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USING BRANCHING PROCESSES TO SIMULATE COSMIC RAYS CASCADES

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The cosmic rays (CR) cascades are one of the most famous examples of branching processes in physics. They consist of many different types of secondary high energy particles which are offsprings of leading primary one, usually high energy nucleon or gamma photon after collision with atmosphere. The rate of expansion of cascade depends on multiple different conditional probabilities as the chance of survival in atmosphere without interactions, particle lifetime, the number of daughter particles, etc. The paper presents quantitative simulated results with a specially written in R software for parallel simulation of distributions of multiple types of daughter particles. The processes are based on simplified models of Hadron cascades simulated by age-dependent and imbedded Markov Galton-Watson branching processes. For the sake of simplicity in modelling the probability dependencies on particles energy and free path in atmosphere are not always constrained strictly to the available experimental results. Moreover, the scattering angles also are not considered in this version of software.

1. Introduction

The atmosphere of the Earth is continually penetrated by high energy particles, a flux of relativistic ones originated mainly outside of Solar system. They consist predominantly from galactic nucleus of hydrogen (90%) and heavier elements.

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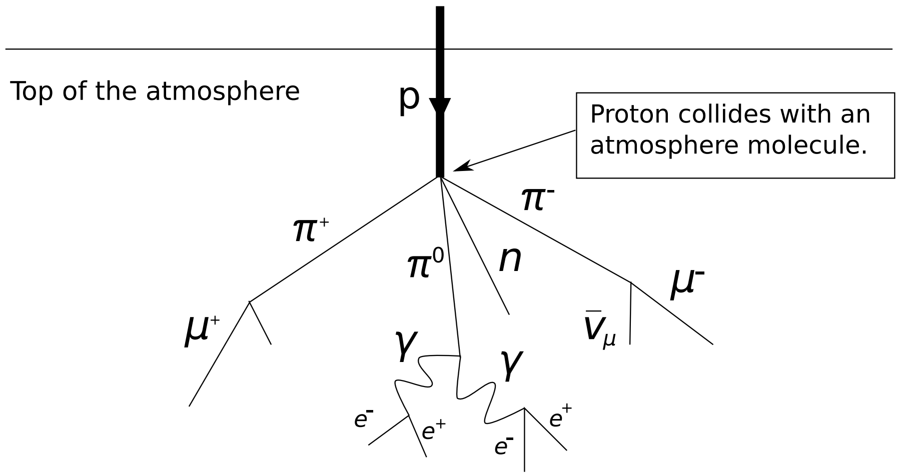

Figure 1: Primary cosmic particle collides with a molecule of atmosphere. Source Wikipedia.com

There are also high energy photons, electrons and solar particles with lower energies. After their penetration in atmosphere, the cosmic rays cascades are induced due to interactions between primary particles and atmosphere molecules. These cascades, in both electron-photon and nucleon part, are the biggest ever uninterrupted natural example of branching processes in nature. That is why simulations of similar process are the most intuitive method of study very complex multi-type branching processes (Fig. 1).

For the current simulations of cosmic rays cascades two different models are used – continuous-time Markov chain and approximated continuous-time age-dependent branching processes with several types of particles. The size of the particle flux is described with a random quantity function $N(E, x)$, yielded from particle interactions and transformation of many type particles $T_0, T_1, \dots, T_n$ with different energies E and atmosphere thickness x. Every particle of type T_i has a random lifetime τ_i, which depends on probabilities of decay and interactions with atmosphere. At the end of its lifetime, the particle gives birth of many particles of either the same or another type with lower energies. They are regarded as immigrants to the flux of already available particles of this type. For them, it is possible, one or more offsprings to be of the same particle type as their parent, but with different energy. For example, the leading proton induces multiple cascades of many pions and kaons in addition to proton with lower energy.

Having a permanent process of energy loss and birth-death transformations, the intensity of flux of different daughter particles in cascade follows the critical behaviour with very quick growth in the beginning to quick extinctions and the end. These intensity properties mainly depend on probabilities of interactions and decay for every particle. The probability is mainly dependent on the type and energy of particles and the density of atmosphere, which are also completely random. Moreover, in addition to these complexities of the branching process, the lack of proved strict values of probabilities for very high energies particles makes the precise analytic solution infeasible and only computational modelling is possible for theoretical investigations.

The process is induced initially by a leading nucleon. The particles traverse through the atmosphere until collide with air molecules. The possible outcomes are due to either elastic or inelastic scattering. The first one produces the same number and type of particles with the kinetic energy conserved in centre-of-mass frame, but only direction is modified. With inelastic scattering, the leading proton produces number n of secondary particles, as pions and kaons in addition to number of m nucleons, as proton and neutron, with lower energy than the parent.

Two constraints must be considered for the process. Firstly, the proton is not allowed to be produced from any daughter particle. Secondly, during the process the particle energies are in decreasing order with the highest on in the very beginning, i.e. – $E_0 \geq E_1 \geq \cdots \geq E_n$.

In addition to scattering, there are another two processes that yield offsprings – particle decay and resonances. The main difference to scattering is that the parent particle vanishes and new ones are born. During the decay, this is direct process - the parent particle vanishes and the new ones are born. However, the final output of resonances is similar to decay, but it pass through intermediate very short lived Hadron particle state which decay to new particles. Because the lifetime of resonant particle is close almost to 0, the resonance process would be approximated as decay.

2. Cosmic Rays as a Branching Process

2.1. Definition

For the CR cascade development, the main probabilities of branching are defined for every particle independently. Each one after any close to 0 time period $o(h) \rightarrow 0$, could undergo interaction with another particle or molecule, decay or pass the path x, without any change and interfere. The notations for all possible particle states are as follows:

$$\begin{aligned} p_{el}(E,x)h + o(h) &\; - \text{ probabiliy of elastic interaction} \\ p_{inel}(E,x)h + o(h) &\; - \text{ probabiliy of inelastic interaction} \\ p_d(E,x)h + o(h) &\; - \text{ probabiliy of decay} \\ \lambda = 1-(p_{inel}(E,x)+p_{el}(E,x)+p_d)h+o(h) &\; - \text{ free mean path } \lambda \end{aligned}$$

Thus, the probability of particle to survive without decay or interactions over its path through matter depth x is equal to (see [1]):

$$P(E,x) = \frac{\exp(-x/\lambda)}{dx/\lambda} \tag{2.1}$$

If the particle could not survive over the path dx, then random number of new particles is born. Let the probability for birth of particle of type $k, k = 1, 2, \ldots, n$ is denoted as $w_k(E', x)$ with energy E' less than parent's energy. Thus, the flux of all offsprings constitutes the array of energies $E_1, E_2, \ldots, E_n$ with such intervals (see [2]):

$$(E_0, E_0 + \Delta_0), (E_1, E_1 + \Delta_1), \ldots, (E_n, E_n + \Delta_n) \tag{2.2}$$

$$\text{with } \Delta_0^2 + \Delta_1^2 + \cdots + \Delta_n^2 \to 0. \tag{2.3}$$

Then any particle of type T_i with energy in interval j is assumed as distinct particle T_i^j [2]:

$$T_i^j = T_i \frac{E_{j-1} + E_j}{2}. \tag{2.4}$$

The lower index of particles is assumed ordered according the type, as 1 is reserved to nucleon. The upper index is for energy intervals (2.2). Thus, the general model is multi-type branching process with separation on type of particle and their energies. Every particle of type T_i^j from this process has a lifetime τ_i^j derived from (2.1) with distribution $P(\tau_i^j \leq t) = f_i^j(t)$. Then, if the probability of a particle of type T_i within energy group j to give birth of random number $\alpha = (\alpha_1, \ldots, \alpha_k)$ particles with different energy levels, is denoted by $w_\alpha^i(E_j, x)$, where parameters are for energy E and traverse path x of the ancestor particle. The size of k is type dependent and it's dimention is a function of i and j. Thus,the probability generating function (p.g.f) and expected number of different outcomes for $i = 0 \div k$ and decreasing energies $j' \leq j$ are:

$$h_i^j(s) \quad = \sum_\alpha w_\alpha^i(E_j, x) s^\alpha \tag{2.5}$$

$$E(N(T_i^j)) \quad = \frac{\partial h_i^j(1)}{\partial s_j} \tag{2.6}$$

where for $s = (s_1, \ldots, s_k)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ the following notation is used:

$$s^{\alpha} = s_1^{\alpha_1} \cdots s_k^{\alpha_k}. \tag{2.7}$$

There are three very important properties in proposed model. The first one is energy conservation, that states that the aggregated energy of all new particles must be equal to the energy of the parent. The second one is the assumption that $\sum_{\alpha} w_{\alpha}^{i} = 1$. The third property is that for any $i = 1, \ldots, k$, $\alpha_i \geq 0$ and α_i depends only on quantum numbers conservation laws (see [4]).

The survival probability of any type of particles during the branching process without migration is equal to $S = 1 - P(N(T_i^j) \to 0)$ for every i, j and $t \to \infty$. This is valid for primary type of particles. However, the in the case of cosmic rays this is not enough for survival of any particular type of daughter particle due to random migration, which occurs through injection of a number of particles of the same type in random time due to another particle type interaction or decay. The probability that the number Y of new particles is exactly equal to l is $P(Y = l) = h_l$. Then the p.g.f. for imigration in injection time point is (see [5]):

$$t(s) = \sum_{i=0}^{\infty} h_i s^i. \tag{2.8}$$

The distribution $\{h_l\}_{l=0}^{\infty}$ could be approximated to either Homogeneous Poisson process or Nonhomogeneous Poisson process depending on periodicity of the injection [5].

2.2. Implementation

For CR simulation a specially developed in R software is used (see [6]). The implementation follows the model described above, simulating all required particles in parallel at every depth. However, the model in current version is simplified by omitting implementation of some physical realities. The first specific trait of the simulation is that the observations of scattering and delays due to differential cross-section are not considered (see [4]). Thus the particle transverse is assumed only forward directed. This assumption simplifies modelling and computations and allows the whole direction of transport to be sliced discretely to Δx intervals following the function of nonlinear standard atmosphere depth (see [7]).This is the second difference from the real particle cascade – the path is discrete, not continuous. The main reason for that is the implementation of a stepwise procedure for computation of probability of particle interactions and their counting process.

The probabilities of interactions and survival of particles are computed for partial atmosphere depth Δx following the equation (2.1). The decision for cascade development is done by random trial with binomial distribution with the particle states derived from probabilities for depth , particle type and lifetime. For the observation of a particle lifetime, the memory parameter 'age' equals to i is denoted as $\langle i \rangle$), is introduced. Its initial state is equal to 1 when a new particle is born or a process begins. After every next trial when the particle survives the value of age is increasing by 1. The most computational resource consuming part of the process is the cascade expansion through the depth. The total depth simulates 90 km altitude from ground level. Then the simulated total age at point x_i from ground level with Δx_i (in km) interval between trial runs is equal to:

$$\langle i \rangle = \frac{90 - x_i}{\Delta x_i} + 1 \tag{2.9}$$

where the counting begins from 1 at the starting point.

Having implemented branching process expansion based on binomial trials with age parameter, two different type of processes - memoryless continuous (Markov) and age-dependent are available for optional run. For the first one, Markov branching process, the probabilities of survival and interactions are computed only over the last slice of Δx_i and probability density for life-length $\tau = [i-1, i]$ is equal to (see [3])

$$b(t+\tau)e^{-\int_{t_1}^{t_1+\tau} b(x)dx}, \tag{2.10}$$

where b(t) is a function derived from (2.1.). Conversely, when the process is age-dependent, the considered atmosphere depth range over the whole path from the cite where age is equal to 1 to the moment of computation, or $\tau \geq 1$. As a result, for particle description every type is distinguished in software implementation by set of descriptive parameters for type, depth, energy, age. Thus, the number of particles $N(x,t)$ of the same type are aggregated by their location only for particles with equal energy, type and age.

The simplified cascade model is constrained to only interactions of 3 type of particles and their indexes are for: protons (1), charged(3) and neutral pions (4) and their daughter particles as muons (10), electrons (11), photons (12) and neutrino (13). The notation is intentionally left open for future model update with another type of mesons as kaons, for example.

Using the denoted indexes, the possible reaction and resulting states for leading proton are:

$$T_1 : \xrightarrow{scattering} s = (T_1, T_3, T_4) \tag{2.11}$$

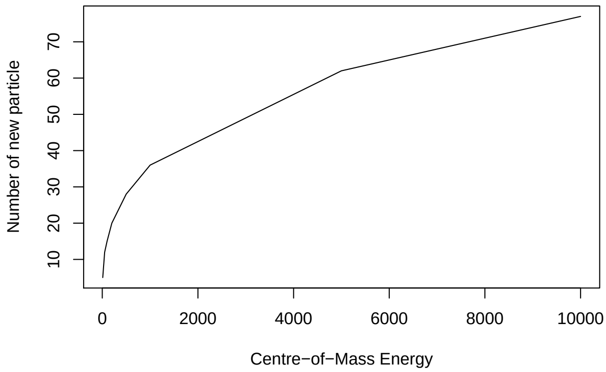

Figure 2: The number of offsprings as a function of centre-of-mass momentum

This process of proton yielded cascades depends strongly on type of particles and number of $Bi(n, w_i^j(E_j, x))$ trials and very weakly on energy. The last one is due to dependence by design of probability of interaction of $w_i^j(E_j, x) \propto ln(E_j)$. However, the initial energy is the most important factor of critical properties of the cascade due to strong dependence of the number of daughter particles (α). The computational values for dependency on energy of mean free path in air and the ratio of multiplication of daughter particles are mainly derived in [8]. Moreover, there is a fixed fraction of 2:1 between the numbers of $T_3 : T_4$. For example, the implemented function of pion numbers after proton scattering depends on the coefficient of elasticity computed over the centre-of-mass s (Fig. 2):

$$\alpha_j = -4.2 + 4.5.s^{0.31}. \tag{2.12}$$

However, the impact of Δx is very important and must be considered in more details in future works by similar simulations. The strong dependency on type and number of iterations is shown from simulation results and survival probability of primary proton (Fig. 3):

$$S(t) = P(\tau_i^j > t) = 1 - f_i^j(t). \tag{2.13}$$

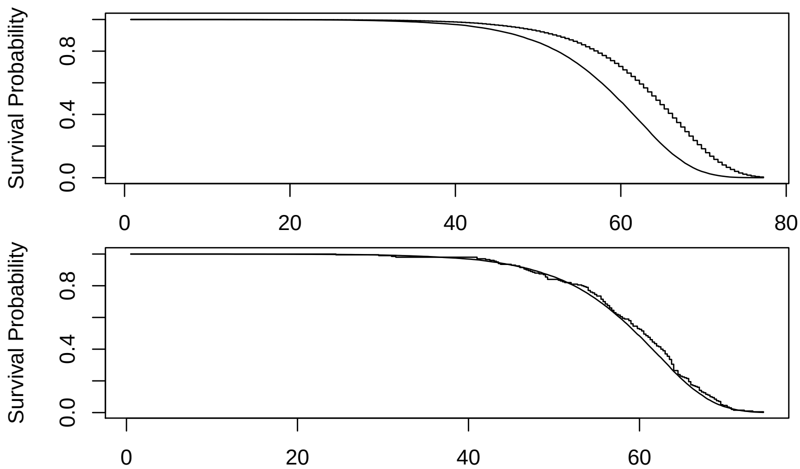

Figure 3: Proton survival. With the straight line are shown simulations of 1TeV primary proton with $\Delta x_i = 250$m for age-dependent process. The comparison graphics with 's'-type lines are for the particles with initial energy 1TeV for age-dependent process, but with $\Delta x_i = 500$m (above) and for Markov process with $\Delta x_i = 250$m (below)

Similarly, for pions the possible states in implemented simulations are:

$$T_3 : \xrightarrow{1.decays;2.scattering} s = (T_3, T_4, T_{10}, T_{11}, T_{12}, T_{13}), \tag{2.14}$$

$$T_4 : \xrightarrow{1.decay;2.scattering} s = (T_{11}, T_{12}). \tag{2.15}$$

The trials for pions are performed in ordered sequence, where decay, which depend only on energy, is first. Then both elastic and inelastic scatterings follow. The multiplicity of pion offsprings during scattering is similar to these of proton, but with reducing coefficient of 2/3. The daughter particles due to decay is fixed and thus their number is only function of probability of their decay.

3. Age-dependent process with injections

The simulations are run for Δx_i equal to 100 meters. This extends the number of x levels to 900, the number large enough to approximate continuous process. The simulated cascade generates $N_1(T_i^j) = 20000$ leading protons with standard normal distributed initial energies of 1 TeV. The function of elasticity of protons is set to 1/2. The required computer resources for simulation of whole cascade are enormous and they are performed on high-performance grid computing infrastructure at the Institute of Information and Communication Technologies BAS in Sofia (see [9]).

The development of age-dependent branching process may be described by family history of particles $w = \langle \tau_1, \nu_1, \ldots, \tau_n, \nu_n \rangle$, where by τ_i is described lifetime and by ν_i number of children. Then the total number of objects alive at moment $t \leq y$ is a function of lifetime and the family of ascendents and descendents is denoted as $Z_t(y, t, w)$, for $0 \leq t \leq \infty$ and $0 \leq y \leq \infty$. If $N_k = N_k(w)$ is the number of objects in the k-th generation,$k = 0, 1, 2, \ldots$, then the process is Galton-Watson process (see [3]).

Having exponential form of lifetime probability (see 2.10) and recurrent binomial trials $l = 1, 2, \ldots, k$ within temporally-homogeneous time, the numerical solution for $E(N)$ is in exponential form at $t \to \infty$ (see [3]):

$$E(N(t)) = \sum A_l e^{\alpha_l t} \tag{3.16}$$

where A_l and α_l are computed recursively. For computation of the simulated results the used form of Malthusian law of population growth is equal to:

$$N(t) = N(0)e^{rt} \tag{3.17}$$

with Malthusian coefficient r for exponential extinsion ($r < 0$) and expantion ($r > 0$).

For the process of proton cascades (Fig. 4), only the initial number $N(0)$ is predominant. Moreover, as the computational results show, the process is permanently in extinction with values of $r \leq 0$ over all process due to lack of multiplication of daughter particles of type 1. The process of extinction accelerates in the last stage due to growing probabilities for interaction.

However, when considering the particles of secondary type, such as pions, their number is strongly dependent on interaction rate of protons because their initial number $N(0) = 0$ and the process is completely born with immigration due to proton-air interactions. Due to very low probabilities for interactions in very first steps of cascade, the Malthusian coefficient is with very strong random fluctuations (Fig. 5). Then, when the number of pions grows and the fraction of newborn pions due to immigration is low the pion's r-coefficient is smoothed. Finally, the strong random fluctuations of Malthusian coefficient with predominantly negative values appear at the end of the process when the number of migration rate of pions begin to shrink quickly due to proton flux extinction (see Fig. 4) altogether with the very high rate of pion decay due to very high atmosphere density.

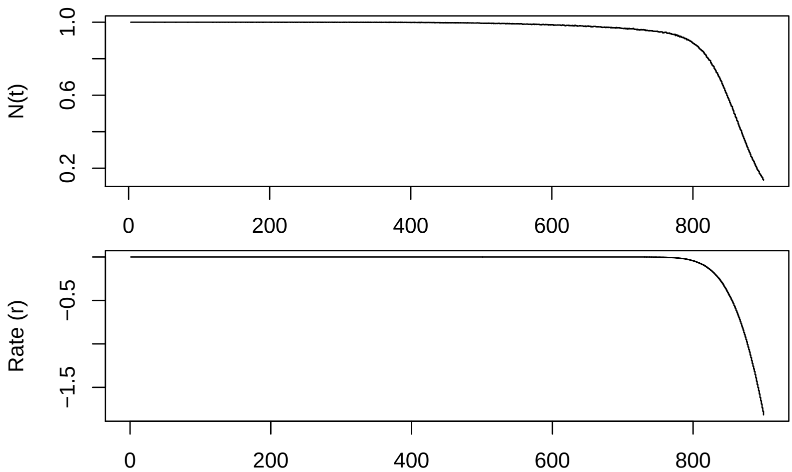

Figure 4: The graphics shows the normalized number of protons $N(t)$ (upper) and their Malthusian coefficient r (below)

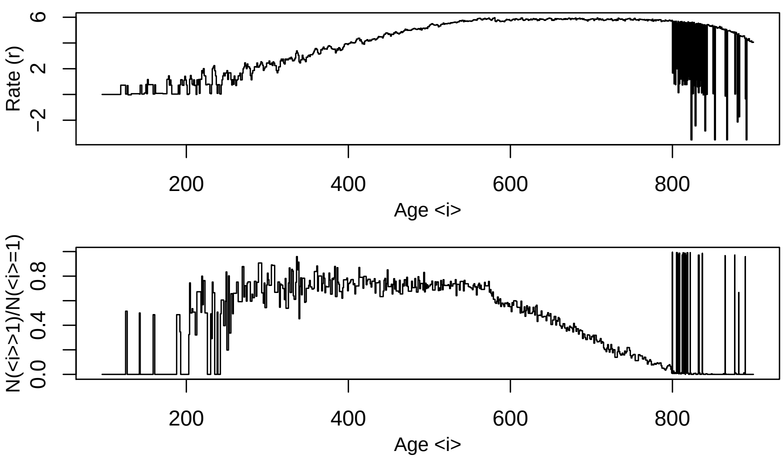

Figure 5: The graphics of dependence on age of Malthusian coefficient r of pions over all energies (upper) and the ratio between pions with age > 1 and these with age 1 (down).

4. Conclusions

With the developed software the basic properties of a simulated CR cascade are demonstrated and implemented as a branching process. These results are focused mainly on computations of the branching process expansion and extinction in CR cascade with small number of particles. However, due to simplification it misses verification with real data. The most important reason for this is its incompleteness as lack of differential cross section, very large intervals between interactions, small number of used particles. These properties must be considered as primary tasks for further development. However, the available version can be used and configured as basic software tool for theoretical research in the field of multi-type branching processes with migrations. This could be easily implemented because the software is developed for easy modification with parameters change and combination. Thus, further development will be only considered as upgrade for enabling the wider range of options and turning the tool to fully functional version of cosmic rays simulator.

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