can assume single or multiple dividends and modify the strike accordingly:

$$\tilde{K} = K + \sum_{0<t_i\leq t} D_i e^{r(t-t_i)}.$$

Details on these modifications can be found at [10].

A similar approach has been proposed by [4]. In a hybrid approximation both the spot and the strike are modified in the following way. We split the expected dividend flows into near payments

$$D_S = \sum_{0<t_i\leq T} \frac{T-t_i}{T-t} D_i e^{r(t_i-t)}$$

and far payments

$$D_K = \sum_{0<t_i\leq T} \frac{t_i}{T-t} D_i e^{r(t_i-t)}.$$

We then define $\bar{S} = S - D_S$ and $\bar{K} = K + D_K e^{r(T-t)}$ and finally obtain

$$Call = \bar{S}e^{-q(T-t)}N(d_1) - \bar{K}e^{-r(T-t)}N(d_2)$$

$$Put = \bar{K}e^{-r(T-t)}N(-d_2) - \bar{S}e^{-q(T-t)}N(-d_1).$$

All modifications to the spot and strike should be reflected in the values of d_1 and d_2.

A common problem for these methods is that volatility remains constant after the spot or strike shifts which leads to underpricing or overpricing the option. Also, it is hard to predict distant dividends.

2.3. Applying an adjustment to volatility based on underlying spot or strike

Beneder and Vorst in [1] define adjusted volatility based on the spot underlying taking the form:

$$\bar{\sigma}_S = \sigma\sqrt{\left(\frac{S}{S-D_1^S}\right)^2 \frac{t_1}{T} + \sum_{1<j<N}\left(\frac{S}{S-D_1^S}\right)^2 \frac{t_j - t_{j-1}}{T} + \frac{T-t_N}{T}}.$$

This expression is used together with the spot approximation already defined. The two adjustments are used directly in the BS formula.

In a similar manner one can define strike-based volatility adjustment:

$$\bar{\sigma}_K = \sigma\sqrt{\frac{t_1}{T} + \sum_{1<j<N}\left(\frac{S}{S+D_j^K}\right)^2 \frac{t_j - t_{j-1}}{T} + \left(\frac{S}{S+D_N^K}\right)^2 \frac{T-t_N}{T}}.$$

This expression can be employed together with the strike modification defined above.

Both approaches suffer from poor performance in case of multiple dividends.

2.4. Closed-form formula from Haug

E. Haug, J. Haug, A. Lewis in [11] define closed form formulae for European and American options with a single dividend. They have integral representation that possesses no explicit solution.

$$C_E(S, D, t_D) = e^{-r(t_D - t)} \int_0^\infty f(S-D, t_D)\phi(S_0, S, t_D)dS,$$

is the price of a call option where t_D denotes the time when a dividend of size D is paid out and $\phi(S_0, S, t_D)$ is the lognormal density. For European options

$$f(S-D, t_D) = C_{BS}(S-D, t_D)$$

and for American options

$$f(S-D, t_D) = \max\{(S-K)^+, C_{BS}(S-D, t_D)\}.$$

2.5. Binomial trees

The original approach has first been proposed by Cox, Ross and Rubinstein [6]. This numerical method is based on building a discretized binomial tree of possible values at each time step for the underlying asset value. At each node, the underlying path can take two different values – it can go either up or down with a certain probability. After reaching maturity, one solves the tree backwards. This approach is widely employed for path-dependent financial contracts such as American options and convertible bonds. For applying discrete dividends one needs to shift the entire tree with the dividend size. This new binomial tree is non-recombining which makes it computationally inefficient.

2.6. Finite Difference Methods

These schemes aim in approximating the solution of a PDE by solving a set of discretized equations. Partial derivatives are approximated by taking differences (forward, backward or central) for a predefined discrete time step Δt and asset step ΔS. The obtained set of equations is then solved backwards. A discussion on different methods can be found in [14].

The most famous schemes are the explicit, implicit and Crank-Nicolson. They all rely on taking central differences when discretizing the partial derivatives with respect to the underlying asset S, i.e.

$$\frac{\partial V}{\partial S} = \frac{V^i_{j+1} - V^i_{j-1}}{2h}, \qquad \frac{\partial^2 V}{\partial S^2} = \frac{V^i_{j+1} + V^i_{j-1} - 2V^i_j}{h^2}.$$

Here, V^i_j denotes the value of the option at i-th point in time and j-th point in the discretized interval $[0, S_{\max}]$. The time step has a size of k and the asset step has a size of h.

The *Crank-Nicolson method* ([7]) takes the average of the explicit and implicit schemes (which take the time derivative as a forward and backward difference, correspondingly). It is accurate at order $O(k^2, h^2)$. As the implicit method, Crank-Nicolson is unconditionally stable. However, in the presence of points of discontinuity in the initial conditions or with special boundary values, small asset steps may introduce spurious oscillations. The latter may even lead to obtaining negative option prices. A discussion on these effects can be found in [8], for example.

3. Duffy scheme with dividends – smoothness and performance

The scheme introduced by Duffy [9] is an interesting alternative to the previously described methods. It is an implicit, exponentially fitted scheme, based on a hyberbolic cotangent function. Consider the operator L defined as:

$$LV = -\frac{\partial V}{\partial t} + \mu(S,t)\frac{\partial V}{\partial S} + \sigma(S,t)\frac{\partial^2 V}{\partial S^2} + b(S,t)V,$$

where $\mu(S,t) = rS$, $\sigma(S,t) = \frac{1}{2}\sigma^2 S^2$ and $b(S,t) = -r$.

Replacing the derivatives, the fitted operator is defined by

$$L_k^h U_j^i = -\frac{U_j^{i+1} - U_j^i}{k} + \mu_j^{i+1}\frac{U_{j+1}^{i+1} - U_{j-1}^{i+1}}{2h} + \rho_j^{i+1}\frac{\delta_x^2 U_j^{i+1}}{h^2} + b_j^{i+1}U_j^{i+1},$$

where h and k are the space and time step, respectively. The factor ρ is defined as

$$\rho_j^{i+1} = \frac{\mu_j^{i+1}h}{2}\coth\frac{\mu_j^{i+1}h}{2\sigma_j^{i+1}}.$$

From here we obtain the matrix equation

$$AU^{i+1} = U^i$$

where A is a tridiagonal iterative matrix such that

$$A = [a_{i,j}] = \text{tridiag}\left\{\left(-\frac{\rho_j^i}{h^2} + \frac{\mu_j^i}{2h}\right)k; \left(\frac{2\rho_j^i}{h^2} - b_j^i + \frac{1}{k}\right)k; -\left(\frac{\rho_j^i}{h^2} + \frac{\mu_j^i}{2h}\right)k\right\}.$$

A thorough analysis of the favourable properties of this scheme can be found in [13] and [9]. It is shown that the numerical solution is always positive. Convergence to the true solution is ensured regardless of the volatility size. For $\sigma \to 0$ we have that

$$\lim_{\sigma\to 0}\rho = \lim_{\sigma\to 0}\frac{\mu h}{2\sigma} = \begin{cases} \frac{\mu h}{2}, & \text{if } \mu > 0 \\ -\frac{\mu h}{2}, & \text{if } \mu < 0 \end{cases}$$

In this border case we have a first-order implicit scheme which is stable and convergent.

The chosen space and time step sizes h and k do not affect the solution stability. It does not suffer from spurious oscillations for extreme parameter values and behaves good for special conditions. Discontinuities in the initial or boundary conditions do not affect the solution quality. These properties make the Duffy scheme a strong candidate for pricing options with more complex structures that would challenge the performance of other numerical methods.

In what follows we apply dividends to the Duffy finite difference method. We build this model extension which according to our knowledge has not yet been performed. We based our implementation on [5] employing the Gauss-Seidel method with successive over-relaxation (SOR). Applying this method on a system of equations $Ax = b$ when solving for x we build an iterative scheme based on some known initial $x^{(0)}$ and it takes the form

$$x_i^{(k+1)} = x_i^{(k)} + \frac{\omega}{a_{ii}}\Big(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i}^{N} a_{ij}x_j^{(k)}\Big) \quad i = 1, \dots, N.$$

Here ω is the over-relaxation parameter, $1 < \omega < 2$ and k is a counter. The scheme also takes a criterion for convergence ϵ as an input parameter and the iteration continues until we fall within the tolerance, i.e. until

$$\|x^{(k+1)} - x^{(k)}\| < \epsilon.$$

The key point when dealing with dividends is that we have to make a shift with the expected cash flow size at the point in time when the dividend will be paid out. Using the SOR implementation we ensure a smooth transition at the discontinuity caused by the dividend payment.

An example is run on an American put option with current underlying value of 40, strike = 40, rate = 0.1, volatility = 0.05, time to maturity of one year, and an expected dividend of size 5 in 0.6 years from evaluation date. The volatility level is considered low ($\sigma^2 \ll r$) In our example $\omega = 1.2$ and $\epsilon = 0.0001$.

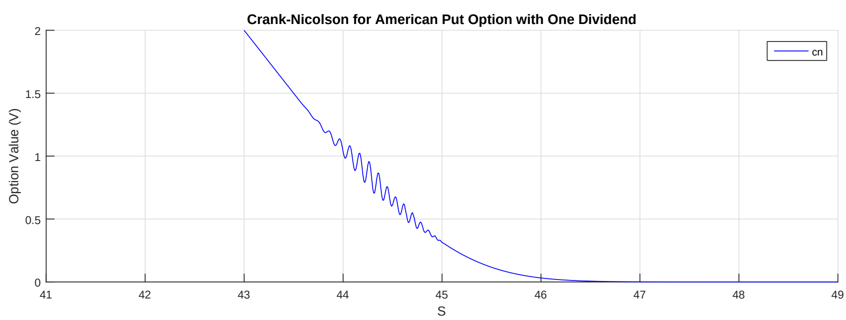

Figure 1: Crank-Nicolson scheme for an American put option

Figure 1 plots the result obtained by the Crank-Nicolson finite difference scheme using the SOR method. As one can easily see, the price function suffers from spurious oscillations. In certain extreme cases these can even result in a negative solution. Then, Figure 2 provides the outcome of employing the Duffy scheme on the very same option. The undesired price fluctuations are gone and the result is a smooth function.

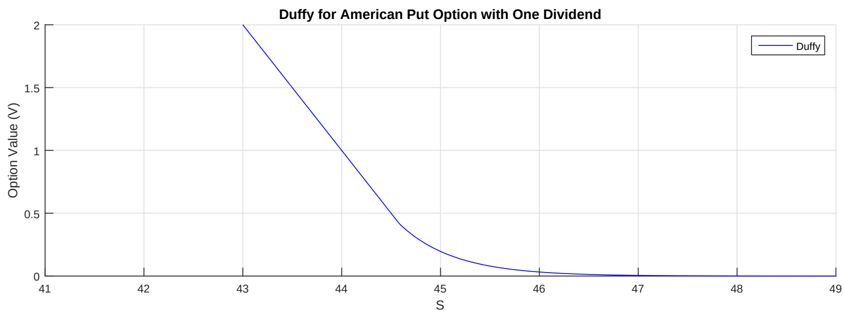

Figure 2: Duffy scheme for an American put option

4. Conclusion

This paper presented a brief outline of different approaches towards dividend effects on option pricing, together with the main problems in their application. Starting from some classical and oversimplifying methods we went through different argument adjustments and extensions of existing numerical methods. The Duffy finite difference scheme is suitable for applying discrete dividends and we showed this by comparing its solution to the one obtained by the Crank-Nicolson method. The latter suffers from spurious oscillations as opposed to the smooth solution obtained by our extension of the Duffy scheme. That is, given that we are equipped with relevant information about future dividends size and timing, we can effectively use the Duffy finite difference scheme for finding the fair value of options written on the underlying equity.

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Dessislava Koleva-Petkova
Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
Sofia, Bulgaria
e-mail: koleva_dn@yahoo.com

Mariyan Milev
Faculty of Economics
University of Food Technology
Plovdiv, Bulgaria
e-mail: marianmilev2002@gmail.com