PLISKA
STUDIA MATHEMATICA

ПЛИСКА
МАТЕМАТИЧЕСКИ
СТУДИИ

Pliska Stud. Math. **27** (2017), 91–102

PLISKA
STUDIA MATHEMATICA

POWER OF EXCEEDANCE-TYPE TESTS AGAINST LOCATION SHIFT ALTERNATIVE

Eugenia Stoimenova

This paper deals with a class of nonparametric two-sample tests for ordered alternatives. The test statistics proposed are based on the number of observations from one sample that precede or exceed a threshold specified by the other sample, and they are extensions of Šidák's test. We study their power properties against the location-shift alternative for distributions from the uniform, normal and exponential families. We give the corresponding power functions, obtained by Monte Carlo simulation, and make some comparative comments.

1. Introduction

Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$ be random samples from continuous distribution functions F and G, respectively. Denote the ordered X's and Y's by $X_{(1)} < \cdots < X_{(m)}$, and $Y_{(1)} < \cdots < Y_{(n)}$, respectively. For $0 \le r < n$, define the exceedance statistics based on thresholds from the both samples.

$$(1.1)\qquad \begin{array}{llll} A_s = & \text{the number of} & Y\text{-observations} & \text{larger than} & X_{(m-s)}, \\ B_r = & \text{the number of} & X\text{-observations} & \text{smaller than} & Y_{(1+r)}, \end{array}$$

These statistics are potentially useful for testing whether the two random samples are from the same population. For example classic precedence test [3]

2010 *Mathematics Subject Classification:* 62G10, 62E15, 62G20.

Key words: Two-sample problem, Exceedance statistics, Precedence statistics, Lehmann alternative; Location-shift alternative, Stochastic ordering.

is based on the number of observations in the X-sample that are smaller than $Y_{(1+r)}$. Large values of this statistic lead to rejection of the null hypothesis about equality of the two distributions.

For testing the hypothesis $H_0: \ F(x) = G(x)$ against the alternative

$$H_A: \ F(x) > G(x), \tag{1.2}$$

we consider the test statistic

$$V_\rho = A_s + B_r, \tag{1.3}$$

where the threshold statistics $X_{(m-s)}$ and $Y_{(1+r)}$ are determined as $s = [\rho m]$ and $r = [\rho n]$ for some $0 \leq \rho < 1$, with $[\cdot]$ denoting the integer part. Various values of ρ yield a family of test statistics which we refer to as Šidák-type tests [10].

Evidently, large values of V_ρ lead to the rejection of H_0 in favor of the stochastically ordered alternative in H_A. It is reasonable for ρ to be small since we want to reduce the possible influence of a small number of potential outliers.

The test based on the sum of A_0 and B_0, that is the number of observations in the Y-sample above all observations in the X-sample, and the number of observations in the X-sample below all observations in the Y-sample, appears as the earliest work of Šidák on nonparametric statistics [8]. The null distribution of this test statistic was studied in [12] and tables of critical values were produced by these authors. A slight modification of the test statistic based on the sum became popular as Tukey's Quick Test (see [7] and [4]).

The proposed tests are a subclass of a general family of tests based on precedences and/or exceedances of a random level specified by the order statistics from the samples. Some basic references include [9], [11], [3], [1] and [2].

The distribution of the V_ρ test orders under H_0 is not affected by the underlying distributions F and G [10]. However, the power of the test is not easy to obtain due to the generality of the alternative H_A. A simple subclass of this alternative suggested by parametric theory is the location-shift alternative. However, in this case the distribution of rank statistics will depend not only on θ, but also on F and G.

The present paper deals with the power of one of the V_ρ tests against location-shift alternatives H_1. The power of the tests is estimated through Monte Carlo simulations as a function of the shift parameter θ for uniform, normal, exponential, lognormal, and gamma distributions. We also compare the power of the V_ρ tests against Lehmann alternatives $H_{LE}: \ G(x) = 1 - (1 - F(x))^{1/\eta}, \eta > 1$. Alternatives of this form are a subclass of stochastically ordered alternatives [10].

Table 1: Critical values for $m = 40$ and $n = 20(4)40$ and different choices of s and r at 5% level of significance

ρ	m	s	n	r	c.v.	α_1	α_2
0	40	0	20	0	7	0.043	0.068
0	40	0	24	0	7	0.030	0.050
0	40	0	28	0	6	0.034	0.069
0	40	0	32	0	6	0.034	0.061
0	40	0	36	0	6	0.032	0.058
0	40	0	40	0	6	0.032	0.058
0.05	40	2	20	1	12	0.041	0.059
0.05	40	2	24	1	11	0.043	0.064
0.05	40	2	28	1	11	0.034	0.053
0.05	40	2	32	1	10	0.048	0.075
0.05	40	2	36	1	10	0.048	0.075
0.05	40	2	40	2	12	0.041	0.062
0.1	40	4	20	2	16	0.044	0.063
0.1	40	4	24	2	15	0.044	0.062
0.1	40	4	28	2	15	0.037	0.053
0.1	40	4	32	3	16	0.046	0.066
0.1	40	4	36	3	16	0.044	0.064
0.1	40	4	40	4	18	0.036	0.052
0.15	40	6	20	3	20	0.044	0.060
0.15	40	6	24	3	19	0.042	0.058
0.15	40	6	28	4	20	0.048	0.067
0.15	40	6	32	4	20	0.044	0.061
0.15	40	6	36	5	22	0.037	0.051
0.15	40	6	40	6	23	0.041	0.056
0.2	40	8	20	4	24	0.044	0.057
0.2	40	8	24	4	23	0.041	0.055
0.2	40	8	28	5	24	0.043	0.058
0.2	40	8	32	6	25	0.047	0.064
0.2	40	8	36	7	27	0.039	0.052
0.2	40	8	40	8	28	0.043	0.056
0.25	40	10	20	5	28	0.042	0.053
0.25	40	10	24	6	28	0.049	0.065
0.25	40	10	28	7	29	0.048	0.062
0.25	40	10	32	8	30	0.049	0.064
0.25	40	10	36	9	32	0.040	0.051
0.25	40	10	40	10	33	0.043	0.055

2. Critical values of exceedance tests

The exact null distribution of the V_ρ is proven in [10]. For small sample sizes, critical values of the V_r-tests are presented in Table 1. These calculations have been carried out on a PC computer by using the statistical package R. The code can be provided by the author upon request.

The chi-square approximation is also quite reasonable in the practical range of sample sizes (between 25 to 100) as long as n does not differ too much from m. In Table 2, we provide an example of the exact significance probabilities for the V_ρ-statistics (close to 5% level) for the choices of the sample size $m = n = 40$ and 100. It is given by a chi-square distribution with degrees of freedom $[\rho m] + 1$.

Table 2: Values of $P(\chi^2_\nu > c)$ (near 5% critical values)

m	ρ	c.v.	χ^2-approx.	m	ρ	c.v.	χ^2-approx.
40	0	6	0.0497	100	0	6	0.0489
	0.05	12	0.0571		0.05	20	0.0649
	0.1	18	0.0496		0.1	33	0.0587
	0.15	23	0.0535		0.15	45	0.0585
	0.2	28	0.0538		0.2	58	0.0475
	0.25	33	0.0529		0.25	69	0.0513

3. Location shift alternative

In this section, we compare the power of the V_ρ-tests against the location-shift alternative of the form

$$H_1 : \; G(x) = F(x - \theta), \qquad \text{for some } \theta > 0. \tag{3.4}$$

This class of alternatives, specified by θ, is a subclass of the general ordered alternative H_A in (1.2). It is a simple suggestion arising from parametric theory, although the distribution of test statistics in this case will depend not only on θ, but also on F.

To make meaningful comparison of the power of different tests, we calculated power functions at prescribed exact level of significance α as follows. First, for any V_ρ-test, we determine two values α_1 and α_2 so that

$$P(V_\rho \geq c) = \alpha_1 \quad \text{and} \quad P(V_\rho \geq c - 1) = \alpha_2,$$

where c is given by $P(V_\rho \geq c|H_0) \leq \alpha$ and therefore, the interval (α_1, α_2) contains the critical level, say $\alpha = 0.05$. Next, we calculate the power values corresponding to the two critical values c and $c-1$

$$\beta_1 = P(V_\rho \geq c|H_{LE}) \quad \text{and} \quad \beta_2 = P(V_\rho \geq c-1|H_{LE}),$$

Then, the power of the test at exact level α is estimated by

$$\beta = \pi\beta_2 + (1-\pi)\beta_1,$$

where $\pi = \dfrac{\alpha - \alpha_1}{\alpha_2 - \alpha_1}$ is the adjusting factor used in the randomized procedure in (3.5).

$$P_i = \begin{cases} 1, & \text{if } V_\rho \geq c \\ \dfrac{\alpha-\alpha_1}{\alpha_2-\alpha_1}, & \text{if } V_\rho = c-1 \\ 0, & \text{otherwise,} \end{cases} \tag{3.5}$$

The power functions were estimated through Monte Carlo simulations as a function of the shift parameter θ for uniform, normal, exponential and gamma distributions. For different choices of sample sizes, we generated 10000 sets of data in order to obtain the estimated power against specific values of the location-shift parameter. All power values were adjusted for the fixed level of significance $\alpha = 0.05$ in a manner similar to the adjustment made earlier for Lehmann alternative (see Subsection 4.).

3.1. Uniform distribution

Take $F(x)$ to be uniform distribution in $[0;\ 1]$ and consider four alternative distributions $G(x) = F(x-\theta)$, specified by $\theta = 0.05, 0.1, 0.2$ and 0.3. Table 3 provides the estimated power values of the Šidák-type tests specified by ρ, for $m = 40$ and different choices of n in this case. Threshold values s and r are determined as $s = [\rho m]$ and $r = [\rho n]$.

For any ρ, the power increases when n approaches m. All tests are most powerful when $m = n$. Looking at the power of different tests ($\rho = 0$ to 0.25), we see that the power of the V_ρ-test with $\rho > 0$ is less than the power of the corresponding V_0-test (Figure 1). This is because the V_0-test is locally most powerful in this case for testing H_0 against a shift close to 0 in the uniform distribution, as shown by [5] (see the highlighted values in Table 3). When ρ is small, 5% or 1%, the V_ρ-test can still retain good power to be useful against small shifts.

Table 3: Power of V_ρ-tests against location shift alternative in the case of uniform distribution for $m = 40$, $n = 20(4)40$ and $\alpha = 0.05$ level of significance

ρ	m	s	n	r	c.v.	shift 0	0.05	0.1	0.2	0.3
0	40	0	24	0	6	0.0555	**0.3546**	**0.8086**	**0.9961**	**1.0000**
0	40	0	32	0	6	0.0483	**0.3622**	**0.8208**	**0.9989**	**1.0000**
0	40	0	36	0	6	0.0496	**0.4506**	**0.8314**	**0.9992**	**1.0000**
0	40	0	40	0	6	0.0408	**0.4646**	**0.8964**	**0.9995**	**1.0000**
0.05	40	2	24	1	11	0.0444	0.2336	0.5301	0.9338	0.9988
0.05	40	2	32	1	10	0.0704	0.2811	0.6360	0.9857	1.0000
0.05	40	2	36	1	10	0.0727	0.3252	0.7302	0.9932	0.9999
0.05	40	2	40	2	12	0.0489	0.2876	0.6263	0.9853	1.0000
0.1	40	4	24	2	15	0.0571	0.1550	0.4149	0.8467	0.9940
0.1	40	4	32	3	16	0.0615	0.1772	0.4589	0.9329	0.9970
0.1	40	4	36	3	16	0.0609	0.2072	0.5251	0.9416	0.9990
0.1	40	4	40	4	18	0.0431	0.1832	0.4387	0.9168	0.9985
0.15	40	6	24	3	19	0.0474	0.1338	0.3206	0.7698	0.9709
0.15	40	6	32	4	20	0.0570	0.1481	0.3927	0.8628	0.9886
0.15	40	6	36	5	21	0.0494	0.1509	0.3715	0.8316	0.9913
0.15	40	6	40	6	23	0.0456	0.1701	0.3696	0.8388	0.9925
0.2	40	8	24	4	23	0.0430	0.1121	0.2859	0.6552	0.9363
0.2	40	8	32	6	25	0.0538	0.1574	0.3078	0.7794	0.9698
0.2	40	8	36	7	27	0.0433	0.1510	0.3300	0.7616	0.9752
0.2	40	8	40	8	28	0.0542	0.1556	0.3086	0.7815	0.9801
0.25	40	10	24	6	28	0.0557	0.1249	0.2595	0.5951	0.8910
0.25	40	10	32	8	31	0.0510	0.1100	0.2392	0.6511	0.9255
0.25	40	10	36	9	32	0.0403	0.1201	0.2629	0.7059	0.9526
0.25	40	10	40	10	33	0.0458	0.1460	0.2719	0.7306	0.9634

3.2. Normal distribution

Take $F(x)$ to be standard normal distribution and consider five alternative distributions $G(x) = F(x - \theta)$, specified by $\theta = 0.2, 0.3, 0.5, 1$ and 2. For equal sample sizes, the parameters s and r, specifying the threshold positions, are equal and in this case the contiguous order statistics determine the family of test statistics. For simplicity, let us denote the family of test statistics in this case by $V_r = A_r + B_r$ with $r = 0, 1, 2, \ldots$.

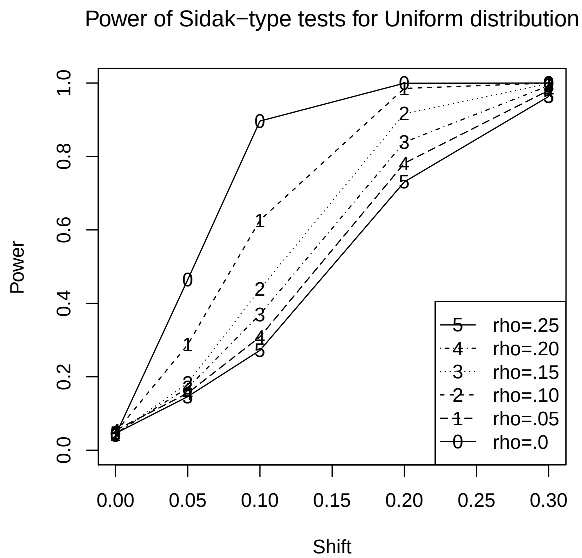

Figure 1: Power curves of exceedance tests against shift in Uniform distribution

Table 4 provides the estimated power values of the Šidák-type tests V_r for equal sample sizes with $r = 0, 1, 2, \ldots$, in this case.

From Table 4, upon comparing the power values of different V_r-tests, we find that the V_r-test with $r > 0$ gives better power performance than the original V_0 test of Šidák when the underlying distribution is normal. The tests with $r > 4$ do not give significant improvement (if any) of the power. For example, in Table 4, for $m = 15$ and $\theta = 0.5$, the power of the V_r-test with $r = 1, 2$ or 3, is about 0.33-0.34 while the corresponding power for $r = 0$ is 0.2726; the power of all other tests with $4 \leq r \leq 7$ is about 0.30-0.31. As a “rule”, the optimal threshold position seems to be close to 20% of the sample size, i.e. $r = 0.2m - 1$ (see the highlighted values in Table 4). Four power curves on Figure 1 compare the power of the optimal statistics: V_1 from sample of size 10, V_2 from sample of size 15, V_3 from sample of size 15, and V_4 from sample of size 25.

3.3. Exponential distribution

Having data from exponential distribution (or more general from right skewed distribution), we do not expect our tests to detect small shift in the distribution very well. Take $F(x)$ to be exponential distribution with parameter 1 and consider

Table 4: Estimated power of the V_r-test against location shift in the case of normal distribution for $m = n$ and $\alpha = 0.05$ level of significance

m	r	shift 0	0.2	0.3	0.5	1	2
10	0	0.0516	0.1022	0.1489	0.2358	0.5814	0.9752
	1	0.0500	**0.1069**	**0.1512**	**0.2599**	**0.6297**	**0.9875**
	2	0.0523	0.0996	0.1414	0.2357	0.5896	0.9806
	3	0.0453	0.0952	0.1305	0.2261	0.5676	0.9724
	4	0.0478	0.1032	0.1325	0.2472	0.6018	0.9821
15	0	0.0509	0.1150	0.1562	0.2726	0.6746	0.9910
	1	0.0505	0.1243	0.1760	0.3403	0.7715	0.9989
	2	0.0529	**0.1262**	0.1797	**0.3370**	**0.7902**	0.9993
	3	0.0510	0.1243	**0.1836**	0.3339	0.7764	0.9996
	4	0.0497	0.1165	0.1595	0.3031	0.7363	0.9971
	5	0.0494	0.1100	0.1627	0.3019	0.7091	0.9969
20	0	0.0493	0.1203	0.1747	0.3148	0.7367	0.9957
	1	0.0486	0.1304	0.1987	0.3728	0.8343	0.9996
	2	0.0485	0.1373	0.2071	0.3963	0.8659	0.9999
	3	0.0496	**0.1439**	**0.2142**	**0.4177**	**0.8816**	**1.0000**
	4	0.0526	0.1390	0.2112	0.4013	0.8722	0.9999
	5	0.0503	0.1364	0.2062	0.3881	0.8619	0.9999
	6	0.0502	0.1296	0.1885	0.3735	0.8354	0.9997
25	0	0.0513	0.1225	0.1804	0.3377	0.7656	0.9971
	1	0.0489	0.1352	0.2198	0.3983	0.8729	0.9998
	2	0.0493	0.1501	0.2281	0.4474	0.9049	1.0000
	3	0.0470	0.1572	0.2363	0.4640	0.9279	1.0000
	4	0.0445	**0.1590**	**0.2520**	**0.4823**	**0.9368**	1.0000
	5	0.0515	0.1507	0.2383	0.4652	0.9326	1.0000
	6	0.0499	0.1535	0.2348	0.4602	0.9302	1.0000

five alternative distributions $G(x) = F(x - \theta)$, specified by $\theta = 0.2, 0.3, 0.5, 1$ and 2.

Table 5 provides the estimated power values of the Šidák-type tests for equal sample sizes, and $\alpha = 0.05$ level of significance, in this case. Figure 2 plots the statistics with highest power for $n = 10, 15, 20$ and 25. We might conclude that the proposed V_ρ-tests would be useful when the underlying distribution is close

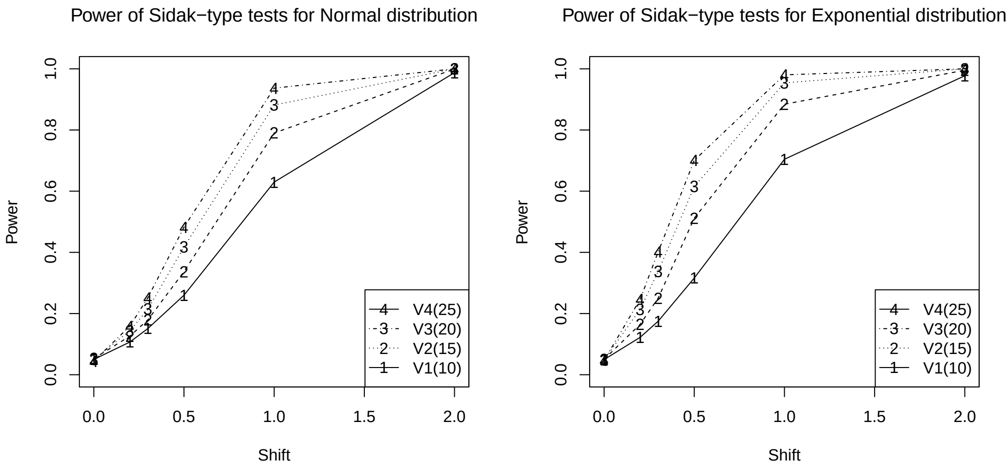

Figure 2: Power curves of exceedance tests against shift in Normal distribution and Exponential distribution

Table 5: Estimated power of the V_r-test against location shift in the case of exponential distribution for $m = n$ and $\alpha = 0.05$ level of significance

m	r	shift 0	0.2	0.3	0.5	1	2
10	1	0.04952	0.12235	0.1750	0.3161	0.7043	0.9773
15	2	0.0503	0.1648	0.2506	0.5117	0.8842	0.9954
20	3	0.0482	0.2141	0.3374	0.6151	0.9540	0.9999
25	4	0.0467	0.2466	0.4017	0.7008	0.9803	1.0000

to symmetric or when there is small or moderate skewness in the underlying distributions.

4. Power against Lehmann alternative

In this section, we express the distribution of V_ρ under the Lehmann alternative given by

$$H_{LE} : G(x) = 1 - (1 - F(x))^{1/\eta}, \tag{4.6}$$

for some $\eta > 0$. When $\eta = 1$, the resulting distributions satisfy the null hypothe-

Table 6: Power comparison of V_r-tests for $m = n = 20$ at 5% level of significance

V_r-test	$\eta = 2$	$\eta = 3$	$\eta = 4$	$\eta = 5$	$\eta = 6$	$\eta = 7$
V_0	0.4566	0.7859	0.9207	0.9705	0.9894	0.9952
V_1	0.5061	0.8292	0.9436	0.9808	0.9931	0.9974
V_2	0.5230	0.8379	0.9476	0.9818	0.9928	0.9969
V_3	0.5182	0.8355	0.9445	0.9795	0.9918	0.9957
V_4	0.5149	0.8262	0.9416	0.9774	0.9901	0.9956

sis H_0, while $\eta > 1$ yields various distributions in the alternative hypothesis H_{LE}, with larger values of η indicating stronger attraction towards $H_A : F(x) \geq G(x)$; see [6] for further discussion on this class of alternatives.

Figure 3 illustrates the gain in power of using any of the first five V_ρ-tests with $\rho > 0$ instead of Šidák's V_0-test.

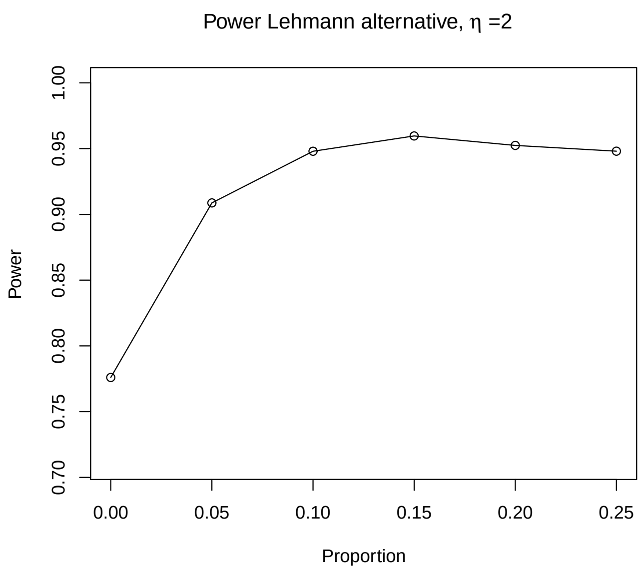

Figure 3: Power functions of V_ρ-tests for $m = 40$ and $n = 40$ against the Lehmann alternative with $\eta = 2$ at 5% level of significance.

For any $0 \leq s \leq m$ and $0 \leq r \leq n$, the joint probability mass function of A_s and B_r, under H_{LE} in (4.6), is given derived in [10]. Consequently, the distribution of V_ρ-statistic under H_{LE} is distribution free.

For $m = n = 20$ and $\eta = 2(1)7$, the power values of the V_ρ-tests corresponding to $r = 0, \ldots, 4$, against the Lehmann alternative H_{LE} in (4.6), are presented in Table 6, where the significance level is set as $\alpha = 0.05$.

From Table 6, we see that the power values of all tests increase with increasing η. For each of the six fixed values 2 (1) 7 of η, the power increases up to the third V_r-test, showing that the V_0-test, based on the extremal thresholds, is less powerful than the tests based on the next extremal thresholds pairs $(Y_{(2)}, X_{(m-1)})$ and $(Y_{(3)}, X_{(m-2)})$.

Acknowledgments. The author acknowledge funding by the Bulgarian fund for scientific investigations Project I02/19.

REFERENCES

[1] I. Bairamov. Advances in exceedance statistics based on ordered random variables. In: Recent Developments in Ordered Random Variables (Eds M. Ahsanullah and M.Z. Raqab) New York, Nova Science Publishers, 2006, 97–117.

[2] N. Balakrishnan, A. Dembinska, A. Stepanov. Precedence-type tests based on record values. *Metrika*, **68**, No 2 (2008), 233–255.

[3] N. Balakrishnan, H. K. Tony Ng. Precedence-Type Tests and Applications. Wiley Series in Probability and Statistics. Hoboken, NJ, John Wiley & Sons, 2006.

[4] D. J. Gans. Corrected and extended tables for Tukey's quick test. *Technometrics*, **23**, No 2 (1981), 193–195.

[5] J. Hájek, Z. Šidák. Theory of Rank Tests. New York-London: Academic Press; Prague: Academia, Publishing House of the Czechoslovak Academy of Sciences, 1967.

[6] E. L. Lehmann. The power of rank tests. *Ann. Math. Stat.*, **24** (1953), 23–43.

[7] H. R. Neave. A development of Tukey's quick test of location. *J. Amer. Stat. Assoc.*, **61** (1966), 949–964.

[8] J. Seidler, J. Vondráček, I. Saxl. The life and work of Zbyněk Šidák (1933–1999). *Appl. Math., Praha*, **45**, No 5 (2000), 321–336.

[9] P. K. Sen. On some asymptotic properties of a class of non-parametric tests based on the number of rare exceedances. *Ann. Inst. Stat. Math.*, **17** (1965), 233–255.

[10] E. Stoimenova, N. Balakrishnan. Šidak-type tests for the two-sample problem based on precedence and exceedance statistics. *Statistics*, **51**, No 2 (2017), 247–264.

[11] P. van der Laan, S. Chakraborti. Precedence tests and Lehmann alternatives. *Statist. Papers*, **42**, No 3 (2001), 301–312.

[12] Z. Šidák, J. Vondráček. A simple nonparametric test of the difference of location of two populations. *Appl. Math., Praha*, **2** (1957), 215–221.

Eugenia Stoimenova
Institute of Information and Communication Technologies
and
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G.Bontchev Str., Bl. 25A
1113 Sofia, Bulgari
e-mail: `jeni@parallel.bas.bg`