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LIMIT THEOREMS FOR SOME CLASSES OF ALTERNATING REGENERATIVE BRANCHING PROCESSES

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In this paper we propose and study three new classes of alternating regenerative (AR) branching processes. Limiting distributions are obtained for AR Sevastyanov processes, for AR Sevastyanov processes with non-homogeneous Poisson immigration and for AR randomly indexed branching processes. All these processes are investigated applying renewal and regenerative methods developed in Mitov and Yanev [16, 17].

1. Introduction

In general the alternating regenerative branching processes (ARBP) can be interpreted as follows: the process stays at zero a random time (called "waiting period" following given distribution), after that it jumps to a positive random level (with given distribution) starting then some "branching mechanism" up to the moment when the process hits zero (it is called a "life-period"). Then the process continues in the same way, i.e. as independent copies of the first part. The positive jump at the state zero can be interpreted as an immigration of new individuals (objects, particles, cells, and so on) which become ancestors of some branching processes. In this paper we consider three new classes of ARBP based on the following three models of branching processes: Sevastyanov age-dependent processes (Sevastyanov [23], [24]), Sevastyanov processes with non-homogeneous Poisson immigration (Mitov and Yanev [20], Hyrien et al. [9]) and randomly indexed Galton-Watson processes (Mitov et al. [13]).

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Comprehensive reviews of branching processes, including their applications to biology, can be found in the following monographs: Harris [8], Sevastyanov [24], Mode [21], Athreya and Ney [3], Jagers [10], Asmussen and Hering [2], Yakovlev and Yanev [25], Kimmel and Axelrod [11], Haccou et al. [7] and Ahsanullah and Yanev [1].

As particular cases of the ARBP one obtains the branching processes with state-dependent immigration, i.e. which admit an instant immigration only in the state zero. The first model in the case of Galton-Watson process was proposed and investigated by Foster [6] and Pakes [22]. After that Yamazato [25] considered continuous time Markov case. Mitov and Yanev [14, 15] investigated Bellman-Harris processes. In all these papers the limiting results were obtained using typical "branching methods": p.g.f. and the relevant equations for them (functional equations for the Galton-Watson processes, non-linear differential equations for Markov branching processes and non-linear integral equations for the Bellman-Harris processes).

In contrast to that, the present paper develops the application of the "regenerative and renewal methods" presented in Mitov and Yanev [16, 17]. Note that these methods were applied by Li [12] for continuous-state branching processes, by Mitov and Yanev [18] for Bellman-Harris processes with infinite offspring variance and state-dependent immigration and by Yanev et al. [27, 28] for branching processes with random migration. For more details one can consider the review chapter by Mitov and Yanev [19].

The paper is organized as follows. The general definition of the alternating regenerative processes is given in Section 2 with some general conditions. In Section 3 the branching mechanism is governed by Sevastyanov branching process and limiting distributions for alternating regenerative Sevastyanov processes are presented in the critical case combining all general conditions for waiting and life periods. Section 4 deals with a complicated case of alternating regenerative Sevastyanov branching processes allowing a nonhomogeneous Poisson immigration and in the subcritical case discrete type limiting distributions are obtained. Finally in Section 5 alternating regenerative randomly indexed branching processes are introduced and limiting theorems are obtained in the critical case with finite or infinite offspring variance.

2. Alternating regenerative processes

We consider a class of non-negative processes $\mathbf{Z} = \{Z(t), t \geq 0\}$ for which $\tau = \inf\{t : Z(t) = 0\}$ is the life-period and $\mathbf{P}(\tau < \infty) = 1$. Assume also that $\zeta = \{\zeta_i\}$ are nonnegative independent identically distributed (i.i.d.) random variables (r.v.) with a distribution function (d.f.) $\Delta(x) = \mathbf{P}(\zeta_i \leq x)$. The sets ζ and $\mathbf{Z}$ are assumed independent.

Let $\mathbf{Z}_k = \{Z_k(t)\}$ be the i.i.d. copies of $\mathbf{Z} = \{Z(t)\}$ with corresponding life-periods τ_k and a d.f. $\Lambda(x) = \mathbf{P}(\tau_k \leq x)$.

We will use the sequence of the random vectors $\{(\zeta_1, \tau_1), (\zeta_2, \tau_2), \dots\}$ to define the renewal epochs $S_0 = 0, S_n = S_{n-1} + \eta_n$, where $\eta_n = \zeta_n + \tau_n$, $n = 1, 2, \dots$. Let $N(t) = \max\{n : S_n \leq t\}$ be the corresponding renewal process. Consider also the alternating renewal epochs $\{(S_n, S'_{n+1}) : S'_{n+1} = S_n + \zeta_{n+1}, n = 0, 1, 2, \dots\}$ and introduce the process $\chi(t) = t - S'_{N(t)+1}$.

Then the alternating regenerative process $\mathbf{X} = \{X(t), t \geq 0\}$ is defined as follows:

$$X(t) = Z_{N(t)+1}(\chi(t))\mathbf{1}\{\chi(t) \geq 0\}. \tag{2.1}$$

Here and later $\mathbf{1}\{A\}$ denotes the indicator of the event A.

Note that ζ is interpreted as a set of waiting periods. If $\chi(t) \geq 0$ then it is called a "spent lifetime" and if $\chi(t) < 0$ then $|\chi(t)|$ is called a "residual waiting time".

The process $X(t)$ develops as follows: $X(t) = 0$ during the waiting period $S'_1 = \zeta_1$, then $X(t) = Z_1(t - S'_1)$, $S'_1 \leq t \leq S_1 = \zeta_1 + \tau_1$ (during the life period τ_1), after that $X(t) = 0$, $S_1 \leq t < S'_2$ (i.e. during the waiting period $\zeta_2 = S'_2 - S_1$), then $X(t) = Z_2(t - S'_2)$, $S'_2 \leq t \leq S_2 = \zeta_2 + \tau_2$, and so on. In general, $X(t)$ is defined as zero during the waiting periods $[S_{n-1}, S'_n)$ and $X(t)$ coincides with the process $Z_n(t - S'_n)$ during the life periods $[S'_n, S_n], n = 1, 2, \dots$.

Further on we will consider the case when $\Lambda(x)$ and $\Delta(x)$ are non-lattice distribution functions, $\Lambda(0) = \Delta(0) = 0$ and there exists

$$\lim_{x \to \infty} \{[1 - \Delta(x)]/[1 - \Lambda(x)]\} = c, \quad 0 \leq c \leq \infty. \tag{2.2}$$

For $\Delta(x)$ we will assume one of the following basic conditions:

$$m_\Delta = \int_0^\infty x d\Delta(x) < \infty; \tag{2.3}$$

$$m_\Delta = \infty, 1 - \Delta(x) = x^{-\alpha} L_\Delta(x), 1/2 < \alpha \leq 1, \tag{2.4}$$

where $L_\Delta(x)$ is a s.v.f. as $x \to \infty$ and for each $h > 0$ fixed $\Delta(t) - \Delta(t - h) = O(1/t)$, $t \to \infty$.

Remark 2.1. *When we consider branching processes we will assume additionally that* $Z_k(0) = I_k$ *with a p.g.f.* $g(s) = \mathbf{E}\{s^{I_k}\} = \sum_{i=1}^{\infty} g_i s^i$, $|s| \leq 1$. *For* $g(s)$

we will consider the following possibilities:

$$m_I = g'(1) = \mathbf{E}\{I_k\} < \infty; \tag{2.5}$$

$$\mathbf{E}\{I_k\} = \infty, g(s) = 1 - (1-s)^{\theta} L_I(1/(1-s)), 0 < \theta < 1, \tag{2.6}$$

and $L_I(x)$ *is a s.v.f. as* $x \to \infty$.

Note that the random variables $\{I_k\}$ can be interpreted as the immigrants at the state zero after the corresponding waiting periods.

In the general case one of the following conditions holds:

$$m_\Lambda = \int_0^\infty x d\Lambda(x) < \infty; \tag{2.7}$$

$$m_\Lambda = \infty, 1 - \Lambda(x) = x^{-\kappa} L_\Lambda(x), 1/2 < \kappa \le 1, \tag{2.8}$$

$L_\Lambda(x)$ is a s.v.f. as $x \to \infty$.

The asymptotic behaviour of the processes on the corresponding cycles is described by the following condition:

$$\lim_{t\to\infty} \mathbf{P}\{Z_k(t)/V(t) \le x \mid \tau_k > t\} = \Omega(x) \tag{2.9}$$

where $V(t)$ is a positive, non-decreasing function, regularly varying at infinity with exponent $\omega \ge 0$, and $\Omega(x)$ is a proper distribution function on $\mathbf{R}^+$.

Further on we will apply the following "Basic regeneration theorem" proved in Mitov and Yanev [17]:

Theorem 2.1. *Assume* (2.1), (2.2) *and* (2.9).

1. *Let* (2.8) *hold with* $1/2 < \kappa < 1$:

(i) If (2.3) *or* (2.4) *is fulfilled and* $0 \le c < \infty$, *then for* $x \ge 0$

$$\lim_{t\to\infty} \mathbf{P}\{X(t)/V(t) \le x\} = (\mathcal{F}_1(x) + c)/(1+c), \tag{2.10}$$

$\mathcal{F}_1(x) = \pi^{-1} \sin \pi\kappa \int_0^1 \Omega(xu^{-\omega})(1-u)^{\kappa-1} u^{-\kappa} du$ *is a distribution function on* $\mathbf{R}^+$.

(ii) If (2.4) *with* $c = \infty$ *is fulfilled, then for* $x \ge 0$

$$\lim_{t\to\infty} \mathbf{P}\{X(t)/V(t) \le x \mid X(t) > 0\} = \mathcal{F}_2(x), \tag{2.11}$$

where $\mathcal{F}_2(x) = B^{-1}(1-\kappa, \alpha) \int_0^1 \Omega(xu^{-\omega})(1-u)^{\alpha-1} u^{-\kappa} du$ *is a distribution function on* $\mathbf{R}^+$.

2. *Let* (2.8) *hold with* $\kappa = 1$ *and* $\Omega(0) = 0$:

(*iii*) *If* (2.3) *or* (2.4) *is fulfilled and* $0 \le c < \infty$, *then for* $0 < x < 1$

$$\lim_{t\to\infty} \mathbf{P}\{m_\Lambda(V^{-1}(X(t)))/m_\Lambda(t) \le x\} = (x+c)/(1+c), \tag{2.12}$$

where $m_\Lambda(t) = \int_0^t [1-\Lambda(x)]dx$ *and* $V^{-1}(\cdot)$ *is the inverse function of* $V(\cdot)$.

(*iv*) *If* (2.4) *with* $c = \infty$ *is fulfilled, then for for* $0 < x < 1$

$$\lim_{t\to\infty} \mathbf{P}\{m_\Lambda(V^{-1}(X(t)))/m_\Lambda(t) \le x \mid X(t) > 0\} = x. \tag{2.13}$$

3. AR Sevastyanov processes

Let now $\mathbf{Z} = \{Z(t), t \ge 0\}$ be a Sevastyanov branching process. Recall that in this case (see Sevastyanov [23, 24]) every individual (object, particle, cell, and so on) evolves independently of the others and the joint distribution of the lifespan η and the offspring ξ is specified as

$$\mathbf{P}\{\eta \in B, \xi = k\} = \int_B p_k(u)dG(u), \tag{3.1}$$

for every Borel set $B \subset \mathbf{R}$, where $\sum_{k=0}^{\infty} p_k(u) = 1$ and $G(u) = \mathbf{P}(\eta \le u)$, $u \ge 0$.

Put $h(u;s) = \sum_{k=0}^{\infty} p_k(u)s^k$, $|s| \le 1$, for the collection of the associated probability generating functions (p.g.f.). These assumptions define a (G,h)-Sevastyanov process and $\Phi(t;s) = \mathbf{E}\{s^{Z(t)}|Z(0) = 1\}$ is the unique solution (in the class of the p.g.f.) of the equation (see Sevastyanov [23, 24])

$$\Phi(t;s) = 1 - G(t) + \int_0^t h(u;\Phi(t-u;s))dG(u), \qquad \Phi(0;s) = s. \tag{3.2}$$

Note that if $h(u;s) \equiv h(s)$ then η and ξ are independent and one obtains the Bellman-Harris branching process. If additionally $G(x) = 1 - e^{-x/\mu}, \mu > 0$, then $\mathbf{Z}$ will be a Markov branching process, but if $G(x) = \mathbf{1}_{[1,\infty)}(x)$ then $\mathbf{Z}$ will be the classical Galton-Watson process.

Introduce the moments $(k, n = 1, 2, \dots)$

$$a_k(u) = \frac{\partial^k}{\partial s^k} h(u;s)|_{s=1}, \quad a_k = \int_0^\infty a_k(u)dG(u),\ \mu_k = \int_0^\infty u^k dG(u),$$
$$M_n^{a_k} = \int_0^\infty u^n a_k(u)dG(u), M_n \equiv M_n^{a_1}.$$

Further on in this section we will consider only the **critical case** $a_1 = 1$ and will investigate the asymptotic behaviour of the process $X(t)$ defined by (2.1) where now $\{Z_k(t)\}$ are i.i.d. copies of the considered Sevastyanov model $Z(t)$.

Theorem 3.1. *Assume* (2.2) *and let* $a_3, \mu_3, M_3, M_3^{a_2}$ *and* $M_3^{a_3}$ *be finite.*

I. Let (2.5) *be fulfilled and* $0 < x < 1$.

(i) *If* (2.3) *is additionally supposed then* $c = 0$ *and*

$$\lim_{t\to\infty} \mathbf{P}\{\log X(t)/\log t \le x\} = x. \tag{3.3}$$

(ii) *If* (2.4) *is additionally supposed then* $c = \infty$ *and*

$$\lim_{t\to\infty} \mathbf{P}\{\log X(t)/\log t \le x \mid X(t) > 0\} = x. \tag{3.4}$$

II. Let (2.6) *be fulfilled with* $1/2 < \theta < 1$ *and* $A = \mu_1 a_2/2M_1^2$.

(iii) *If either* (2.3) *or* (2.4) *with* $\alpha > \theta$ *is additionally supposed then* $c = 0$ *and*

$$\lim_{t\to\infty} \mathbf{P}\{X(t)/At \le x\} = K_1(x), \tag{3.5}$$

where

$$K_1(x) = (\pi^{-1}\sin\pi\theta)\int_0^1 H(xu^{-1})(1-u)^{\theta-1}u^{-\theta}du \tag{3.6}$$

and the d.f. $H(x)$ *has a Laplace transform*

$$\varphi(\lambda) = \int_0^\infty e^{-\lambda x}dH(x) = 1 - \lambda^\theta/(1+\lambda)^\theta, Re\lambda \ge 0. \tag{3.7}$$

(iv) *If* (2.4) *with* $\alpha < \theta$ *is additionally supposed then* $c = \infty$ *and*

$$\lim_{t\to\infty} \mathbf{P}\{X(t)/Qt \le x \mid X(t) > 0\} = K_2(x), \tag{3.8}$$

where

$$K_2(x) = [1/B(1-\theta,\alpha)]\int_0^1 H(xu^{-1})(1-u)^{\alpha-1}u^{-\theta}du, \tag{3.9}$$

$B(\cdot,\cdot)$ *is the classical beta function and the d.f.* $H(x)$ *has* (3.7) *as a Laplace transform.*

(v) *If* (2.4) *with* $\alpha = \theta$ *is additionally supposed then* $c \in [0,\infty]$ *depending of the relevant s.v.f. For* $c = \infty$ *we have* (3.8) *with* (3.9) *but for* $0 \le c < \infty$ *one has*

$$\lim_{t\to\infty} \mathbf{P}\{X(t)/At \le x\} = (K_1(x)+c)/(1+c), \tag{3.10}$$

where $K_1(x)$ *is defined by* (3.6) *and* (3.7).

P r o o f. If $Z(0) = 1$ then we have (see [23] and [24])

$$Q_1(t) = \mathbf{P}\{Z(t) > 0\} = \mathbf{P}\{\tau > t\} \sim 2M_1/a_2 t, t \to \infty, \tag{3.11}$$

$$\lim_{t\to\infty} \mathbf{P}\{Z(t)/At \leq x \mid Z(t) > 0\} = 1 - e^{-x}, x \geq 0, A = \mu_1 a_2/2M_1^2. \tag{3.12}$$

In the case (2.5) one has to mention only that $g(s) \sim 1 - m_I(1-s)$ as $s \uparrow 1$ and $Q_I(t) = 1 - \Lambda(t) \sim m_I Q_1(t)$ as $t \to \infty$. Then under the condition (2.3) one can obtain from (2.2) and (3.11) that $c = 0$. Note that $V(t) = At$ and as $t \to \infty$ one has

$$m_\Lambda(t) = \int_0^t Q_I(x)dx \sim (2m_I M_1/a_2)\log t, m_\Lambda(V^{-1}(X(t))) \sim (2m_I M_1/a_2)\log X(t).$$

Therefore by (2.12) from Theorem 2.1 (*iii*) one obtains (3.3).

Assume now (2.4) and (2.5). Then one can prove from (2.2) that $c = \infty$. Hence (3.4) follows by (2.13) of Theorem 2.1 (*iv*).

In the case (2.6) instead of (3.11) one has as $t \to \infty$

$$1 - \Lambda(t) = 1 - g(1 - Q_1(t)) = Q_1^\theta(t) L_I(1/Q_1(t)). \tag{3.13}$$

Then from (2.2) and (3.13) under the conditions of the case (*iii*) one obtains $c = 0$ while in the case (*iv*) one has $c = \infty$.

Now from (2.6) and (3.12) one can prove that for $Re\lambda \geq 0$ and $t \to \infty$

$$\begin{aligned} \mathbf{E}\{e^{-\lambda Z(t)/At} \mid Z(t) > 0\} &= 1 - [1 - g(\Phi(t; e^{-\lambda/At}))]/[1 - \Lambda(t)] \\ \sim\ 1 - \{[1 - \Phi(t; e^{-\lambda/At})]/Q_1(t)\}^\theta &\to 1 - \{\lambda/(1+\lambda)\}^\theta = \varphi(\lambda). \end{aligned} \tag{3.14}$$

Therefore we are able to apply Theorem 2.1 (i) (with $\omega = 1$ and $\kappa = \theta$) and to obtain from (2.10) the relation (3.5) with (3.6) and (3.7). Similarly one can apply Theorem 2.1 (ii) (with $\omega = 1$ and $\kappa = \theta$) to obtain (3.9). The final case (v) can be proved in the same way. □

Corollary 3.1. *From* (3.7), (3.6) *and* (3.9) *applying the Tauberian theorem (see e.g.* [4, 5]*) one obtains as* $x \to \infty$ *that*

$$\begin{aligned} 1 - H(x) &\sim x^{-\theta}/\Gamma(1-\theta), \\ 1 - K_1(x) &\sim (1-\theta)x^{-\theta}/\Gamma(1-\theta), \\ 1 - K_2(x) &\sim (1-\theta)x^{-\theta}/(1-\theta+\alpha)\Gamma(1-\theta), \end{aligned}$$

i.e. for the three d.f. we have a normal domain of attraction of a stable law with parameter θ.

Remark 3.1. *The process* $X(t)$ *can be interpreted as a generalization of the processes with state-dependent immigration, i.e. with instantaneous immigration* *only in the state zero, considered by Foster* [6] *and Pakes* [22] *in the case of Galton-Watson process, by Yamazato* [26] *for Markov branching process and by Mitov and Yanev* [14, 15] *for Bellman-Harris processes. Especially, the limiting result* (3.3) *was first proved by Foster, and after that by Yamazato and by Mitov and Yanev (for the cited above processes).*

4. AR Sevastyanov processes with non-homogeneous Poisson immigration

Let us now assume that along with the (G, h)-Sevastyanov process $\mathbf{Y} = \{Y(t)\}$ there is a sequence of random vectors (U_k, ν_k), $k = 0, 1, 2, \ldots$, independent of $\mathbf{Y}$, where $0 = U_0 < U_1 < U_2 < \ldots$, are the jump points of a non-homogeneous Poisson process $\Pi(t)$ (of course independent of $\mathbf{Y}$) and $\{\nu_k\}$ are i.i.d. positive integer-valued r.v. with $b = \mathbf{E}\nu_k < \infty$.

Let $r(t)$ be the intensity of $\Pi(t)$, i.e. $\mathbf{P}\{\Pi(t) = n\} = e^{-R(t)}R^n(t)/n!$ for $n = 0, 1, 2, \ldots$ and $R(t) = \int_0^t r(x)dx$.

Suppose that at every jump point U_n, a random number ν_n of new individuals immigrate into the process $\mathbf{Y}$, and they participate in the evolution as the other individuals. Let $\{Y^{\nu_i}(t)\}_{i=1}^{\infty}$ be independent and identically distributed processes which have the same branching mechanism as $\mathbf{Y}$ but they started with the random number of ancestors ν_i, i.e. $Y^{\nu_i}(0) = \nu_i$, $i = 1, 2, \ldots$. Then the Sevastyanov process with non-homogeneous Poisson immigration $\mathbf{Z} = \{Z(t)\}$ can be defined as follows (see Mitov and Yanev [20]):

$$Z(t) = \sum_{i=1}^{\Pi(t)} Y^{\nu_i}(t - U_i) \text{ if } \Pi(t) > 0; Z(t) = 0 \text{ if } \Pi(t) = 0. \tag{4.1}$$

Note that the Sevastyanov process $Y^{\nu_i}(t - U_i)$ started with ν_i ancestors but at the random moment U_i.

We will investigate the asymptotic behaviour of the process $X(t)$ defined by (2.1) where now $\{Z_k(t)\}$ are i.i.d. copies of the proposed Sevastyanov model (4.1) with a non-homogeneous Poisson immigration.

Consider first the **subcritical case** $a_1 < 1$ and assume additionally that there exists the Malthusian parameter $\beta < 0 : \int_0^{\infty} e^{-\beta u} a_1(u)dG(u) = 1.$

Introduce the moments $(k, n = 1, 2, \dots)$

$$a_k(\beta) = \int_0^\infty e^{-\beta u} a_k(u) dG(u), \quad \mu_k(\beta) = \int_0^\infty e^{-\beta u} u^k dG(u),$$
$$M_n^{a_k}(\beta) = \int_0^\infty e^{-\beta u} u^n a_k(u) dG(u).$$

Theorem 4.1. *Assume* (2.2), (2.6) *and* $r(t) = \overline{r}(t+1)^{-\rho}$, $1/2\theta < \rho < 1/\theta$, $\overline{r} > 0$. *Let* $\mu_1(\beta)$, $a_2(\beta)$, $M_1^{a_1}(\beta)$, *and* $M_1^{a_2}(\beta)$ *be finite.*

(*a*) *If additionally* (2.3) *is fulfilled then* $c = 0$ *and*

$$\lim_{t\to\infty} \mathbf{P}\{X(t) \le x\} = D(x) = \sum_{k=1}^{[x]} q_k, \quad x \ge 0, \tag{4.2}$$

where $\Psi(s) = \sum_{k=1}^{\infty} q_k s^k = 1 - Q(s)/Q(0)$, $\Psi(1) = 1$, *and*

$$Q(s) = \lim_{t\to\infty} e^{-\beta t}\{1 - \Phi(t; s)\}, \quad s \in [0, 1].$$

(*b*) *Suppose additionally* (2.4).

(*i*) *If* $\rho\theta < \alpha$ *then* $c = 0$ *and one has* (4.2).

(*ii*) *If* $\rho\theta > \alpha$ *then* $c = \infty$ *and*

$$\lim_{t\to\infty} \mathbf{P}\{X(t) \le x \mid X(t) > 0\} = D(x) = \sum_{k=1}^{[x]} q_k, \ x \ge 0. \tag{4.3}$$

(*iii*) *If* $\rho\theta = \alpha$ *then* $c \in [0, \infty]$ *depending of the s.v.f.* $L_\Delta(x)$. *If* $L_\Delta(x) \to \infty$ *then* $c = \infty$ *and we have* (4.3). *If* $L_\Delta(x) \to 0$ *then* $c = 0$ *and we have* (4.2). *If* $L_\Delta(x) \to C \in (0, \infty)$ *then* $0 < c = \overline{r}CQ(0)\mathbf{E}\nu_k/(-\beta) < \infty$ *and one has*

$$\lim_{t\to\infty} \mathbf{P}\{X(t) \le x\} = (D(x) + c)/(1 + c). \tag{4.4}$$

P r o o f. Introduce the p.g.f. $\Psi(t; s) = \mathbf{E}\{s^{Z(t)} \mid Z(0) = 0\}, |s| \le 1$. As it is given in [9] and [20]

$$\Psi(t; s) = \exp\{-\int_0^t r(t-u)[1 - g(\Phi(u; s)]du\}, \Psi(0; s) = 1,$$

where the p.g.f. $\Phi(t; s) = \mathbf{E}\{s^{Y(t)} \mid Y(0) = 1\}, |s| \le 1$, satisfies the non-linear integral equation (3.2) (see [23, 24]).

Under the assumptions of the theorem it was shown by Sevastyanov (see [24], Section IX.3, Theorem 2) that for every $s \in [0,1]$

$$\lim_{t\to\infty}[1-\Phi(t;s)]e^{-\beta t} = Q(s), \tag{4.5}$$

where $1-Q(s)/Q(0)$ is a p.g.f.

On the other hand, for every $\rho > 0$ and other conditions of the theorem it was proved in [9], Theorem 5.3, that for $C = \overline{r}bQ(0)/(-\beta)$

$$1-\Psi(t;0) = \mathbf{P}\{Z(t) > 0\} \sim Ct^{-\rho}, \quad t\to\infty, \tag{4.6}$$

$$\lim_{t\to\infty}\mathbf{P}\{Z(t)=k \mid Z(t)>0\} = q_k, k=1,2,\dots, \tag{4.7}$$

where $\Psi(s) = \sum_{k=1}^{\infty} q_k s^k = 1-Q(s)/Q(0)$, $\Psi(1)=1$, and $Q(s)$ is well determined by (4.5).

Now from (2.6) and (4.6) it follows that

$$1-\Lambda(t) \sim C^{\theta}t^{-\rho\theta}L_I([1-\Psi(t;0)]^{-1}), t\to\infty. \tag{4.8}$$

It means that under conditions of the theorem $\kappa = \rho\theta \in (1/2,1)$ and the condition 1 of Theorem 2.1 is fulfilled. If additionally one uses (2.3) then it is not difficult to see that by (2.2) one obtains $c=0$. Now from (4.7) it follows that in our case (2.9) is fulfilled with $\omega = 0$ and $V(t)\equiv 1$. Hence $\Omega(x)\equiv D(x)$ from (4.2). Therefore by Theorem 2.1, (2.10), one obtains now the case (a) of the theorem. The case (b) follows in the same way. One has to take into account only that in (2.2) from (2.4) and (4.8) one has: $c=\infty$ if $\rho\theta>\alpha$, $c=0$ if $\rho\theta<\alpha$ and $0\le c\le\infty$ if $\rho\theta=\alpha$. $\square$

Corollary 4.1. *Consider the Markov case* $G(x) = 1-e^{-x/\mu}, \mu > 0$, *and* $h(u;s)\equiv h(s)$. *Assume additionally that*

$$0 < \int_0^1 \{[\beta x + f(1-x)]/[xf(1-x)]\}dx < \infty,$$

where $f(s) = [h(s)-s]/\mu$ *is the infinitesimal generating function. Then*

$$\Psi(s) = \sum_{k=1}^{\infty} q_k s^k = 1-\exp\{\beta\int_0^s dx/f(x)\}, \Psi(1)=1.$$

P r o o f. Under the conditions one can use the following relations (see [24], Ch.II.2, Th.1 and Ch.II.4, Th.1) as $t \to \infty$

$$1 - \Phi(t,0) \sim Q(0)e^{\beta t}, Q(0) > 0,$$
$$[1 - \Phi(t,s)]/[1 - \Phi(t,0)] \to \exp\{\beta \int_0^s dx/f(x)\}.$$

Then from the proof of Theorem 5.3, [9], one has $Q(s) = Q(0)\exp\{\beta \int_0^s dx/f(x)\}$. Therefore $\Psi(s) = 1 - Q(s)/Q(0) = 1 - \exp\{\beta \int_0^s dx/f(x)\}$. □

Remark 4.1. *Consider the* ***critical case*** $a_1 = 1$ *(and then* $\beta = 0$*). Assume additionally that* $0 < R = \int_0^\infty r(x)dx < \infty$, $r(t) \downarrow 0$, *and there exists a function* $k(t)$ *such that* $k(t) \to \infty$, $k(t) = o(t)$, *and* $r(k(t)) = o(1/(t \log t))$, $t \to \infty$. *Then under the conditions of the Theorem* 3.1 *we have all asymptotic results* (3.3)–(3.10). *The proof is similar to Theorem* 3.1 *where one has to use Theorem* 5.2 *from* [20] *instead of Theorem* 2 *from* [24], *Section* IX.3.

5. AR randomly indexed branching processes

Let now $\mathbf{W} = \{W_n\}$ be the classical Galton-Watson branching process defined recurrently as follows

$$W_0 = I, W_{n+1} = \sum_{k=1}^{W_n} V_k(n), n = 0, 1, 2, \ldots,$$

where $\{V_k(n)\}$ are i.i.d. random variables with a p.g.f. $h(s) = \sum_{k=0}^{\infty} p_k s^k$, $|s| \le 1$ (interpreted as an offspring p.g.f.). As usually $h_n(s)$ denotes the n-th iteration of $h(s)$.

Introduce the set $\mathbf{J} = \{J_k\}$ of i.i.d. non-negative random variables with a d.f. $F(x) = \mathbf{P}\{J_k \le x\}$ and let $\nu(t) = \max\{n : \sum_{k=1}^{n} J_k \le t\}$, $t \ge 0$, be the corresponding renewal process.

Then the process $Z(t) = W_{\nu(t)}, t \ge 0$, is called a randomly indexed GW branching process (see Mitov et al. [13]). Introduce the corresponding p.g.f. $\phi(t; s) = \mathbf{E}\{s^{Z(t)}|Z(0) = 1\}, |s| \le 1$. Let $\{Z_k(t)\}$ be i.i.d. copies of $Z(t)$ and define the regenerative process $X(t)$ by (2.1.

Further on we will investigate the **critical case** $h'(1) = 1$. More precisely we will assume that

$$h(s) = s + (1-s)^{1+\gamma}\pounds(1/(1-s)),\ \gamma \in (0,1], \tag{5.1}$$

where $\pounds(x)$ is a s.v.f. as $x \to \infty$. In fact (5.1) means that $h'(1) = 1$ but the variance is infinite for $0 < \gamma < 1$ or $\gamma = 1$ and $\pounds(x) \to \infty$. The variance will be finite if $\gamma = 1$ and $\pounds(x) \to b$ for some $b \in (0,\infty)$. In this case the offspring variance is $2b = h''(1)$ and $h(s)$ takes the following form:

$$h(s) = s + b(1-s)^2 + o((1-s)^2), \quad s \uparrow 1. \tag{5.2}$$

These "branching properties" will be combined with the following "renewal" conditions:

$$0 < d = \int_0^\infty x dF(x) < \infty; \tag{5.3}$$

$$1 - F(x) = x^{-\delta}L_F(x), \delta \in [0,1), L_F(x) \text{ is a s.v.f. as } x \to \infty. \tag{5.4}$$

Let $M(t)$ is a s.v.f. such that

$$\gamma M^\gamma(t)\pounds(M(t)t^{-1/\gamma}) \to 1, t \to \infty. \tag{5.5}$$

Introduce the s.v.f. $L(t) = M^\theta(t)L_I(t^{1/\gamma}/M(t))L_\Delta(t)$. Let us denote by T the time to extinction of the process $\{W_n\}$, i.e.

$$T = \min\{n : W_0 = 1, W_n > 0, n = 2,3,\dots,T-1, W_T = 0\}.$$

Note that $\mathbf{P}\{T > n\} = \mathbf{P}\{W_n > 0\}$ and in the case (5.1)

$$\mathbf{E}T = \sum_{n=1}^\infty [1 - h_n(0)] = K \in (0,\infty).$$

Introduce the s.v.f. $L^*(t) = L_\Delta(t)/[K^\theta L_I(t^\delta/L_F(t))L_F^\theta(t)]$.

Theorem 5.1. *Assume conditions* (2.2), (2.5), (5.2), *and* (5.3) *in the case that* $1 - F(t) = o(t^{-2})$, $t \to \infty$.

(*a*) *If additionally* (2.3) *is fulfilled then* $c = 0$ *and if* (2.4) *with* $\alpha = 1$ *is supposed then* $c = 0$ *for* $L_\Delta(x) \to 0$ *and* $0 < c < \infty$ *for* $L_\Delta(x)$ *converging to some constant. Then*

$$\lim_{t\to\infty} \mathbf{P}\{\log X(t)/\log t \le x\} = (x+c)/(1+c), x \in (0,1). \tag{5.6}$$

(*b*) *If additionally* (2.4) *is fulfilled then* $c = \infty$ *for* $1/2 < \alpha < 1$ *and for* $\alpha = 1$ *with* $L_\Delta(x) \to \infty$. *Then*

$$\lim_{t\to\infty} \mathbf{P}\{\log X(t)/\log t \le x \mid X(t) > 0\} = x, x \in (0,1). \tag{5.7}$$

P r o o f. Note first that under the conditions of the theorem one has (see [13], Theorem 4, (i))

$$1-\Lambda(t)=\mathbf{P}\{Z(t)>0 \mid Z(0)=I\}\sim m_I d/bt,\ t\to\infty. \tag{5.8}$$

On the other hand, one has by Theorem 6 from [13] that

$$\lim_{t\to\infty}\mathbf{P}\{Z(t)[1-\Lambda(t)]\le x \mid Z(t)>0\}=1-e^{-x}, x\ge 0. \tag{5.9}$$

Therefore from (5.8) and (2.3) one can see by (2.2) that $c=0$. Similarly from (5.8) and (2.4) with $\alpha=1$ one obtains that $c=0$ for $L_\Delta(x)\to 0$ and $0<c<\infty$ for $L_\Delta(x)\to$ constant. It follows from (5.8) that (2.8) is fulfilled with $\kappa=1$. The relation (5.9) means that (2.9) holds with $V(t)=m_I d/bt, \omega=1$ and $\Omega(x)=1-e^{-x}$.

Note that $m_\Lambda(t)=\displaystyle\int_0^t[1-\Lambda(x)]dx\sim(m_I d/b)\log t, t\to\infty$. Hence by Theorem 2.1, (2.12), one obtains (5.6).

Now from conditions (b) and (5.8) it is not difficult to see that in (2.2) one obtains $c=\infty$. Therefore the limiting distribution (5.7) follows from (2.13) of Theorem 2.1. □

Theorem 5.2. *Assume conditions* (2.2), (2.6), (5.1) *and* (5.3) *in the case that* $1-F(t)=o(M(t)t^{-1-1/\gamma})$, *where* $M(t)$ *is defined by* (5.5). *Suppose also that* $1/2<\theta/\gamma\le 1$.

(a) *If* (2.3) *or* (2.4) *with* $\alpha>\theta/\gamma$ *is fulfilled then* $c=0$ *but if* (2.4) *with* $\alpha=\theta/\gamma$ *is supposed then* $0<c<\infty$ *in the case when* $L(t)$ *converges to some positive constant. Then*

$$\lim_{t\to\infty}\mathbf{P}\{X(t)M(t)/(t/d)^{1/\gamma}\le x\}=(E_1(x)+c)/(1+c),$$

where

$$E_1(x)=[\pi^{-1}\sin(\pi\theta/\gamma)]\int_0^1 D_{\theta,\gamma}(xu^{-1/\gamma})(1-u)^{\theta/\gamma-1}u^{-\theta/\gamma}du \tag{5.10}$$

and the d.f. $D_{\theta,\gamma}(x)$ *has a Laplace transform*

$$\varphi_{\theta,\gamma}(\lambda)=\int_0^\infty e^{-\lambda x}dD_{\theta,\gamma}(x)=1-\lambda^\theta/(1+\lambda^\gamma)^{\theta/\gamma}, Re\lambda\ge 0. \tag{5.11}$$

(b) *If suppose additionally* (2.4) *with* $\alpha<\theta/\gamma$ *or* $\alpha=\theta/\gamma$ *with* $L(t)\to\infty$ *then* $c=\infty$ *and*

$$\lim_{t\to\infty}\mathbf{P}\{X(t)M(t)/(t/d)^{1/\gamma}\le x \mid X(t)>0\}=E_2(x), \tag{5.12}$$

where

$$(5.13)\qquad E_2(x) = [1/B(1-\theta/\gamma,\alpha)]\int_0^1 D_{\theta,\gamma}(xu^{-1/\gamma})(1-u)^{\alpha-1}u^{-\theta/\gamma}du,$$

$B(\cdot,\cdot)$ *is the classical beta function and the d.f.* $D_{\theta,\gamma}(x)$ *has* (5.11) *as a Laplace transform.*

P r o o f. Under the conditions of the theorem it was proved in Theorem 4, (ii), of [13] that

$$(5.14)\qquad Q(t) = \mathbf{P}\{Z(t)>0 \mid Z(0)=1\} \sim (d/bt)^{1/\gamma}M(t), t\to\infty,$$

where the s.v.f. $M(t)$ is determined by (5.5). On the other hand, by Theorem 7 of [13] one can see that

$$(5.15)\qquad \lim_{t\to\infty}\mathbf{P}\{Z(t)Q(t)\le x \mid Z(t)>0; Z(0)=1\} = D_\gamma(x), x\ge 0,$$

where the d.f. $D_\gamma(x)$ has the following Laplace transform

$$(5.16)\qquad \varphi_\gamma(\lambda) = \int_0^\infty e^{-\lambda x}dD_\gamma(x) = 1-\lambda/(1+\lambda^\gamma)^{1/\gamma}, Re\lambda\ge 0.$$

Since (2.6) is supposed then from (5.14) one obtains as $t\to\infty$

$$(5.17)\qquad \begin{aligned} 1-\Lambda(t) &= \mathbf{P}\{Z(t)>0 \mid Z(0)=I\} = 1-g(1-Q(t))\\ &= Q^\theta(t)L_I(1/Q(t)) \sim (d/bt)^{\theta/\gamma}M^\theta(t)L_I((d/bt)^{-1/\gamma}M^{-1}(t)).\end{aligned}$$

Now using (5.15)–(5.17) and (2.6) one can obtain for $t\to\infty$

$$(5.18)\quad \begin{aligned} &\mathbf{E}\{e^{-\lambda Z(t)Q(t)} \mid Z(t)>0; Z(0)=I\} = 1-[1-g(\phi(t;e^{-\lambda Q(t)}))]/[1-\Lambda(t)]\\ &\sim\ 1-\{[1-\phi(t;e^{-\lambda/At})]/Q(t)\}^\theta \to 1-\{\lambda/(1+\lambda^\gamma)^{1/\gamma}\}^\theta = \varphi_{\theta,\gamma}(\lambda).\end{aligned}$$

Then from (2.2), (2.3) and (5.17) with $1/2<\theta/\gamma\le 1$ it is not difficult to see that $c=0$. Similarly in the case (2.4) with $\alpha>\theta/\gamma$. But if (2.4) and (5.17) with $\alpha=\theta/\gamma$ are fulfilled, and $L(t)\to K\in(0,\infty)$, then in (2.2) one can observe that $0<c<\infty$.

Note that (2.8) is fulfilled with $\kappa=\theta/\gamma$ because of (5.17). It follows from (5.18) and (5.14) that (2.9) holds with $V(t)=1/Q(t)$ and $\Omega(x)=D_{\theta,\gamma}(x)$, where $\varphi_{\theta,\gamma}(\lambda) = \int_0^\infty e^{-\lambda x}dD_{\theta,\gamma}(x)$. Therefore by Theorem 2.1, (2.10), one obtains (a) with (5.10) and (5.11).

If (2.4) with $\alpha<\theta/\gamma$ holds then in (2.2) one obtains that $c=\infty$ and similarly in the case $\alpha=\theta/\gamma$ with $L(t)\to\infty$. Hence by Theorem 2.1, (2.11), the relations (5.12) and (5.13) follow. $\square$

Corollary 5.1. *From* (5.10), (5.11) *and* (5.13) *applying the Tauberian theorem (see e.g.* [4, 5]*) one obtains as* $x \to \infty$ *that*

$$1 - D_{\theta,\gamma}(x) \sim x^{-\theta}/\Gamma(1-\theta),$$
$$1 - E_1(x) \sim x^{-\theta}[\sin(\pi\theta/\gamma)]/[(\pi\theta/\gamma)\Gamma(1-\theta)]$$
$$1 - E_2(x) \sim x^{-\theta}/[\alpha B(1-\theta/\gamma,\alpha)\Gamma(1-\theta)],$$

i.e. for the three d.f. we have a normal domain of attraction of a stable law with parameter θ*.*

Theorem 5.3. *Assume conditions* (2.2), (5.1) *and* (5.4) *with* $\delta \in (1/2, 1)$.

(i) *If* (2.3) *is fulfilled then* $c = 0$;

(ii) *If* (2.4) *and* (2.5) *are fulfilled then* $c = 0$ *for* $\delta < \alpha$, $c = \infty$ *for* $\delta > \alpha$ *and* $c = \lim_{x\to\infty} \{L_\Delta(x)/[m_I K L_F(x)]\}, c \in [0,\infty]$ *for* $\delta = \alpha$.

(iii) *If* (2.4) *and* (2.6) *are fulfilled and* $\delta\theta \in (1/2, 1)$ *then* $c = 0$ *for* $\delta\theta < \alpha$, $c = \infty$ *for* $\delta\theta > \alpha$ *and* $c = \lim_{x\to\infty} L^*(x), c \in [0,\infty]$ *for* $\delta\theta = \alpha$.

(A) *Then in the case* $0 \le c < \infty$ *one has*

$$\lim_{t\to\infty} \mathbf{P}\{X(t) \le x\} = (D(x) + c)/(1 + c), x \ge 0, \tag{5.19}$$

where

$$D(x) = \int_0^\infty \mathbf{P}\{W_{[y]} \le x \mid W_{[y]} > 0\} dF_2(y), \tag{5.20}$$

and

$$\Pi(y) = \mathbf{E}\{\min(T, [y] + 1)\}/\mathbf{E}T, \quad y \ge 0. \tag{5.21}$$

(B) *In the case* $c = \infty$ *one has*

$$\lim_{t\to\infty} \mathbf{P}\{X(t) \le x \mid X(t) > 0\} = D(x), \ x \ge 0. \tag{5.22}$$

P r o o f. Under the conditions of the theorem the following result holds (see Theorem 5, (ii), from [13])

$$Q(t) = \mathbf{P}\{Z(t) > 0 \mid Z(0) = 1\} \sim K L_F(t) t^{-\delta}, t \to \infty. \tag{5.23}$$

By the conditions of the theorem we have also the following limiting distribution (see Theorem 9 from [13])

$$\lim_{t\to\infty} \mathbf{P}\{Z(t) \le x \mid Z(t) > 0; Z(0) = 1\} = D(x), x \ge 0, \tag{5.24}$$

where $D(x)$ is defined by (5.20) and (5.21).

Then as $t \to \infty$ from (5.23) one has when (2.5) is fulfilled

$$1 - \Lambda(t) \sim m_I Q(t) \sim m_I K L_F(t) t^{-\delta}, \tag{5.25}$$

while for (2.6) one obtains

$$1 - \Lambda(t) \sim Q^\theta(t) L_I(1/Q(t)) \sim [K L_F(t)]^\theta L_I(1/Q(t)) t^{-\delta\theta}. \tag{5.26}$$

Now from (2.2) using (5.25) and (5.26) it is not difficult to check that all relations (i), (ii), and (iii) are correct. From the limiting relationship (5.25) and (5.26) it follows that the condition (2.8) is fulfilled with $\kappa = \delta$ or $\kappa = \delta\theta$ and in both cases $\kappa \in (1/2, 1)$. On the other hand, the limiting distribution (5.24) shows that (2.9) is fulfilled with $V(t) \equiv 1$. Therefore the limiting distributions (5.19) and (5.22) with (5.20) and (5.21) follow by applying Theorem 2.1, (2.10) and (2.11), where $\Omega(x) \equiv D(x)$, $\omega = 0$ and therefore $\mathcal{F}_1(x) \equiv D(x)$ and $\mathcal{F}_2(x) \equiv D(x)$. $\square$

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