Pliska Stud. Math. **28** (2017), 7–127

PLISKA
STUDIA MATHEMATICA

SUMMATION BY EULER'S TRANSFORM OF THE SERIES OF DIRICHLET, FACTORIAL SERIES AND THE SERIES OF NEWTON

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0. Introduction

Let the series

$$a_0 + a_1 + a_2 + \dots \tag{0.1}$$

be given, for which at first it is not suppose anything about its convergence. From the series (0.1) Euler obtains a new one by means of the formal expansion

$$\sum_{n=0}^{\infty} a_n \left(\frac{z}{2-z}\right)^{n+1} = \sum_{n=0}^{\infty} A_n z^{n+1}.$$

From this equality it follows that

$$A_n = \frac{1}{2^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu.$$

The series

$$A_0 + A_1 + A_2 + \dots \tag{0.2}$$

is called E-transform of (0.1). If the series (0.1) is convergent, then it is easily verified that the series (0.2) is convergent with sum equal to that of (0.1). But, there are cases when the series (0.2) is convergent without (0.1) to be convergent. If the series (0.2) is convergent with sum s, then they say that (0.1) is summable by the method of Euler, or shortly E-summable, with sum s.

By iterating the E-transform K. Knopp [1] comes to its generalization, namely: from the series (0.1) he gets a new series by using the following expansion

$$\sum_{n=0}^{\infty} a_n \left(\frac{z}{q+1-qz}\right)^{n+1} = \sum_{n=0}^{\infty} A_n^k z^{n+1}, \quad q = 2^k - 1, \; k \geq 0,$$

whence

$$A_n^k = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} a_\nu, \quad n = 0, 1, 2, \ldots .$$

The series

$$A_0^k + A_1^k + A_2^k + \cdots \tag{0.3}$$

is called E_k-transform of (0.1). If the series (0.3) is convergent with sum s, then the series (0.1) is called E_k-summable with sum s. The E_0-summation is the ordinary convergence, and if the series (0.1) is E_k-summable, then it is E_{k_1}-summable for each $k_1 > k$ with the same sum. The study of this summation as well as its application to the analytical continuation of Taylor's series is due to Knopp.

Another summation with a wide application is that of Cesàro. The series (0.1) is summable by the method of Cesàro, shortly (C, k)-summable, if the expression

$$\frac{1}{C_n^k} \sum_{\nu=0}^{n} C_{n-\nu}^k a_\nu, \quad C_n^k = \binom{n+k}{n},$$

tends to a definite limit when $n \to \infty$. This summation has an important significance by studying the trigonometric series of Fourier as well as of the series of Taylor on the boundary of their disks of convergence. H. Bohr applied this method to the ordinary Dirichlet series

$$\sum_{n=0}^{\infty} \frac{a_n}{(n+1)^s}. \tag{0.4}$$

M. Riesz generalized the (C, k)-summation so that it to be applied to the general series of Dirichlet

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}.$$

In a paper of the author of this issue it was given a more general summation as well as its applications to the theory of the series just mentioned and all the results already known were obtained as particular cases.

In this paper is studied the summation of the series of Dirichlet, Newton and the factorial one by the method of Euler-Knopp. It turns out that this method is a powerful tool for their investigation, more powerful than the known till now. The following theorem is established:

Let the series (0.4) *be E_k-summable for $s = s_0$, then it is E_k-summable for each s with $\Re s > \Re s_0$. If $\sum_{n=0}^{\infty} A_n^k(s_0)$ is the E_k-transform of the series* (0.4) *for $s = s_0$, then the E_k-transform of this series for $\Re s > \Re s_0$ is a holomorphic function $f(s)$ in this half-plane given by the expression*

$$f(s) = \frac{1}{\Gamma(s-s_0)} \int_0^{\infty} t^{s-s_0-1} \sum_{n=0}^{\infty} S_n^k(s_0) \left(\frac{q+1}{q+e^t}\right)^{n+1} \frac{e^t-1}{e^t+q}\, dt,$$

where

$$S_n^k(s_0) = \sum_{\nu=0}^{n} A_n^k(s_0).$$

From here one obtains that

$$f(s) = O\left(\frac{\Gamma(\sigma-\sigma_0)}{|\Gamma(s-s_0)|}\right)$$

for $\Re s = \sigma \geq \sigma_0 + \varepsilon > \sigma_0 = \Re s_0$.

From this theorem it follows also that there is a real number e_k with the following property: the series (0.4) is E_k-summable for each s with $\Re s > e_k$ and it is not E_k-summable for $\Re s < e_k$. This number is called abscissa of the E_k-summability for the series (0.4) and, moreover:

If $e_k \geq 0$, then

$$e_k = \limsup_{n\to\infty} \frac{\log |A_0^k + A_1^k + \cdots + A_n^k|}{\log(n+1)},$$

and

$$e_k = \limsup_{n\to\infty} \frac{\log |A_n^k + A_{n+1}^k + \dots|}{\log n}$$

when $e_k < 0$.

If the series (0.3) is absolutely convergent, then they say that the series (0.1) is absolutely summable by the method of Euler, or shortly, $|E_k|$-summable. First, in this issue are proved some simple basic properties of this summation which were not pointed out till now, namely:

If the series (0.1) *is absolutely convergent, then it is* $|E_k|$*-summable for each* $k > 0$*. Moreover, the* $|E_k|$*-summability implies* $|E_{k_1}|$*-summability for each* $k_1 > k$.

If the series

$$a_0 + a_1 + a_2 + \dots$$

is $|E_k|$*-summable, then the series*

$$0 + a_0 + a_1 + \dots$$

is also $|E_k|$*-summable and conversely.*

Let the series $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *be* $|E_k|$*-summable, then their Cauchy's convolution* $\sum_{n=0}^{\infty} c_n$*, i.e.*

$$c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0,$$

is $|E_k|$*-summable too.*

Each $|E_k|$*-summable series is absolutely summable by the method of Borel.*

Further, this summation is applied to the series (0.4) and the following properties are obtained:

If the series (0.4) *is* $|E_k|$*-summable for* $s = s_0$*, then it is* $|E_k|$*-summable for each* s *with* $\Re s > \Re s_0$.

If $\overline{e}_k$ *is the abscissa of the* $|E_k|$*-summability of the series* (0.4), *then*

$$\overline{e}_k = \limsup_{n \to \infty} \frac{\log(|A_0^k| + |A_1^k| + \dots + |A_n^k|)}{\log(n+1)}$$

when $\overline{e}_k \geq 0$*, and*

$$\overline{e}_k = \limsup_{n \to \infty} \frac{\log(|A_n^k| + |A_{n+1}^k| + \dots)}{\log(n+1)},$$

provided that $\overline{e}_k < 0$*. Moreover, one always has that* $\overline{e}_k - e_k \leq 1$.

It has to be pointed out that the proofs of these properties require rather complicated calculations.

For the factorial series

$$\sum_{\nu=0}^{\infty} \frac{a_\nu \nu!}{s(s+1)\dots(s+\nu)}, \quad s \in \mathbb{C} \setminus \{0, -1, -2, \dots\}, \tag{0.5}$$

the following theorem is proved:

Let the series (0.5) *be* E_k*-summable for* $s = s_0$*, then, it is* E_k*-summable for each* s*, such that* $\Re s > \Re s_0$*. Of course, the points* $0, -1, -2, \dots$ *are excluded. Let* $\sum_{n=0}^{\infty} A_n^k(s_0)$ *denote the* E_k*-transformed series* (0.5) *with* $s = s_0$*, then its* E_k*-sum is holomorphic function* $f(s)$ *in the half-plane* $\Re s > \Re s_0$ *except possible simple poles at the points* $0, -1, -2, \dots$*. If* $\Re s_0 \geq 0$*, then*

$$f(s) = \frac{\Gamma(s)}{\Gamma(s-s_0)\Gamma(s_0)} \sum_{\mu=0}^{\infty} S_\mu^k(s_0) \int_0^1 t^{s-s_0}(1-t)^{\mu+s_0} \left(\frac{q+1}{q+1-qt}\right)^{\mu+1} \frac{dt}{q+1-qt},$$

where $S_n^k(s_0) = \sum_{\mu=0}^{n} A_\mu^k(s_0)$*, and if* $\Re s_0 < 0$*, then*

$$f(s) = \sum_{\nu=0}^{p-1} \frac{a_\nu \nu!}{s(s+1)\dots(s+\nu)}$$

$$+\frac{\Gamma(s)}{\Gamma(s-s_0)\Gamma(s_0)} \sum_{\mu=0}^{\infty} S_\mu^{(p)}(s_0) \int_0^1 t^{s-s_0}(1-t)^{\mu+s_0+p} \left(\frac{q+1}{q+1-qt}\right)^{\mu+1} \frac{dt}{q+1-qt},$$

where p *is an integer greater than* $-\Re s_0$*,* $S_n^{(p)}(s_0) = \sum_{\mu=0}^{n} A_\mu^{(p)}(s_0)$ *and*

$$A_\mu^{(p)}(s_0) = \frac{1}{(q+1)^{\mu+1}} \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} q^{\mu-\nu} \frac{a_{p+\nu}(p+\nu)!}{s_0(s_0+1)\dots(s_0+p+\nu)}.$$

This theorem is established by another method which leads to the following form of the function $f(s)$, namely

$$f(s) = \Gamma(s) s_0 \sum_{\mu=0}^{\infty} A_\mu^k(s_0)(A_\mu^k(s_0))^{s_0} \sum_{\tau=0}^{\infty} \frac{\Gamma(\mu+\tau+1)}{\Gamma(\mu+\tau+s+1)} (A_\tau^k(s_0))^{s-s_0-1} \left(\frac{q}{q+1}\right)^{\tau},$$

which holds for all admissible s_0 and s, such that $\Re s > \Re s_0$.

It is also proved that if the series (0.5) is $|E_k|$-summable for $s = s_0$, then it is $|E_k|$-summable for each s with $\Re s > \Re s_0$. From here it comes out a representation of $f(s)$ by means of the Laplace integral provided that $\Re s > 0$, namely:

If the series (0.5) *is* E_k*-summable for* $s = s_0$*, then*

$$f(s) = \int_0^1 (1-x)^{s-1} \psi(x)\, dx$$

when $\Re s > 0$, where the function

$$\psi(x) = \sum_{n=0}^{\infty} a_n x^n$$

is holomorphic in the disk

$$\left| x - \frac{q}{2q+1} \right| \leq \frac{q+1}{2q+1}$$

and moreover, it is of a finite order in the sense of Hadamard on it.

This theorem is invertible. Indeed, by a well-known theorem of Pincherle and Nörlund, in a case of convergence the function $\psi(x)$ is holomorphic in the larger disk $|x| \leq 1$.

Further, for the series of Newton

$$a_0 + \sum_{\nu=1}^{\infty} (-1)^{\nu} a_{\nu} \frac{(s-1)(s-2)\dots(s-\nu)}{\nu!} \tag{0.6}$$

it is proved completely analogous theorem, namely:

If the series (0.6) is E_k-summable for $s = s_0$, then it is E_k-summable for each s such that $\Re s > \Re s_0$, then function $f(s)$ represented by it is holomorphic when $\Re s > \Re s_0$ and if $\Re s \leq 0$, then

$$\frac{\Gamma(s-s_0)\Gamma(1-s_0)}{\Gamma(1-s_0)} f(s)$$

$$= \sum_{n=0}^{\infty} S_n^k \int_0^1 t^{s-s_0} (1-t)^{n-s} \left(\frac{q+1}{q+1-qt} \right)^{n+1} \frac{dt}{q+1-qt},$$

where $S_n^k(s_0) = \sum_{\nu=0}^{n} A_n^k(s_0)$, and $\sum_{n=0}^{\infty} A_n^k$ is the E_k-transform of the series (0.6) for $s = s_0$. If $\Re s > 0$, then

$$f(s) = \sum_{\nu=0}^{p-1} (-1)^{\nu} a_{\nu} \frac{(s-1)(s-2)\dots(s-\nu)}{\nu!}$$

$$+ \frac{\Gamma(1-s_0)}{\Gamma(s-s_0)\Gamma(1-s)} \sum_{\nu=0}^{\infty} S_n^{(p)}(s_0) \int_0^1 \frac{t^{s-s_0}(1-t)^{n-s+p}}{q+1-qt} \left(\frac{q+1}{q+1-qt} \right)^{n+1} dt,$$

where p is an integer greater than $\Re s$, $S_n^{(p)}(s_0) = \sum_{\nu=0}^{n} A_\nu^{(p)}(s_0)$ and

$$A_\nu^{(p)}(s_0) = \frac{1}{(q+1)^{\nu+1}} \sum_{\mu=0}^{\nu} (-1)^{\mu+p} \binom{\nu}{\mu} q^{\nu-\mu} \frac{(s_0-1)\dots(s_0-\mu)}{(\mu+p)!}.$$

With another method it is obtained an expression for $f(s)$ which holds for each admissible s and s_0 such that $\Re s > \Re s_0$, namely:

$$\frac{f(s)}{\Gamma(s-s_0)\Gamma(1-s_0)}$$

$$= \sum_{n=0}^{\infty} A_n^k(s_0) b_n(-s) \sum_{\tau=0}^{\infty} \frac{\Gamma(n+\tau+1)}{\Gamma(n+\tau+1-s_0)} b_\tau(s-s_0-1) \left(\frac{q}{q+1}\right)^\tau,$$

where

$$b_0(s) = a_0, \quad b_n(s) = \frac{a_n}{(s+1)(s+2)\dots(s+n)}, \quad n = 1, 2, 3, \dots.$$

If the series (0.6) is $|E_k|$-summable for $s = s_0$, then it is $|E_k|$-summable for each s with $\Re s > \Re s_0$.

At the end the Borel summation is applied to the factorial series. The series (0.1) is B-summable if

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$$

is an entire function and the integral

$$\int_0^\infty e^{-x} \varphi(x)\, dx$$

is convergent. The following theorem is proved:

Let the series (0.5) *be B-summable for $s = s_0$, then it is B-summable for each s such that $\Re s > \Re s_0$. If $\Re s_0 > 0$, then its sum $f(s)$ is given by*

$$f(s) = \frac{\Gamma(s)}{\Gamma(s_0)\Gamma(s-s_0)} \int_0^1 (1-u)^{s-s_0} u^{s_0-1}\, du \int_0^\infty h(tu) e^{-t(1-u)}\, dt,$$

where

$$h(x) = \int_0^x e^{-t} \Phi(t)\, dt, \ \ \Phi(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{s_0(s_0+1)\dots(s_0+n)}.$$

If $\Re s_0 < 0$ *and* m *is a positive integer such that* $m > -\Re s_0$*, then*

$$f(s) = \sum_{n=0}^{m-1} \frac{a_n n!}{s(s+1)\dots(s+n)}$$

$$+\frac{\Gamma(s)}{\Gamma(s_0)\Gamma(s-s_0)} \int_0^1 (1-u)^{s-s_0} u^{s_0-1}\, du \int_0^\infty H(ut) e^{-(1-u)t}\, dt,$$

where

$$H(x) = \int_0^x e^{-t}\psi(t)\, dt, \quad \psi(t) = \sum_{n=0}^{\infty} \frac{a_n t^n}{s_0(s_0+1)\dots(s_0+n)}.$$

1. On the absolute summation by Euler's transform

By Knopp, the series

$$\sum_{n=0}^{\infty} a_n \tag{1.1}$$

is absolutely summable by Euler's transform of order k, shortly $|E_k|$-summable, if the series

$$\sum_{n=0}^{\infty} A_n^k, \; A_n^k = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} a_\nu, \; q = 2^k - 1, \; k > 0,$$

is absolutely convergent.

This summation is studied in the first part of the paper, its new basic properties are established and thus the results of Knopp are filled up. The following theorems are proved:

Theorem 1. *If the series* (1.1) *is absolutely convergent, then it is* $|E_k|$*-summable for each* $k > 0$*.*

Indeed,

$$|A_n^k| \leq \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} |a_\nu|,$$

$$\sum_{n=0}^{m} |A_n^k| \leq \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} q^{n-\nu} |a_n|$$

$$= \sum_{\nu=0}^{m} |A_\nu^k| \sum_{n=\nu}^{m} \frac{1}{(q+1)^{n+1}} \binom{n}{\nu} q^{n-\nu}.$$

But

$$\sum_{n=\nu}^{m} \frac{1}{(q+1)^{n+1}} \binom{n}{\nu} q^{n-\nu} < \sum_{n=\nu}^{\infty} \frac{1}{(q+1)^{n+1}} \binom{n}{\nu} q^{n-\nu}$$

$$= \frac{1}{(q+1)^{\nu+1}} \cdot \frac{1}{\left(1 - \dfrac{q}{q+1}\right)^{\nu+1}} = 1.$$

Hence,

$$\sum_{n=0}^{\infty} |A_n^k| < \sum_{n=0}^{\infty} |a_n|$$

and thus the theorem is established.

Theorem 2. *If the series* (1.1) *is* $|E_k|$*-summable, then it is* $|E_{k_1}|$*-summable for each* $k_1 > k$.

This theorem is an immediate corollary of the preceding one, since if $\delta > 0$, then the $E_{k+\delta}$-transform of the series (1.1) is the E_δ-transform of its E_k-transform. Moreover, the sum $\sum_{n=0}^{\infty} |a'_n|$ decreases by increasing of k.

Theorem 3. *If the series*

$$a_0 + a_1 + a_2 + \ldots$$

is $|E_k|$*-summable, then the series*

$$0 + a_0 + a_1 + a_2 + \ldots \tag{1.2}$$

is also $|E_k|$*-summable, and conversely.*

Let, as before, $\sum_{n=0}^{\infty} A_n^k$ be the E_k-transform of the series (1.1) and $\sum_{n=0}^{\infty} \widetilde{A}_n^k$ be that of (1.2), then

$$\sum_{n=0}^{\infty} a_n \left(\frac{z}{q+1-qz}\right)^{n+1} = \sum_{n=0}^{\infty} A_n^k z^{n+1},$$

$$\sum_{n=1}^{\infty} a_{n-1} \left(\frac{z}{q+1-qz}\right)^{n+1} = \sum_{n=0}^{\infty} \widetilde{A}_n^k z^{n+1},$$

whence one easily gets that

$$\frac{1}{q+1-qz}\sum_{n=0}^{\infty} A_n^k z^{n+1} = \sum_{n=0}^{\infty} \widetilde{A}_n^k z^n. \tag{1.3}$$

This relation yields that

$$\widetilde{A}_n^k = \frac{1}{q+1}\sum_{\nu=0}^{n-1} A_\nu^k \left(\frac{q}{q+1}\right)^{n-1-\nu}.$$

Hence,

$$\begin{aligned}\sum_{n=0}^{m} |\widetilde{A}_n^k| &\le \frac{1}{q+1}\sum_{n=0}^{m}\sum_{\nu=0}^{n-1} |A_\nu^k| \left(\frac{q}{q+1}\right)^{n-1-\nu} \\ &= \frac{1}{q+1}\sum_{\nu=0}^{m-1} |A_\nu^k| \sum_{n=\nu+1}^{m} \left(\frac{q}{q+1}\right)^{n-1-\nu} \\ &< \frac{1}{q+1}\sum_{\nu=0}^{m-1} |A_\nu^k| \sum_{\mu=0}^{\infty} \left(\frac{q}{q+1}\right)^{\mu} = \sum_{\nu=0}^{m-1} |A_\nu^k|,\end{aligned}$$

and thus the first part of the theorem is established. The relation (1.3) gives that $A_n = (q+1)\widetilde{A}_{n+1}^k - q\widetilde{A}_n^k$ whence the second part immediately follows.

Theorem 4. *If the series* $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ *are* $|E_k|$*-summable, then their Cauchy's convolution* $\sum_{n=0}^{\infty} c_n$, $c_n = a_o b_n + a_1 b_{n-1} + \cdots + a_n b_0$ *is also* $|E_k|$*-summable.*

If $\sum_{n=0}^{\infty} A_n$, $\sum_{n=0}^{\infty} B_n$, $\sum_{n=0}^{\infty} C_n$ are their E_k-transforms, then [1, p. 131]

$$C_n = (q+1)(A_0 B_n + \cdots + A_n B_0) - q(A_0 B_{n-1} + \cdots + A_{n-1} B_0),$$

and since the series $\sum_{n=0}^{\infty} |A_n|$, $\sum_{n=0}^{\infty} |B_n|$ are convergent, the same holds for the series $\sum_{n=0}^{\infty} |C_n|$.

Theorem 5. *If the series* $\sum_{n=0}^{\infty} a_n$ *is* $|E_k|$*-summable, then it is also* $|B|$*-summable.*

It is well-known that the series $\sum_{n=0}^{\infty} a_n$ is absolutely summable by Borel's method ([2]) when the integral

$$\int_0^{\infty} e^{-x}|u^{(\lambda)}(x)|\,dx, \quad u(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n, \tag{1.4}$$

is convergent for each $\lambda = 0, 1, 2, \ldots$.

If we denote

$$s_n^{(k)} = \frac{1}{(q+1)^n} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} s_\nu, \quad s_n = \sum_{\nu=0}^{n} a_\nu, \tag{1.5}$$

then

$$e^{-(q+1)x} \sum_{n=0}^{\infty} \frac{s_n^{(k)}}{n!} ((q+1)x)^n = e^{-x} \sum_{n=0}^{\infty} \frac{s_n x^n}{n!} = h(x), \tag{1.6}$$

whence by the substitution of $(q+1)x$ by x, one gets that

$$e^{-x} \sum_{n=0}^{\infty} \frac{s_n^{(k)}}{n!} \frac{x^n}{(q+1)^n} = g(x).$$

Then after differentiation, it follows that

$$e^{-x} \sum_{n=0}^{\infty} \frac{a_{n+1}^{(k)}}{n!} x^n = g'(x),$$

where $a_n^{(k)} = s_n^{(k)} - s_{n-1}^{(k)}$. Therefore,

$$\int_0^A |g'(x)|\, dx \le \sum_{n=0}^{\infty} \frac{|a_{n+1}^{(k)}|}{n!} \int_0^A e^{-x} x^n \, dx$$

$$< \sum_{n=0}^{\infty} \frac{|a_{n+1}^{(k)}|}{n!} \int_0^{\infty} e^{-x} x^n \, dx = \sum_{n=0}^{\infty} |a_{n+1}^{(k)}|,$$

and since

$$\int_0^{\infty} |g'(x)|\, dx = \frac{1}{q+1} \int_0^{\infty} |h'(x)|\, dx,$$

the integral

$$\int_0^{\infty} e^{-x} \left| \sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!} \right| dx$$

is convergent. But according to the previous theorems it is possible to increase or decrease the indices of the terms of the series (1.1) with an arbitrary integer and, hence, the integrals(1.4) are convergent, i.e., the theorem is established.

The series (1.1) is summable by the method of Abel and Poisson, shortly A-summable, if the function

$$f(x)=\sum_{n=0}^{\infty}a_nx^n=(1-x)\sum_{n=0}^{\infty}s_nx^n,\;\; s_n=a_0+a_1+\cdots+a_n, \tag{1.7}$$

tends to a definite limit when $x\to 1-0$. By Whittaker [3] the series (1.1) is absolutely summable by this method, or $|A|$-summable, if the integral

$$\int_0^1 |f'(x)|\,dx$$

is convergent.

Theorem 6. *If a series is $|E_k|$-summable, then it is also $|A|$-summable.*

Let the Laplace transform

$$L(\varphi)=\int_0^{\infty}e^{-sx}f(x)\,dx$$

be applied to both parts of (1.7). Then after simple calculations one gets that

$$\sum_{n=0}^{\infty}s_n^{(k)}\frac{(q+1)^n}{(s+q+1)^{n+1}}=\sum_{n=0}^{\infty}\frac{s_n}{(s+1)^{n+1}}. \tag{1.8}$$

Both series in it are convergent for $s>0$. If $\dfrac{1}{s+1}$ is replaced by x, then $x<1$ when $s>0$ and $x\to 1$ when $s\to 0$. The function $f(x)$, defined by (1.6), is exactly equal to

$$s\sum_{n=0}^{\infty}\frac{s_n}{(s+1)^{n+1}}.$$

But if $s>0$, then (1.8) yields that

$$f(x)=s\sum_{n=0}^{\infty}\frac{(q+1)^n}{(s+q+1)^{n+1}}\sum_{\nu=0}^{n}a_\nu^{(k)}=s\sum_{\nu=0}^{\infty}a_\nu^{(k)}\sum_{n=\nu}^{\infty}\frac{(q+1)^n}{(s+q+1)^{n+1}}$$

$$=s\sum_{\nu=0}^{\infty}a_\nu^{(k)}\frac{(q+1)^\nu}{(q+1+s)^{\nu+1}}\cdot\frac{1}{1-\dfrac{q+1}{q+1+s}}=\sum_{\nu=0}^{\infty}a_\nu^{(k)}\left(\frac{q+1}{q+s+1}\right)^{\nu}.$$

If $t = \dfrac{q+1}{q+s+1}$, then t tends to 1 when $s \to 0$, hence,

$$f(x) = \sum_{\nu=0}^{\infty} a_\nu^{(k)} t^\nu = \varphi(t).$$

From here it is seen that also the function $\varphi(t)$ corresponds to the series $\sum_{\nu=0}^{\infty} a_\nu^{(k)}$, which is the E_k-transform of (1.1). If this series is convergent, i.e., the series (1.1) is E_k-summable, then $\varphi(t)$ tends to a definite limit when $t \to 1$ and the series (1.1) is A-summable. If (1.1) is $|E_k|$-summable, i.e., the series $\sum_{n=0}^{\infty} |a_n^{(k)}|$ is convergent, then by a theorem of Fekete [4] the series (1.1) is $|A|$-summable.

2. Summation of Dirichlet's series by the method of Euler-Knopp

Let the series

$$a_0 + a_1 + a_2 + \dots \tag{2.1}$$

be given.

This series is summable by Euler's method generalized by Knopp [1] of arbitrary order $k > 0$, shortly E_k-summable, if the transformed series

$$\sum_{n=0}^{\infty} A_n^k, \tag{2.2}$$

where, as before,

$$A_n^k = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} a_\nu, \quad q = 2^k - 1, \ k > 0, \tag{2.3}$$

is convergent. The theory of this summation method is given by Knopp in the papers already cited, as well as its application to the analytic continuation of a function given by its Taylor's series.

In the present work this method is applied for the first time to the ordinary series of Dirichlet by creating a full theory of this problem. It turns out that the method in question is a very convenient tool for studying the series

$$\sum_{n=0}^{\infty} \frac{a_n}{(n+1)^s}, \tag{2.4}$$

which are mostly used in the applications.

Theorem 7. *Let the series (2.4) be E_k-summable for $s = s_0$. Then it is E_k-summable for each s such that $\Re\, s > \Re\, s_0$. Moreover, the E_k-transform of the series (2.4) is a holomorphic function for $\Re\, s > \Re\, s_0$ which is given by the expression*

$$f(s) = \frac{1}{\Gamma(s-s_0)} \int_0^\infty t^{s-s_0-1} \sum_{n=0}^\infty S_n^k \left(\frac{q+1}{q+e^t}\right)^{n+1} \frac{e^t-1}{e^t+q}\, dt, \tag{2.5}$$

where

$$S_n^k = \sum_{\nu=0}^n A_\nu^k.$$

First, the substitution of s by $s - s_0$ allows to suppose that $s_0 = 0$. From (2.3) it follows that

$$a_n = (-1)^n (q+1) q^n \sum_{\nu=0}^n (-1)^\nu \binom{n}{\nu} \left(\frac{q+1}{q}\right)^\nu A_\nu^k. \tag{2.6}$$

Let

$$\sum_{n=0}^\infty A_n^k(s) \tag{2.7}$$

be the E_k-transform of the series (2.4), i.e.,

$$A_n^k(s) = \frac{1}{(q+1)^{n+1}} \sum_{nu=0}^n \binom{n}{\nu} q^{n-\nu} (\nu+1)^{-s} a_\nu. \tag{2.8}$$

Then, substituting of a_ν by its expression from (2.6), one obtains

$$A_n^k(s) = \left(\frac{q}{q+1}\right)^n \sum_{\nu=0}^n (-1) \binom{n}{\nu} (\nu+1)^{-s} \sum_{\mu=0}^\nu \binom{\nu}{\mu} \left(\frac{q+1}{q}\right)^\mu A_\mu^k$$

and the exchange of order of summations gives that

$$A_n^k(s) = \left(\frac{q}{q+1}\right)^n \sum_{\mu=0}^n (-1)^\mu A_\mu^k \left(\frac{q+1}{q}\right)^\mu \sum_{\nu=\mu}^n (-1)^\nu \binom{n}{\nu} \binom{\nu}{\mu} (\nu+1)^{-s}.$$

By taking in view that

$$\binom{n}{\nu}\binom{\nu}{\mu} = \binom{n}{\mu}\binom{n-\mu}{\nu-\mu}$$

and by setting $\nu = \mu + \tau$, one gets that

$$A_n^k(s) = \left(\frac{q}{q+1}\right)^n \sum_{\mu=0}^{n} A_\mu^k \binom{n}{\mu} \left(\frac{q+1}{q}\right)^\mu \sum_{\tau=0}^{n-\mu} (-1)^\tau \binom{n-\mu}{\tau} (\mu+\tau+1)^{-s}.$$

But if $\Re\ s > 0$, then

$$(\mu+\tau+1)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(\mu+\tau+1)t}\, dt,$$

i.e.,

$$A_n^k(s) = \left(\frac{q}{q+1}\right)^n \sum_{\mu=0}^{n} A_\mu^k \left(\frac{q+1}{q}\right)^\mu \binom{n}{\mu} b_{n,\mu}, \tag{2.9}$$

where

$$b_{n,\mu}(s) = \sum_{\tau=0}^{n-\mu} (-1)^\tau \binom{n-\mu}{\tau} (\mu+\tau+1)^{-s}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(\mu+1)t} \sum_{\tau=0}^{n-\mu} (-1)^\tau \binom{n-\mu}{\tau} e^{-\tau t}\, dt$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(\mu+1)t} (1-e^{-t})^{n-\mu}\, dt.$$

Thus the representation

(2.10) $A_n^k(s)$

$$= \frac{1}{\Gamma(s)} \left(\frac{q}{q+1}\right)^n \sum_{\mu=0}^{n} A_\mu^k \binom{n}{\mu} \left(\frac{q+1}{q}\right)^\mu \int_0^\infty t^{s-1} e^{-(\mu+1)t} (1-e^{-t})^{n-\mu}\, dt$$

is obtained. From (2.8) and (2.9) it follows that

$$\sum_{n=0}^{m} A_n^k(s) = \sum_{n=0}^{m} \left(\frac{q}{q+1}\right)^n \sum_{\mu=0}^{n} A_\mu^k \binom{n}{\mu} \left(\frac{q+1}{q}\right)^\mu b_{n\mu}(s)$$

$$= \sum_{\mu=0}^{m} A_\mu^k \left(\frac{q+1}{q}\right)^\mu \sum_{n=\mu}^{m} \binom{n}{\mu} \left(\frac{q}{q+1}\right)^n b_{n\mu}(s)$$

$$= \frac{1}{\Gamma(s)} \sum_{\mu=0}^{m} A_\mu^k \left(\frac{q+1}{q}\right)^\mu \sum_{n=\mu}^{m} \binom{n}{\mu} \left(\frac{q}{q+1}\right)^n \int_0^\infty t^{s-1} e^{-(\mu+1)t} (1-e^{-t})^{n-\mu}\, dt.$$

By setting $n = \mu + \tau$, one gets that

$$\sum_{n=0}^{m} A_n^k(s)$$

$$= \sum_{\mu=0}^{m} \frac{A_\mu^k}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(\mu+1)t} \sum_{\tau=0}^{n-\mu} \binom{\mu+\tau}{\tau} \left(\frac{q}{q+1}\right)^\tau (1-e^{-t})^\tau \, dt.$$

The use, for the sake of simplicity, of the notations

$$l_{m,\mu}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(\mu+1)t} \sum_{\tau=0}^{m-\mu} \binom{\mu+\tau}{\tau} \left(\frac{q}{q+1}\right)^\tau (1-e^{-t})^\tau \, dt \tag{2.11}$$

leads to the relation

$$S_m^k(s) = \sum_{n=0}^{m} A_n^k(s) = \sum_{\mu=0}^{m} A_\mu^k l_{m,\mu}(s). \tag{2.12}$$

Recall that if $S_n^k(s)$ denotes the sum

$$S_n^k(s) = A_0^k(s) + A_1^k(s) + \cdots + A_n^k(s),$$

then (2.12) yields

$$S_m^k(s) = S_0^k h_{m,0}(s) + S_1^k h_{m,1}(s) + \cdots + S_{m-1}^k h_{m,m-1}(s) + S_m^k l_{m,m}(s), \tag{2.13}$$

where

$$h_{m,i}(s) = l_{m,i}(s) - l_{m,i+1}(s), \quad i = 0, 1, 2, \ldots, m-1.$$

It follows now an application of a theorem of Toeplitz. This theorem says that if it is given an infinite matrix $(a_{n,m})$ which transforms a sequence s_n into the sequence

$$t_n = \sum_{i=0}^{\infty} a_{ni} s_i,$$

then, the necessary and sufficient conditions which ensure that each convergence sequence s_n to be transformed into convergent sequence t_n are the following:

1) $\sum_{i=0}^\infty |a_{ni}| < M, \quad n = 0, 1, 2, \ldots,$

2) $\lim_{n\to\infty} a_{ni} = a_i$ to exist for $i = 0, 1, 2, \ldots,$

3) $\lim_{n\to\infty}\sum_{i=0}^{\infty} a_{ni} = A$ to exist.

Then,

$$\lim_{n\to\infty} t_n = \left(A - \sum_{i=0}^{\infty} a_i\right) \lim_{n\to\infty} s_n + \sum_{i=0}^{\infty} s_i a_i.$$

It is easily seen that if $\Re s > 0$, then the term $S_m^k l_{m,m}(s)$ in (2.13) tends to zero when $m \to \infty$. Indeed, by assumption S_m^k tends to a finite limit and

$$l_{m,m}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(m+1)t}\, dt$$

obviously tends to zero when $m \to \infty$.

It remains the theorem of Toeplitz to be applied for the expression

$$S_0^k h_{m,0}(s) + S_1^k h_{m,1}(s) + \cdots + S_{m-1}^k h_{m,m-1}(s).$$

From (2.11) it is immediately seen that if $\Re s > 0$ and μ is fixed, then $l_{m,\mu}$ tends to the limit

$$(2.14)\qquad l_\mu(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(\mu+1)t} \sum_{\tau=0}^{\infty} \binom{\mu+\tau}{\tau} (1-e^{-t})^\tau \left(\frac{q}{q+1}\right)^\tau dt$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(\mu+1)t} \left(1 - (1-e^{-t})\frac{q}{q+1}\right)^{-\mu-1} dt$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\frac{q+1}{q+e^t}\right)^{\mu+1} dt,$$

when $m \to \infty$ and, hence, $h_{m,i}(s)$ tends to $h_i = l_i(s) - l_{i+1}(s)$. Since

$$\sum_{i=0}^{m-1} h_{m,i}(s) = l_{m,0}(s) - l_{m,m}(s),$$

the left side tends to

$$l_0(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{q+1}{q+e^t}\, dt$$

when $m \to \infty$ and from (2.14) it is seen that $h_0(s) = l_0(s)$.

It is at hand now to be studied whether the condition 1) of Toeplitz's theorem is fulfilled, i.e. whether the sequence

$$\{|h_{m,0}(s)| + |h_{m,1}(s)| + \cdots + |h_{m,m-1}(s)|\}_{m=1}^{\infty}$$

is bounded. To that end the function

$$g_\mu(z) = \sum_{\tau=0}^{m-\mu} \binom{\mu+\tau}{\tau} z^\tau$$

has to be considered. It is easily seen that

$$g_\mu(z) = \frac{1}{\mu!}\frac{d^\mu}{dz^\mu}(1+z+z^2+\dots+z^m) = \frac{1}{\mu!}\frac{d^\mu}{dz^\mu}\{(1-z^{m+1})(1-z)^{-1}\}$$

and Leibnitz's rule yields that

$$g_\mu(z) = \frac{1}{\mu!}\sum_{k=0}^{\mu}\binom{\mu}{k} u^{(k)}(z) v^{(\mu-k)}(z),$$

where $u(z) = 1 - z^{m+1}, v(z) = (1-z)^{-1}$.

The general term in (2.12), i.e.

$$\frac{1}{\mu!}\binom{\mu}{k} u^{(k)}(z) v^{(\mu-k)}(z),$$

is equal to

$$-\frac{1}{\mu!}\binom{\mu}{k}(m+1)m(m-1)\dots(m-k+2)z^{m+1-k}\frac{(\mu-k)!}{(1-z)^{\mu-k+1}}$$

$$= -\frac{1}{\mu!}\cdot\frac{\mu!}{k!(\mu-k)!}\cdot\frac{(m+1)!}{(m-k+1)!}(\mu-k)!\frac{z^{m+1-k}}{(1-z)^{\mu-k+1}}$$

$$= -\binom{m+1}{k}\frac{z^{m+1-k}}{(1-z)^{\mu-k+1}},$$

and hence,

$$g_n(z) = -\sum_{k=0}^{\mu}\binom{m+1}{k}\frac{z^{m+1-k}}{(1-z)^{\mu-k+1}}. \tag{2.15}$$

If

$$z = (1-e^{-t})\frac{q}{q+1},$$

then $l_{m,\mu}(s)$ can be written as follows

$$l_{m,\mu} = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(\mu+1)t}g_\mu(z)\,dt.$$

From here one gets that

$$h_{m,\mu}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t} \{e^{-\mu t} g_\mu(z) - e^{-(\mu+1)t} g_{\mu+1}(z)\}\, dt,$$

whence

$$|h_{m,\mu}(s)| \le \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} e^{-t} |e^{-\mu t} g_\mu(z) - e^{-(\mu+1)t} g_{\mu+1}(z)|\, dt, \ \sigma = \Re s.$$

Then,

$$(2.16) \quad |h_{m,\mu}(s)| \le \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} e^{-2t} \left| \frac{(1-z)e^t - 1}{1-z} \right| \sum_{k=0}^{\mu} \binom{m+1}{k} \frac{z^{m-k+1} e^{-\mu t}}{(1-z)^{\mu-k+1}}\, dt + \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} \frac{e^{-(\mu+2)t}}{1-z} \binom{m+1}{\mu+1} z^{m-\mu}\, dt = \alpha_\mu(s) + \beta_\mu(s),$$

and hence,

$$\sum_{\mu=0}^{m-1} |h_{m,\mu}(s)| \le \sum_{\mu=0}^{m-1} \alpha_\mu(s) + \sum_{\mu=0}^{m-1} \beta_\mu(s) = P_m(s) + Q_m(s).$$

Further,

$$|Q_m(s)| = \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} \sum_{\mu=0}^{m+1} \frac{e^{-(\mu+2)t}}{1-z} \binom{m+1}{\mu+1} z^{m-\mu}\, dt < \frac{1}{|\Gamma(s)|} \int_0^\infty \frac{t^{\sigma-1} e^{-t}}{1-z} \sum_{\mu=0}^{m+1} \binom{m+1}{\mu} z^{m-\mu+1} e^{-\mu t}\, dt = \frac{1}{|\Gamma(s)|} \int_0^\infty \frac{t^{\sigma-1} e^{-t}}{1-z} (e^{-t} + z)^{m+1}\, dt.$$

But

$$z = (1 - e^{-t}) \frac{q}{q+1} \le \frac{q}{q+1}, \quad 0 \le t < \infty,$$

i.e., $1 - z \ge (q+1)^{-1},\ 0 \le t < \infty$, and hence, for the function

$$\psi(t) = e^{-t} + z = e^{-t} + (1 - e^{-t}) \frac{q}{q+1} = \frac{q + e^{-t}}{q+1}$$

it follows that $\psi(t) \leq 1$, $0 \leq t < \infty$. Therefore,

$$|Q_m(s)| \leq \frac{q+1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} e^{-t}\, dt = \frac{(q+1)\Gamma(\sigma)}{|\Gamma(s)|},$$

i.e., the sequence $\{|Q_m(s)|\}_{m=0}^\infty$ is bounded.

For the sum $P_m(s)$ one has that

$$(2.17) \qquad |P_m(s)|$$

$$= \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} e^{-1} \frac{1-z-e^{-t}}{1-z} \sum_{\mu=0}^{m-1} \sum_{k=0}^{\mu} \binom{m+1}{k} \frac{z^{m+1-k} e^{-\mu t}}{(1-z)^{\mu+1-k}}\, dt$$

$$= \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} e^{-t} \frac{1-z-e^{-t}}{1-z} \sum_{k=0}^{m-1} \sum_{\mu=k}^{m-1} \binom{m+1}{k} \frac{z^{m+1-k} e^{-kt}}{(1-z)^{\mu+1-k}}\, dt$$

$$= \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} e^{-t} \sum_{k=0}^{m-1} \binom{m+1}{k} z^{m+1-k} e^{-kt}\, dt$$

$$- \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} e^{-t} \sum_{k=0}^{m-1} e^{-kt} \binom{m+1}{k} \frac{e^{-mt} z^{m+1-k}}{(1-z)^{m+1-k}}\, dt$$

$$< \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} e^{-t} \sum_{k=0}^{m+1} \binom{m+1}{k} z^{m+1-k} e^{-kt}\, dt$$

$$< \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} e^{-t} (\psi(t))^{m+1}\, dt,$$

hence

$$|P_m(s)| < \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} e^{-t}\, dt = \frac{\Gamma(\sigma)}{|\Gamma(s)|}.$$

Thus it is proved that the sequence (2.16) is bounded, i.e., the conditions of Toeplitz's theorem are completely satisfied. But if μ is fixed, then it follows that

$$\lim_{m\to\infty} h_{m,\mu}(s) = l_\mu(s) - l_{\mu+1}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^t - 1}{e^t + q} \left(\frac{q+1}{q+e^t} \right)^{\mu+1} dt.$$

Hence, the series (2.4) is E_k-summable for each s such that $\Re s > 0$ with sum $f(s)$ given by (2.5) with $s_0 = 0$, i.e.,

$$f(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^t - 1}{e^t + q} \sum_{\mu=0}^\infty S_\mu \left(\frac{q+1}{q+e^t} \right)^{\mu+1} dt.$$

From the proof it is seen that the E_k-summation of the series (2.4) is uniform on each compact subset of the half-plane $\{s : \Re s > 0\}$, i.e., the representation (2.5) holds uniformly on each such subset of the half-plane $\{s : \Re s > \Re s_0\}$.

From the latter theorem it follows that there exists a real number e_k with the property that the series (2.1) is E_k-summable for each s such that $\Re s > e_k$ and it is not E_k-summable if $\Re s < e_k$. This number is called, as it is assumed, the abscissa of the E_k-summation for the series (2.4).

Theorem 8. *If the series (2.4) is E_k-summable for $s = s_0 = \sigma_0 + it_0$ and ε is an arbitrary positive number, then*

$$f(s) = O\left(\frac{\Gamma(\sigma - \sigma_0)}{|\Gamma(s - s_0)|}\right), \quad s = \sigma + it, \tag{2.18}$$

provided that $\sigma \geq \sigma_0 + \varepsilon$.

It can be assumed that $s_0 = 0$ and then (2.18) is a consequence of the representation (2.13). Indeed, the sequence $\{S_n^k\}_{n=0}^{\infty}$ is bounded since it is convergent, i.e., $|S_n^k| \leq A < \infty,\ n \in \mathbb{N}$. Then,

$$|f(s)| \leq \frac{A}{|\Gamma(s)|} \int_0^{\infty} t^{\sigma-1} \frac{e^t - 1}{e^t + q} \sum_{\mu=0}^{\infty} \left(\frac{q+1}{q+e^t}\right)^{\mu+1} dt.$$

If $t > 0$, then

$$\frac{e^t - 1}{e^t + q} \sum_{\mu=0}^{\infty} \left(\frac{q+1}{q+e^t}\right)^{\mu+1} = \frac{e^t - 1}{e^t + q} \frac{q+1}{q+e^t} \frac{1}{1 - \dfrac{q+1}{q+e^t}} = \frac{q+1}{q+e^t},$$

whence

$$|f(s)| \leq \frac{A}{|\Gamma(s)|} \int_0^{\infty} t^{\sigma-1} \frac{q+1}{q+e^t}\, dt = \frac{A}{|\Gamma(s)|} \int_0^{\infty} t^{\sigma-1} e^{-t} \frac{q+1}{qe^{-t}+1}\, dt$$
$$\leq \frac{A(q+1)}{|\Gamma(s)|} \int_0^{\infty} t^{\sigma-1} t^{\sigma-1}\, dt = A(q+1) \frac{\Gamma(\sigma)}{|\Gamma(s)|}.$$

The series (2.1) is called absolutely E_k-summable, shortly $|E_k|$-summable, if the series (2.2) is absolutely convergent.

Theorem 9. *If the series (2.4) is $|E_k|$-summable for $s = s_0$, then it is $|E_k|$-summable for each s with $\Re s > \Re s_0$.*

It can be supposed that $s_0 = 0$, i.e., that the series (2.7) is convergent for $s_0 = 0$ which means that the series $\sum_{n=0}^{\infty} |A_n^k|$ is convergent. One has to prove that the series $\sum_{n=0}^{\infty} |A_n^k(s)|$ is convergent provided that $\Re s = \sigma > 0$. From (2.10) it follows that

$$|A_n^k(s)| \\ \leq \frac{1}{|\Gamma(s)|}\left(\frac{q}{q+1}\right)^n \sum_{\mu 0}^{n} |A_\mu^k| \left(\frac{q+1}{q}\right)^\mu \binom{n}{\mu} \int_0^\infty t^{\sigma-1} e^{-(\mu+1)t}(1-e^{-t})^{n-\mu}\,dt,$$

whence, as before,

$$\begin{aligned}
&\sum_{n=0}^{m} |A_n^k(s)| \\
&\quad \leq \frac{1}{|\Gamma(s)|}\sum_{\mu=0}^{m} |A_\mu^k| \int_0^\infty t^{\sigma-1} e^{-(\mu+1)t} \sum_{\tau=0}^{m-\mu} \binom{\mu+\tau}{\tau}(1-e^{-t})^\tau \left(\frac{q}{q+1}\right)^\tau dt \\
&\quad < \int_0^\infty t^{\sigma-1} e^{-(\mu+1)t} \sum_{\tau=0}^{\infty} \binom{\mu+\tau}{\tau}(1-e^{-t})^\tau \left(\frac{q}{q+1}\right)^\tau dt \\
&\quad = \int_0^\infty t^{\sigma-1}\left(\frac{q+1}{q+e^t}\right)^{\mu+1} dt \\
&\quad \leq \int_0^\infty t^{\sigma-1}\frac{q+1}{q+e^t}\,dt = \int_0^\infty t^{\sigma-1} e^{-t}\frac{(q+1)e^t}{q+e^t}\,dt \\
&\quad \leq (q+1)\int_0^\infty t^{\sigma-1} e^{-t}\,dt = (q+1)\Gamma(\sigma).
\end{aligned}$$

Hence,

$$\sum_{n=0}^{m} |A_n^k(s) \leq \frac{(q+1)\Gamma(\sigma)}{|\Gamma(s)|}\sum_{\mu=0}^{m} |A_\mu^k| \leq \frac{(q+1)\Gamma(\sigma)}{|\Gamma(s)|}\sum_{\mu=0}^{\infty} |A_\mu^k|,$$

i.e., the series (2.7) is absolutely convergent when $\Re s > 0$ and thus the theorem is proved.

In this case, i.e., when $s_0 = 0$, another representation can be given to the function (2.5). First, from (2.11) it follows that

$$|l_{n,\mu}(s)| \leq \frac{1}{|\Gamma(s)|}\int_0^\infty t^{\sigma-1}\sum_{\tau=0}^{\infty}\binom{\mu+\tau}{\tau}(1-e^{-t})\left(\frac{q}{q+1}\right)^\tau dt \leq (q+1)\frac{\Gamma(\sigma)}{|\Gamma(s)|}\,.$$

Let ε be an arbitrary positive number and λ be such that

$$\sum_{n=\lambda+1}^{\infty} |A_n^k| < \varepsilon$$

and let

$$S_n^k(s) = \sum_{\mu=0}^{\lambda} A_\mu^k l_{n,\mu}(s) + \sum_{\mu=\lambda+1}^{\infty} = A_\mu^k l_{n,\mu} = P_{n,\lambda}(s) + Q_{n,\lambda}(s).$$

Then,

$$|Q_{n,\lambda}(s)| \le (q+1)\frac{\Gamma(\sigma)}{|\Gamma(s)|} \sum_{\mu=\lambda+1} |A_\mu^k| < \varepsilon(q+1)\frac{\Gamma(\sigma)}{|\Gamma(s)|}.$$

Further, if λ is fixed, then

$$\lim_{n\to\infty} P_{n,\lambda}(s) = \sum_{\mu=0}^{\lambda} A_\mu^k l_\mu(s)$$

and, hence,

$$\lim_{n\to\infty} S_n^k(s) = \sum_{\mu=0}^{\infty} A_\mu^k l_\mu(s),$$

i.e.,

$$f(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{\mu=0}^{\infty} A_\mu^k \left(\frac{q+1}{q+e^t}\right)^{\mu+1} dt, \quad \Re s > 0.$$

If s_0 is not equal to zero, then

$$f(s) = \frac{1}{\Gamma(s-s_0)} \int_0^\infty t^{s-s_0-1} \sum_{\mu=0}^{\infty} A_\mu^k(s_0) \left(\frac{q+1}{q+e^t}\right) dt, \quad \Re s > \Re s_0.$$

From the theorem just proved it follows that there exists an abscissa $\overline{e}_k$ for the $|E_k|$-summation. That means $\overline{e}_k$ is a real number with the property that the series (2.4) is $|E_k|$-summable if $\Re s > \overline{e}_k$ and it is not $|E_k|$-summable when $\Re s < \overline{e}_k$. Since each E_k-summable series is $|E_{k_1}|$-summable if $k_1 > k$ and each $|E_k|$-summable series is $|E_{k_1}|$-summable when $k_1 > k$, as it was already established, it follows that $e_k \le \overline{e}_k$ and $\overline{e}_{k_1} \le \overline{e}_k$, $k_1 > k$.

Theorem 10. *If* $e_k \ge 0$, *then*

$$e_k = \limsup_{n\to\infty} \frac{\log |A_0^k + A_1^k + \cdots + A_n^k|}{\log n}. \tag{2.19}$$

Let α be the right-hand side of (2.19) and let $\alpha \geq 0$. Then for each $\varepsilon > 0$ there exists $K = K(\varepsilon) > 0$ such that

$$\left|\sum_{\nu=0}^{n} A_\nu^k\right| = |S_n^k| \leq K(n+1)^{\alpha+\varepsilon}, \quad n = 0, 1, 2, \dots . \tag{2.20}$$

A consequence of the last inequalities is that if $\Re s = \sigma > \alpha + \varepsilon$, then the series

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{e^t - 1}{e^t + q} \sum_{\mu=0}^{\infty} S_\mu^k \left(\frac{q+1}{q+e^t}\right) dt \tag{2.21}$$

is absolutely convergent.

This series is majorized by the series

$$\frac{1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} \frac{e^t - 1}{e^t + q} \sum_{\mu=0}^{\infty} (\mu+1)^{\alpha+\varepsilon} \left(\frac{q+1}{q+e^t}\right)^{\mu+1} dt.$$

It is easily seen by induction that if $|x| < 1$, then

$$\sum_{n=1}^{\infty} n^p x^n = R_p(u), \quad u = \frac{1}{1-x}, \quad p = 1, 2, 3, \dots ,$$

where R_p is a polynomial with $\deg R_p = p+2$ and $R_p(0) = 0$. Hence, if $p \geq \alpha + \varepsilon$, then

$$\sum_{\mu=0}^{\infty} (\mu+1)^{\alpha+\varepsilon} \left(\frac{q+1}{q+e^t}\right)^{\mu+1}$$
$$\leq \sum_{\mu=0}^{\infty} (\mu+1)^p \left(\frac{q+1}{q+e^t}\right)^{\mu+1} = R_p\left(\frac{q+e^t}{e^t-1}\right), \ t > 0.$$

But

$$R_p\left(\frac{q+e^t}{e^t-1}\right) = O(t^{-p-2}), \quad t \to 0,$$

and hence,

$$t^{\sigma-1} \frac{e^t - 1}{e^t + q} \sum_{\mu=0}^{\infty} (\mu+1)^{\alpha+\varepsilon} \left(\frac{q+1}{q+e^t}\right)^{\mu+1}$$
$$= O(t^{\sigma-p-1}) = O(t^{\sigma-1-\alpha-\varepsilon}), \quad t \to 0,$$

i.e., the integral in (2.17) really exists when $\sigma > \alpha + \varepsilon$.

It is at hand now to be proved that $S_m^k(s)$ tends to the sum of the series (2.21) when $m \to \infty$ provided that $\Re s > \alpha + 2\varepsilon$ and it is sufficient to establish this when s is real. The starting point is again the representation

$$l_{m,m}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-(m+1)t}\, dt = \frac{1}{(m+1)^s}.$$

From the last equality it follows now that $\lim_{m\to\infty} S_m^k l_{m.m}(s) = 0$ when $\Re s > \alpha + \varepsilon$ and as in the proof of Theorem 1, one has to establish that if $\widetilde{h}_{m,\mu}(s) = (\mu+1)^{\alpha+2\varepsilon} h_{m,\mu}(s)$, $\mu = 0, 1, 2, \ldots, m-1$, then the sequence

$$\{|\widetilde{h}_{m,0}(s)| + |\widetilde{h}_{m,1}(s)| + \cdots + |\widetilde{h}_{m,m-1}(s)|\}_{m=1}^\infty \tag{2.22}$$

is bounded for each s with $\Re s > \alpha + 2\varepsilon$.

From the inequality (2.16) it follows that

$$\sum_{\mu=0}^{m-1} |\widetilde{h}_{m,\mu}(s)| \le \sum_{\mu=0}^{m-1} (\mu+1)^{\alpha+2\varepsilon} \alpha_\mu(s) + \sum_{\mu=0}^{m-1} (\mu+1)^{\alpha+2\varepsilon} \beta_\mu(s) \tag{2.23}$$

$$= \widetilde{P}_m(s) + \widetilde{Q}_m(s).$$

Further,

$$\widetilde{Q}_m(s) \le m^{\alpha+2\varepsilon} Q_m(s) < \frac{m^{\alpha+2\varepsilon}}{|\Gamma(s)|} \int_0^\infty \frac{t^{\sigma-1} e^{-t}}{1-z} (e^{-t} + z)^{m+1}\, dt,$$

where $z = (1 - e^{-t}) \dfrac{q}{q+1}$. Since $1 - z \le \dfrac{1}{q+1}$,

$$\widetilde{Q}_m(s) \le \frac{(q+1) m^{\alpha+2\varepsilon}}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} e^{-t} \left(\frac{e^{-t} + q}{1+q} \right)^{m+1} dt.$$

If $\lambda = \dfrac{1}{e(q+1)}$, then the right-hand side can be written in the form

$$\widetilde{Q}_m(s) \le \widetilde{Q}_m^{(1)}(s) + \widetilde{Q}_m^{(2)}(s),$$

where

$$\widetilde{Q}_m^{(1)}(s) = \frac{(q+1)(m+1)^{\alpha+2\varepsilon}}{|\Gamma(s)|} \int_0^\lambda t^{\sigma-1} e^{-t} \left(\frac{e^{-t} + q}{1+q} \right)^{m+1} dt,$$

$$\widetilde{Q}_m^{(2)}(s) = \frac{(q+1)(m+1)^{\alpha+2\varepsilon}}{|\Gamma(s)|} \int_\lambda^\infty t^{\sigma-1} e^{-t} \left(\frac{e^{-t}+q}{1+q} \right)^{m+1} dt.$$

Since

$$\frac{e^{-t}+q}{1+q} \le \frac{e^{-\lambda}+q}{1+q} < 1, \quad \lambda \le t < \infty,$$

from the inequality

$$\widetilde{Q}_m^{(2)}(s) \le \frac{(q+1)\Gamma(\sigma)}{|\Gamma(s)|}(m+1)^{\alpha+2\varepsilon} \left(\frac{e^{-\lambda}+q}{1+q} \right)^{m+1},$$

it follows that $\lim_{m\to\infty} \widetilde{Q}_m^{(2)}(s) = 0$ for each s with $\Re s > 0$ and, hence, the sequence $\{\widetilde{Q}_m^{(2)}(s)\}_{m=0}^\infty$ is bounded when $\Re s > 0$.

It is easily seen that if $0 \le t \le 1$, then

$$\frac{e^{-t}+q}{1+q} \le 1 - \lambda t.$$

Indeed, the function

$$\varphi(t) = \frac{e^{-t}+q}{1+q} + \lambda t - 1, \quad 0 \le t \le 1,$$

is increasing and therefore $\varphi(t) \le \varphi(0)$, i.e., $\varphi(t) \le 0,\ 0 \le t \le 1$.

Further, since $\lambda < 1/\lambda$,

$$\widetilde{Q}_m^{(1)}(s) \le (m+1)^{\alpha+2\varepsilon} \int_0^{1/\lambda} t^{\sigma-1} e^{-t} (1-\lambda t)^{m+1}\, dt$$

$$\le (m+1)^{\alpha+2\varepsilon} \int_0^{1/\lambda} t^{\sigma-1} (1-\lambda t)^{m+1}\, dt$$

$$= \lambda^{-\sigma} \int_0^1 t^{\sigma-1}(1-t)^{m+1}\, dt = \frac{\lambda^{-\sigma}(m+1)^{\alpha+2\varepsilon}\Gamma(\sigma)\Gamma(m+2)}{\Gamma(\sigma+m+2)},$$

whence

$$\widetilde{Q}_m^{(1)}(s) = O((m+1)^{\alpha+2\varepsilon}(m+1)^{-\sigma}) = o(1), \quad m \to \infty,$$

provided that $\sigma > \alpha + 2\varepsilon$.

In a similar way it can be proved that the sequence $\{\widetilde{P}_m(s)\}_{m=0}^\infty$ is bounded when $\Re s > \alpha + 2\varepsilon$ and then (2.22) yields that the sequence (2.20) is bounded for each such s. But ε is an arbitrary positive number, hence, the series (2.4) is E_k-summable for each s with $\Re s > \alpha$.

Conversely, let the series (2.4) be E_k-summable for $s = \alpha \geq 0$, i.e., if

$$A_n^k(\alpha) = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} a_\nu (\nu+1)^{-\alpha},$$

then the series

$$\sum_{n=0}^{\infty} A_n^k(\alpha)$$

is convergent. Then the equality (2.7) yields that

$$a_\nu = (-1)^\nu (q+1) q^\nu (\nu+1)^\alpha \sum_{\mu=0}^{\nu} (-1)^\mu \binom{\nu}{\mu} \left(\frac{q+1}{q}\right)^\mu A_\mu^k(\alpha),$$

and hence,

$$A_n^k = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} q^{n-\nu} a_\nu$$

$$= \left(\frac{q}{q+1}\right)^n \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (\nu+1)^\alpha \sum_{\mu=0}^{\nu} (-1)^\mu \binom{\nu}{\mu} \left(\frac{q+1}{q}\right)^\mu A_\mu^k(\alpha)$$

$$= \left(\frac{q}{q+1}\right)^n \sum_{\mu=0}^{n} (-1)^\mu A_\mu^k(\alpha) \left(\frac{q+1}{q}\right)^\mu \sum_{\nu=\mu}^{n} (-1)^\nu \binom{n}{\nu} \binom{\nu}{\mu} (\nu+1)^\alpha$$

$$= \left(\frac{q}{q+1}\right)^n \sum_{\mu=0}^{n} \binom{n}{\mu} \left(\frac{q+1}{q}\right)^\mu A_\mu^k(\alpha) \sum_{\tau=0}^{n-\mu} (-1)^\tau \binom{n-\mu}{\tau} (\mu+\tau+1)^\alpha.$$

If $\Delta w, \Delta^2 w, \ldots$ are the difference sequences for the sequence $\{w_m\}_{m=0}^{\infty}$, then

$$\sum_{\tau=0}^{n-\mu} (-1)^\tau \binom{n-\mu}{\tau} (\mu+\tau+1)^\alpha = \Delta^{n-\mu} (\mu+1)^\alpha,$$

and hence,

$$A_n^k = \left(\frac{q}{q+1}\right)^n \sum_{\mu=0}^{n} A_\mu^k(\alpha) \left(\frac{q+1}{q}\right)^\mu \binom{n}{\mu} \Delta^{n-\mu} (\mu+1)^\alpha.$$

From here one gets that

$$S_m^k = \sum_{n=0}^{m} A_n^k = \sum_{n=0}^{m} \left(\frac{q}{q+1}\right)^n \sum_{\mu=0}^{n} A_\mu^k(\alpha) \left(\frac{q+1}{q}\right)^\mu \binom{n}{\mu} \Delta^{n-\mu} (\mu+1)^\alpha$$

$$= \sum_{\mu=0}^{m} A_\mu^k(\alpha) \sum_{\nu=0}^{m-\mu} \left(\frac{q}{q+1}\right)^\nu \binom{\mu+\nu}{\nu} \Delta^\nu[(\mu+1)^\alpha].$$

If α is a positive integer, then $\Delta^\nu(\mu+1)^\alpha = 0$ when $\nu < \alpha$. Hence, in this case

$$S_m^k(\alpha) = \sum_{\mu=0}^{m} A_m^k(\alpha) \sum_{\nu=0}^{\min(\alpha, m-\mu)} \left(\frac{q}{q+1}\right)^\nu \binom{\mu+\nu}{\nu} \Delta^\nu(\mu+1)^\alpha,$$

i.e.,

$$S_m^k(\alpha) = \sum_{\mu=0}^{m-\alpha} A_\mu^k(\alpha) \sum_{\nu=0}^{\alpha} \left(\frac{q}{q+1}\right)^\nu \binom{\mu+\nu}{\nu} \Delta^\nu(\mu+1)^\alpha$$

$$+ \sum_{\mu=m-\alpha+1}^{m} A_m^k(\alpha) \sum_{\nu=0}^{m-\mu} \left(\frac{q}{q+1}\right)^\nu \binom{\mu+\nu}{\nu} \Delta^\nu(\mu+1)^\alpha$$

$$= F_m(\alpha) + G_m(\alpha), \quad m \geq \alpha - 1.$$

But $\Delta^\nu(\mu+1)^\alpha$ is a polynomial of μ of degree $\alpha - \nu$ and, hence, there exists a constant L such that

$$|\Delta^\nu(\mu+1)^\alpha| \leq L\mu^{\alpha-\nu}, \quad \nu = 0, 1, 2, \ldots, \alpha.$$

Since the series $\sum_{n=0}^\infty A_n^k(\alpha)$ is convergent, the sequence

$$\left\{ S_{m-\alpha}^k(\alpha) = \sum_{\mu=0}^{m-\alpha} A_\mu^k(\alpha) \right\}_{m=\alpha}^\infty$$

is bounded, i.e., $|S_{m-\mu}^k(\alpha)| \leq K$ and, hence,

$$|F_m(\alpha)| < K_1 L \sum_{\nu=0}^{\alpha} \left(\frac{q}{q+1}\right)^\nu \binom{\mu+\nu}{\nu} \mu^{\alpha-\nu} < L_1 m^\alpha.$$

Since in $G_m(\alpha)$ there are α summands, i.e., a finite number not depending on α and, moreover, $|A_\mu^k(\alpha)| < K_2$, it holds the inequality $|G_m(\alpha)| < L_2 m^\alpha$. Further, the inequality $|S_m^k| < (L_1 + L_2)m^\alpha$ yields that

$$\limsup_{m\to\infty} \frac{\log |S_m^k|}{\log m} \leq \alpha,$$

and thus the proof when α is a positive integer is finished.

The case when $\alpha > 0$ is not an integer requires considerable efforts. If $p = [\alpha] + 1$, i.e., $\alpha < p < \alpha + 1$, then it is easy to see that the integral

$$g_p(\alpha) = \int_0^\infty \left(e^{-t} - 1 + t - \frac{t^2}{2!} + \cdots + (-1)^p \frac{t^{p-1}}{(p-1)!}\right) t^{-\alpha-1}\, dt \tag{2.24}$$

is absolutely convergent. Indeed, the integrand is $O(t^{p-\alpha-1}), p - \alpha - 1 > -1$, when $t \to 0$ and it is $O(t^{p-\alpha-2}), p - \alpha - 2 < -1$, when $\alpha \to \infty$. After setting nt instead of t, one gets that

$$n^\alpha = \frac{1}{g(\alpha)} \int_0^\infty \left(e^{-nt} - 1 + nt - \cdots + (-1)^{p-1} \frac{n^{p-1}t^{p-1}}{(p-1)!}\right) t^{-\alpha-1}\, dt, \tag{2.25}$$

whence

$$\Delta^\nu(n^\alpha) = \frac{1}{g(\alpha)} \int_0^\infty e^{-nt}(1 - e^{-t})^\nu t^{-\alpha-1}\, dt, \tag{2.26}$$

provided that $\nu \geq p$. Then,

$$\begin{aligned} S_m^k &= \sum_{\mu=0}^{m-p} A_\mu^k(\alpha) \sum_{\nu=0}^{p-1} \left(\frac{q}{q+1}\right)^\nu \binom{\mu+\nu}{\nu} \Delta^\nu(\mu+1)^\alpha \\ &+ \sum_{\mu=0}^{m-p} A_\mu^k(\alpha) \sum_{\nu=p}^{m-\mu} \left(\frac{q}{q+1}\right)^\nu \Delta^\nu(\mu+1)^\alpha \\ &+ \sum_{\mu=m-p+1}^{m} A_\mu^k(\alpha) \sum_{\nu=0}^{m-\mu} \left(\frac{q}{q+1}\right)^\nu \binom{\mu+\nu}{\nu} \Delta^\nu(\mu+1)^\alpha \\ &= K_m(\alpha) + L_m(\alpha) + M_m(\alpha). \end{aligned}$$

Then,

$$\begin{aligned} L_m(\alpha) &= \sum_{\mu=0}^{m-p} \frac{A_\mu^k(\alpha)}{g(\alpha)} \int_0^\infty \sum_{\nu=p}^{m-\mu} \left(\frac{q}{q+1}\right)^\nu \binom{\mu+\nu}{\nu} e^{-(\mu+1)t}(1 - e^{-t}) t^{-\alpha-1}\, dt \\ &= \sum_{\mu=0}^{m-p} \frac{A_\mu^k(\alpha)}{g(\alpha)} \int_0^\infty t^{-\alpha-1} g_{\mu,p}(z) e^{-(\mu+1)t}\, dt, \end{aligned}$$

where

$$g_{\mu,p}(z) = \sum_{\nu=p}^{m-\mu} \binom{\mu+\nu}{\nu} z^{\nu}, \quad z = (1-e^{-t})\frac{q}{q+1}, \tag{2.27}$$

i.e.,

$$g_{\mu,p}(z) = -\sum_{\nu=0}^{\mu} \binom{m+1}{\nu} \frac{z^{m+1-\nu}}{(1-z)^{\mu+1-\nu}} + \sum_{\nu=0}^{\mu} \binom{\mu+p}{\nu} \frac{z^{\mu+p-\nu}}{(1-z)^{\mu+1-\nu}}$$

$$= g^{(1)}_{\mu,p}(z) - g^{(2)}_{\mu,p}(z).$$

It has to be noted that in fact $g^{(2)}_{\mu,p}(z)$ does not depend on m. Further,

$$l_{m,\mu}(\alpha) = \frac{1}{g(\alpha)} \int_0^{\infty} t^{-\alpha-1} e^{-(\mu+1)t} g^{(1)}_{\mu,p}(z)\, dt \tag{2.28}$$

$$-\frac{1}{g(\alpha)} \int_0^{\infty} t^{-\alpha-1} e^{-(\mu+1)t} g^{(2)}_{\mu,p}(z)\, dt = l^{(1)}_{m,\mu}(\alpha) - l^{(2)}_{m,\mu}(\alpha),$$

then

$$L_{m,p}(\alpha) = \sum_{\mu=0}^{m-p} A^k_m(\alpha) l^{(1)}_{m,\mu}(\alpha) - \sum_{\mu=0}^{m-p} A^k_{\mu}(\alpha) A^k_{\mu} l^{(2)}_{m,\mu}(\alpha) \tag{2.29}$$

$$= L^{(1)}_{m,p}(\alpha) - L^{(2)}_m(\alpha),$$

and

$$L^{(1)}_{m,p}(\alpha) = \sum_{\mu=0}^{m-p-1} S^k_{\mu}(\alpha) h^{(1)}_{m,\mu}(\alpha) + S^k_{m-p}(\alpha) l^{(1)}_{m,m-p}(\alpha), \tag{2.30}$$

$$h^{(1)}_{m,\mu}(\alpha) = l^{(1)}_{m,\mu}(\alpha) - l^{(1)}_{m,\mu+1}(\alpha),$$

$$L^{(2)}_m(\alpha) = \sum_{\mu=0}^{m-p-1} S^k_{\mu}(\alpha) h^{(2)}_{m,\mu}(\alpha) + S^k_{m-p}(\alpha) l^{(2)}_{m,m-p}(\alpha), \tag{2.31}$$

$$h^{(2)}_{m,\mu}(\alpha) = l^{(2)}_{m,\mu}(\alpha) - l^{(2)}_{m,\mu+1}(\alpha).$$

Further,

$$S^k_{m-p}(\alpha)l^{(1)}_{m,m-p}(\alpha)+S^k_{m-p}(\alpha)l^{(2)}_{m,m-p}(\alpha)=S^k_{m-p}(\alpha)l_{m,m-p}$$

$$=S^k_{m-p}(\alpha)\binom{m}{m-p}\frac{1}{g(\alpha)}\int_0^\infty t^{-\alpha-1}e^{-(m-p+1)t}z^p\,dt$$

$$=O(m^p)\int_0^\infty t^{p-\alpha-1}e^{-(m-p+1)t}\,dt=O(m^\alpha).$$

For the sum in $L^{(2)}_m(\alpha)$ it holds the estimate

$$\Big|\sum_{\mu=0}^{m-p-1}S^k_\mu(\alpha)h^{(1)}_{m,\mu}(\alpha)\Big|<S\sum_{\mu=0}^{m-p-1}|h^{(1)}_{m,\mu}(\alpha)|,\quad S=Const.$$

For $h^{(1)}_{m,\mu}(\alpha)$ one gets that

$$|h^{(1)}_{m,\mu}(\alpha)|\le\frac{1}{|g(\alpha)|}\int_0^\infty t^{-\alpha-1}e^{-t}|e^{-\mu t}g^{(1)}_{\mu,p}(z)-e^{-(\mu+1)t}g_{\mu+1,p}(z)|\,dt$$

$$\le\frac{1}{|g(\alpha)|}\int_0^\infty t^{-\alpha-1}e^{-t}\Big|\frac{1-z-e^{-t}}{1-z}\Big|\sum_{\nu=0}^{\mu}\binom{m+1}{\nu}\frac{z^{m-\nu+1}e^{-\mu t}}{(1-z)^{\mu-\nu+1}}\,dt$$

$$+\frac{1}{|g(\alpha)|}\int_0^\infty t^{-\alpha-1}\frac{e^{-(\mu+2)t}}{1-z}\binom{m+1}{\mu+1}z^{m-\mu}\,dt$$

$$=A_\mu(\alpha)+B_\mu(\alpha),\quad \mu=0,1,2,\dots,m-p-1,$$

whence

$$\sum_{\mu=0}^{m-p-1}|h^{(1)}_{m,\mu}(\alpha)|\le\sum_{\mu=0}^{m-p-1}A_\mu(\alpha)+\sum_{\mu=0}^{m-p-1}A_\mu(\alpha)=\widetilde{A}_m(\alpha)+\widetilde{B}_m(\alpha).$$

Further,

$$\widetilde{B}_m(\alpha)=\frac{1}{|g(\alpha)|}\int_0^\infty\frac{t^{-\alpha-1}}{1-z}\sum_{\mu=0}^{m-p-1}\binom{m+1}{\mu+1}z^{m-\mu}e^{-(\mu+2)t}\,dt$$

$$=\frac{1}{|g(\alpha)|}\int_0^\infty\frac{t^{-\alpha-1}}{1-z}\sum_{\nu=1}^{m-p}z^{m-\nu+1}e^{-(\nu+1)t}\,dt.$$

But

$$\binom{m+1}{\nu} = O\left(m^{p+1}\binom{m-p}{\nu}\right),$$

so that

$$\begin{aligned}
\widetilde{B}_m &= O\left(m^{p+1}\int_0^\infty t^{-\alpha-1}\sum_{\nu=1}^{m-p}\binom{m-p}{\nu} z^{m-\nu-p} z^{p+1} e^{-(\nu+1)t}\right)\\
&= O\left(m^{p+1}\int_0^\infty z^{p+1} t^{-\alpha-1} e^{-t}(e^{-t}+z)^{m-p}\, dt\right)\\
&= O\left(m^{p+1}\int_0^\infty t^{p-\alpha} e^{-t}\left(\frac{q+e^{-t}}{q+1}\right)^{m-p} dt\right)\\
&= O\left(m^{\alpha}\int_0^\infty t^{p-\alpha} e^{-t/m}\left(\frac{q+e^{-t/m}}{q+1}\right)^{m-p} dt\right).
\end{aligned}$$

If $\lambda \in (0,1)$ and $\tau = -\log\lambda$, then

$$1 - e^{-x} \geq \lambda x, \quad 0 \leq x \leq \tau. \tag{2.32}$$

Indeed, since $e^{-x} - \lambda x \geq 0, 0 \leq x \leq \tau$, the function $\varphi(x) = 1 - e^{-x} - \lambda x$ is increasing in the interval $[0,\tau]$ and, moreover, $\varphi(0) = 0$.

A consequence of the auxiliary statement just proved is that

$$J_m(\alpha) = \int_0^\infty t^{p-\alpha} e^{-t/m}\left(\frac{q+e^{-t/m}}{q+1}\right)^{m-p} dt = O(1), \quad m \to \infty.$$

Let

$$J_m(\alpha) = \int_0^{m\tau} + \int_{m\tau}^\infty = J_m^{(1)}(\alpha) + J_m^{(2)}(\alpha).$$

If $0 \leq t \leq m\tau$, then (2.32) yields that

$$\frac{q+e^{-t/m}}{q+1} = 1 - \frac{1-e^{-t/m}}{q+1} \leq 1 - \frac{\lambda t}{m(q+1)}$$

and the well-known inequality $(1-t/n)^n < e^{-t}, 0 < t < n$, gives that

$$J_m^{(1)}(\alpha) = O\left(\int_0^{m\tau} t^{p-\alpha} e^{-t/m} e^{-(\lambda/(q+1))t}\, dt\right)$$

$$= O\left(\int_0^\infty t^{p-\alpha} e^{-(\lambda/(q+1))t}\right).$$

If $t \geq m\tau$, then

$$\frac{q+e^{-t/m}}{q+1} \leq \frac{q+e^{-\tau}}{q+1} = \gamma < 1,$$

and, hence,

$$J_m^{(2)}(\alpha) \leq \gamma^{m-p} \int_{m\tau}^\infty t^{p-\alpha} e^{-t/m} = \gamma^{m-p} m^{p-\alpha+1} \int_\tau^\infty t^{p-\alpha} e^{-t}\, dt.$$

Therefore, $\lim_{m\to\infty} J_m^{(2)}(\alpha) = 0$ and thus it is established that

$$B_m(\alpha) = O(m^\alpha).$$

Further,

$$A_m(\alpha)$$

$$= \frac{1}{|g(\alpha)|} \int_0^\infty t^{-\alpha-1} e^{-t} \frac{1-e^{-t}}{q+1} \sum_{\mu=0}^{m-p-1} \sum_{\nu=0}^{\mu} \binom{m+1}{\nu} \frac{z^{m-\nu+1} e^{-\mu t}}{(1-z)^{\mu-\nu+1}}\, dt$$

$$= \frac{1}{|g(\alpha)|} \int_0^\infty t^{-\alpha-1} e^{-t} \frac{1-e^{-t}}{q+1} \sum_{\nu=0}^{m-p-1} \binom{m+1}{\nu} \sum_{\mu=\nu}^{m-p-1} \frac{z^{m-\nu} e^{-\mu t}}{(1-z)^{\mu-\nu+1}}\, dt$$

$$< \frac{1}{|g(\alpha)|} \int_0^\infty t^{-\alpha-1} \sum_{\nu=0}^{m-p-1} \binom{m+1}{\nu} \frac{z^{m-\nu+1} e^{-\nu t}}{1-z}\, dt$$

$$< Km^{p+2} \int_0^\infty t^{-\alpha-1} \sum_{\nu=0}^{m-p-1} \binom{m-p-1}{\nu} z^{m-k+p+1}\, dt$$

$$= K\left(\frac{q}{q+1}\right)^{p+2} m^{p+2} \int_0^\infty t^{-\alpha-1} e^{-t} (1-e^{-t})^{p+2} (z+e^{-t})^{m-p-1}\, dt$$

$$< Lm^{p+2} \int_0^\infty t^{p-\alpha+1} \left(\frac{q+e^{-t}}{q+1}\right)^{m-p-1} dt,$$

where K, L are constants not depending on m and then, as above, it follows that

$$A_m(\alpha) = O(m^\alpha).$$

It remains the sum

$$\sum_{\mu=0}^{m-p-1} S_\mu^k(\alpha) h_{m,\mu}^{(2)}(\alpha)$$

to be estimated when $m \to \infty$. Since

$$\Big| \sum_{\mu=0}^{m-p-1} S_\mu^k(\alpha) h_{m,\mu}^{(2)}(\alpha) \Big| \le S \sum_{\mu=0}^{m-p-1} |h_{m,\mu}^{(2)}(\alpha)|, \quad S = Const.,$$

it is sufficient to know the asymptotic behavior of the sum

$$\sum_{\mu=0}^{m-p-1} |h_{m,\mu}^{(2)}(\alpha)|$$

when $m \to \infty$.

Let $0 < \alpha < 1$, then $p = 1$ and

$$-g_{\mu,1}(z) = \sum_{\nu=0}^{\mu} \binom{\mu+1}{\nu} \left(\frac{z}{1-z}\right)^{\mu-\nu+1} = \frac{1}{(1-z)^{\mu+1}} - 1,$$

$$l_{m,\mu}^{(2)}(\alpha) = -\frac{1}{g(\alpha)} \int_0^\infty t^{-\alpha-1} e^{-(\mu+1)t} \left(\frac{1}{(1-z)^{\mu+1}} - 1 \right) dt$$

and

$$h_{m,\mu}^{(2)}(\alpha) = -\frac{1}{g(\alpha)} \int_0^\infty t^{-\alpha-1} e^{-(\mu+1)t} \frac{1-e^{-t}}{(q+1)(1-z)^{\mu+1}}\, dt$$
$$+ \frac{1}{g(\alpha)} \int_0^\infty t^{-\alpha-1} e^{-(\mu+1)t} (1-e^{-t})\, dt.$$

Then,

$$|h_{m,\mu}^{(2)}(\alpha)| \le \frac{1}{|g(\alpha)|} \int_0^\infty t^{-\alpha} e^{-t} \left(\frac{q+1}{q+e^t}\right)^{\mu+2} dt + \frac{1}{|g(\alpha)|} \int_0^\infty t^{-\alpha} e^{-(\mu+1)t}\, dt$$
$$= O((\mu+1)^{\alpha-1}) + O((\mu+1)^{\alpha-1}) = O((\mu+1)^{\alpha-1}),$$

i.e.,

$$|h_{m,\mu}^{(2)(\alpha)}| \le H(\mu+1)^{(\alpha-1)},$$

where H is a constant not depending on m. Hence,

$$\sum_{\mu=0}^{m-p-1} |h_{m,\mu}^{(2)}(\alpha)| = \sum_{\mu=0}^{m-p-1} O((\mu+1)^{\alpha-1}) = O(m^\alpha).$$

From (2.27) it holds that

$$-g_{p,\mu}(z)=\frac{1}{(1-z)^{\mu+1}}-\sum_{\tau=0}^{p-1}z^{p-1-\tau}(1-z)^{\tau}$$

and replacing e^{-t} by $1-\dfrac{q+1}{q}z$, one gets that the function

$$f(z)=-g_{p,\mu}(z)+e^{-t}g_{p,\mu+1}(z)$$

becomes

$$f(z)=\frac{z}{q(1-z)^{\mu+2}}$$

$$-\sum_{\tau=0}^{p-1}z^{p-1-\tau}(1-z)^{\tau}\left\{\binom{\mu+p}{p-1-\tau}-\binom{\mu+p+1}{p-1-\tau}\left(1-\frac{q+1}{q}z\right)\right\}.$$

If

$$\frac{1}{(1-z)^{\mu+2}}=\sum_{\nu=0}^{\infty}a_{\mu,\nu}z^{\nu},$$

then $f(z)$ can be written in the following form

$$f(z)=\varphi(z)+\psi(z),$$

where

$$\varphi(z)=\frac{z}{q}\left(\frac{1}{(1-z)^{\mu+2}}-\sum_{\nu=0}^{p-2}a_{\mu,\nu}z^{\nu}\right),$$

$$\psi(z)=\frac{z}{q}\sum_{\nu=0}^{p-2}a_{\mu,\nu}z^{\nu}$$

$$-\sum_{\tau=1}^{p-1}z^{p-1-\tau}(1-z)^{\tau}\left(\binom{\mu+p}{p-1-\tau}-\binom{\mu+1+p}{p-1-\tau}\right)\left(1-\frac{q+1}{q}z\right).$$

The Maclaurin expansion of the function f begins with a monomial of degree p and that of φ begins with a monomial of degree $p-1$ and therefore, that of ψ has to begin with a monomial of degree p. But ψ is a polynomial of degree not greater than p, i.e.,

$$\psi(z)=a_{\mu,p}z^{p}, \tag{2.33}$$

where

$$a_{\mu,p} = \binom{\mu+1+p}{p-1}\frac{q+1}{q} = O((\mu+1)^{p-1}), \quad \mu \to \infty. \tag{2.34}$$

The function φ is, in fact, the reminder of Maclaurin's formula for the function $(1-z)^{-\mu-2}$ multiplied by $\frac{z}{q}$ and then, the Lagrange formula implies the equality

$$\varphi(z) = \frac{(\mu+2)(\mu+3)\dots(\mu+p)z^p}{(p-1)!q(1-\theta z)^{\mu+p+1}}, \quad 0<\theta<1,$$

whence

$$\varphi(z) < \frac{(\mu+2)(\mu+3)\dots(\mu+p)z^p}{(p-1)!q(1-z)^{\mu+p+1}}. \tag{2.35}$$

Further,

$$\begin{aligned} h^{(2)}_{m,\mu}(\alpha) &= \frac{1}{g(\alpha)}\int_0^\infty t^{-\alpha-1}e^{-(\mu+1)t}\varphi(t)\,dt + \frac{1}{g(\alpha)}\int_0^\infty t^{-\alpha-1}e^{-(\mu+1)t}\psi(t)\,dt \\ &= H^{(1)}_\mu(\alpha) + H^{(2)}_\mu(\alpha). \end{aligned}$$

Then,

$$\begin{aligned} |H^{(2)}_\mu(\alpha)| &< K_1(\mu+1)^{p-1}\int_0^\infty t^{p-\alpha-1}e^{-(\mu+1)t}\,dt \\ &= K_1(\mu+1)^{\alpha-1}\int_0^\infty t^{p-\alpha-2}e^{-t}\,dt = K_2(\mu+1)^{\alpha-1}. \end{aligned}$$

Further,

$$|H^{(1)}_\mu(\alpha)| < K_3(\mu+1)^{p-1}\int_0^\infty t^{p-\alpha-1}e^{-(\mu+1)t}\frac{dt}{(1-z)^{\mu+1}}$$

$$= K_3(\mu+1)^{p-1}\int_0^\infty t^{p-\alpha-1}\left(\frac{q+1}{q+e^t}\right)^{\mu+1} dt = O\left((\mu+1)^{p-1}\frac{1}{(\mu+1)^{p-\alpha}}\right)$$

$$= O((\mu+1)^{\alpha-1}), \quad K_j = Const., \quad j=1,2,3,$$

hence, $h^{(2)}_{m,\mu}(\alpha) = O(\mu^{\alpha-1})$, i.e. $|h^{(2)}_{m,\mu}(\alpha)| \le K\mu^{\alpha-1}$, where K is a constant not depending on m. Then

$$\sum_{\mu=1}^m |h^{(2)}_{m,\mu}(\alpha)| = \sum_{\mu=1}^m O((\mu+1)^{\alpha-1}) = O(m^\alpha)$$

and thus the theorem is proved.

Theorem 11. *If e_k is a negative number, then*

$$e_k = \limsup_{n\to\infty} \frac{\log|A_n^k + A_{n+1}^k + \dots|}{\log(n+1)}, \tag{2.36}$$

where

$$A_n^k = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} a_\nu, \quad n = 0, 1, 2, \dots .$$

Let β be the right-hand side of (2.36), then for each $\varepsilon > 0$ there exists $K = K(\varepsilon)$ such that

$$|A_n^k + A_{n+1}^k + A_{n+2}^k + \dots| \le K(n+1)^{\beta+\varepsilon}, \quad n = 0, 1, 2, 3, \dots .$$

Let $-\alpha = \beta + 2\varepsilon < 0$ and let $\sum_{n=0}^{\infty} A_n^k(-\alpha)$ be the E_k-transform of the series (2.4) for $s = -\alpha$. Then,

$$A_n^k(-\alpha) = A_n^k(\beta + 2\varepsilon)$$

$$= \left(\frac{q}{q+1}\right)^n \sum_{\mu=0}^{n} \left(\frac{q+1}{q}\right)^\mu \binom{n}{\mu} A_\mu^k \Delta^{n-\mu}(\mu+1)^\alpha.$$

Let at first α be an integer. In such a case if $n > \alpha$, then

$$A_n^k(-\alpha) = \left(\frac{q}{q+1}\right)^n \sum_{\mu=n-\alpha}^{n} \left(\frac{q+1}{q}\right)^\mu \binom{n}{\mu} A_\mu^k \Delta^{n-\mu}(\mu+1)^\alpha$$

$$= \sum_{\tau=0}^{n} \left(\frac{q+1}{q}\right)^\tau \binom{n}{\tau} A_{n-\tau}^k \Delta^\tau (n-\tau+1)^\alpha = \sum_{\tau=0}^{n} \left(\frac{q+1}{q}\right)^\tau T_{n,\tau}.$$

But since τ is fixed, then

$$\binom{n}{\tau} \Delta^\tau (n-\tau+1)^\alpha = g_\alpha n^\alpha + O(n^{\alpha-1}), \quad n \to \infty,$$

and hence,

$$\sum_{n=m}^{N} T_{n,\tau} = g_\tau \sum_{n=m}^{N} A_{n-m}^k n^\alpha + \sum_{n=m}^{N} O(|A_{n-\tau}^k| n^{\alpha-1})$$

$$= g_\tau Q_{m,N} + \sum_{n=m}^{N} O(n^{\alpha-1} n^{\beta+\varepsilon}) = g_\tau Q_{n,N} + \sum_{n=m}^{N} O\left(\frac{1}{n^{1+\varepsilon}}\right).$$

Further, if $R_n^k = A_n^k + A_{n+1}^k + \dots$, then

$$Q_{m,N} = \sum_{n=m}^{N} n^\alpha (R_{n-\tau} - R_{n-\tau+1})$$

$$= m^\alpha R_{m-\tau}^k + ((m+1)^\alpha - m^\alpha) R_{m+1-\tau} + \dots$$

$$+((N^\alpha) - N - 1)^\alpha) R_{N-\tau}^k - N^\alpha R_{N-\tau+1}.$$

But since $|R_n^k| < K((n+1)^{\beta+\varepsilon})$ and $n^\alpha - (n-1)^\alpha < K_1 n^{\alpha-1}, n = 1, 2, \dots,$

$$|Q_{m,N} < K_2 \left(\frac{1}{m^\varepsilon} + \frac{1}{(m+1)^{1+\varepsilon}} + \frac{1}{(m+2)^{1+\varepsilon}} + \dots + \frac{1}{N^{1+\varepsilon}} + \frac{1}{N^\varepsilon} \right),$$

whence it follows that the series (2.4) is convergent for each s such that $\Re s > \beta$.

Let now α be not an integer and let $p = [\alpha] + 1$. If $\nu \geq p$, then

$$\Delta^\nu (n^\alpha) = \frac{1}{g(\alpha)} \int_0^\infty t^{-\alpha-1} e^{-nt} (1 - e^{-t})^\nu \, dt.$$

Further, under the assumption that $n \geq p$ it holds the representation

$$A_n^k(-\alpha) = \left(\frac{q}{q+1} \right)^n \sum_{\mu=0}^{n-p} \left(\frac{q+1}{q} \right)^\mu A_\mu^k \binom{n}{\mu} \Delta^{n-\mu} (\mu+1)^\alpha$$

$$+ \left(\frac{q}{q+1} \right)^n \sum_{\mu=n-p+1}^{n} \left(\frac{q+1}{q} \right)^\mu A_\mu^k \binom{n}{\mu} \Delta^{n-\mu} (\mu+1)^\alpha$$

$$= C_n(-\alpha) + D_n(-\alpha).$$

Then,

$$J_m(-\alpha) = \sum_{n=p}^{m} C_n(-\alpha)$$

$$= \frac{1}{g(\alpha)} \sum_{n=p}^{m} \sum_{\mu=0}^{n-p} \left(\frac{q}{q+1} \right)^{n-\mu} A_\mu^k \binom{n}{\mu} \int_0^\infty t^{-\alpha-1} e^{-(\mu+1)t} (1 - e^{-t})^{n-\mu} \, dt$$

$$= \frac{1}{g(\alpha)} \sum_{\mu=0}^{m-p} A_\mu^k \sum_{n=\mu+p}^{m} \left(\frac{q}{q+1} \right)^{n-\mu} \binom{n}{\mu} \int_0^\infty t^{-\alpha-1} e^{-(\mu+1)t} (1 - e^{-t})^{n-\mu} \, dt$$

$$= \frac{1}{g(\alpha)} \sum_{\mu=0}^{m-p} A_\mu^k \sum_{\nu=p}^{m-\mu} \left(\frac{q}{q+1}\right)^\nu \int_0^\infty t^{-\alpha-1} e^{-(\mu+1)t} (1-e^{-t})^\nu \, dt$$

$$= \frac{1}{g(\alpha)} \sum_{\mu=0}^{m-p} \int_0^\infty t^{-\alpha-1} e^{-(\mu+1)t} g_{\mu,p}(z) \, dt.$$

By taking into account the numbers $L_{m,p}^{(j)}(\alpha), l_{m,\mu}^{(j)}(\alpha), h_{m,\mu}^{(j)}(\alpha), j = 1, 2$, defined by the equalities (2.28), (2.29), (2.30) and (2.31), one obtains that

$$J_m(-\alpha) = \sum_{\mu=0}^{m-p} A_\mu^k l_{m,\mu}^{(1)}(\alpha) - \sum_{\mu=0}^{m-p} A_\mu^k l_{m,\mu}^{(2)}(\alpha) = L_{m,p}^{(1)}(\alpha) - L_{m,p}^{(2)}(\alpha),$$

$$L_m^{(1)}(\alpha) = \sum_{\mu=0}^{m-p} (R_\mu - R_{\mu+1}) l_{m,\mu}^{(1)}(\alpha)$$

$$= R_0 l_{m,0}^{(1)}(\alpha) - \sum_{\mu=1}^{m-p} R_\mu (l_{m,\mu-1}^{(1)}(\alpha) - l_{m,\mu}^{(1)}(\alpha)) - R_{m-p+1} l_{m,m-p}^{(1)}(\alpha)$$

$$= R_0 l_{m,0}^{(1)}(\alpha) - \sum_{\mu=1}^{m-p} R_\mu h_{m,\mu-1}^{(1)}(\alpha) - R_{m-p+1} l_{m,m-p}^{(1)}(\alpha),$$

as well as

$$L_m^{(2)}(\alpha) = R_0 l_{m,0}^{(2)}(\alpha) - \sum_{\mu=1}^{m-p} R_\mu h_{m,\mu-1}^{(2)}(\alpha) - R_{m-p+1} l_{m,m-p}^{(2)}(\alpha).$$

Then,

$$R_{m-p+1} l_{m,m-p}^{(1)}(\alpha) - R_{m-p+1} l_{m,m-p}^{(2)}(\alpha) = R_{m,m-p+1} l_{m,m-p}(\alpha)$$

$$= R_{m-p+1} \binom{m}{m-p} \left(\frac{q}{q+1}\right)^p \frac{1}{g(\alpha)} \int_0^\infty t^{-\alpha-1} e^{-(m-p+1)t} (1-e^{-t})^p \, dt$$

$$= \left(m^{\beta+\varepsilon} m^p \int_0^\infty t^{p-\alpha-1} e^{-(m-p+1)t} \, dt \right)$$

$$= O(m^{\beta+\varepsilon} m^p m^{-p+\alpha}) = O(m^\varepsilon) = o(1).$$

Further,

$$R_0 l^{(1)}_{m,0}(\alpha) - R_0 l^{(2)}_{m,0}(\alpha) = R_0 l_{m,0}(\alpha) = \frac{R_o}{g(\alpha)} \int_0^\infty t^{-\alpha-1} e^{-t} g_{0,p}(z)\, dt,$$

where

$$g_{0,p}(z) = \sum_{\nu=p}^{m} z^\nu = z^p \frac{1-z^{m-p+1}}{1-z}$$
$$= (1-e^{-t})^p \left(\frac{q}{q+1}\right)^p \frac{q+1}{qe^{-t}1}(1-z^{m-p+1})$$

whence

$$\lim_{m\to\infty} \frac{R_0}{g(\alpha)} \int_0^\infty t^{-\alpha-1} e^{-t} g_{0,p}(z)\, dt$$
$$= \frac{q+1}{g(\alpha)} R_0 \left(\frac{q}{q+1}\right)^p \int_0^\infty t^{-\alpha-1}(1-e^{-t})^p \frac{dt}{qe^{-t}+1}.$$

Since $|z| \le \dfrac{q}{q+1} < 1$, $l^{(1)}_{m,\mu}(\alpha)$ tends to zero when $m \to \infty$ provided that μ is fixed. Moreover, as it was already mentioned, $l^{(2)}_{m,\mu}(\alpha)$ does not depend on m. Hence, the series

$$\sum_{\mu=1}^{\infty} R_\mu h^{(2)}_{m.\mu-1}(\alpha), h^{(2)}_{m,\mu-1}(\alpha) = l^{(2)}_{m,\mu-1}(\alpha) - l^{(2)}_{m,\mu}(\alpha), \tag{2.37}$$

has to be studied whether is it convergent. Since $\alpha < p < \alpha+1, |h^{(2)}_{m,\mu}(\alpha)| < K\mu^{p-2}$ it holds the inequality $|R_\mu h^{(2)}_{m,\mu-1}(\alpha)| < L\mu^{\beta+\alpha}\mu^{p-2}$. But $\alpha = -\beta - 2\varepsilon$, hence, $\beta+\varepsilon+p-2 < \beta+\varepsilon+\alpha-1 = -1-\varepsilon$, i.e., the series in (2.37) is absolutely convergent which means that there exists

$$\lim_{m\to\infty} \sum_{\mu=1}^{m-p} R_\mu h^{(2)}_{m,\mu-1}(\alpha) = B(\alpha).$$

It remains to be studied the asymptotic of the expression

$$T_{m,p}(\alpha) = \sum_{\mu=1}^{m-p} R_\mu h^{(1)}_{m,\mu-1}(\alpha)$$

when $m \to \infty$. To that end it is used the representation

$$h^{(1)}_{m,\mu-1}(\alpha) = \frac{1}{g(\alpha)} \int_0^\infty t^{-\alpha-1} e^{-\mu t} \sum_{\nu=0}^{\mu-1} \binom{m+1}{\nu} \frac{z^{m+1-\nu}}{(1-z)^{\mu-\nu}} \left(1 - \frac{e^{-t}}{1-z}\right) dt$$

$$-\frac{1}{g(\alpha)}\binom{m+1}{\mu}\int_0^\infty t^{-\alpha-1}e^{-(\mu+1)t}\frac{z^{m+1-\mu}}{1-z}\,dt = A_{m,\mu}(\alpha) - B_{m,\mu}(\alpha).$$

Further,

$$\binom{m+1}{\mu} = \frac{(m+1)m(m-1)\dots(m-p+1)}{(m+1-\mu)(m-\mu)\dots(m+p-1-\mu)}\binom{m-p}{\mu}$$

and if $\gamma = -\beta - \varepsilon$, then

$$\mu^\gamma(m+1-\mu)(m-\mu)\dots(m-p+1-\mu) > \mu^\gamma(m-p+1-\mu)^{p+1}.$$

Since $p > \alpha = \gamma - \varepsilon$, it follows that $p > \gamma$ and it is easily seen that if $1 \le \mu \le m$, then

$$\sum_{\mu=1}^{m-p} |R_\mu B_{m,\mu}|(\alpha) \le K \sum_{\mu=1}^{m-p}\binom{m+1}{\mu}\frac{\mu^{\beta+\varepsilon}}{|g(\alpha)|}\int_0^\infty t^{-\alpha-1}e^{-(\mu+1)t}\frac{z^{m+1-\mu}}{1-z}\,dt$$

$$\le K_1 m^{p+1}\int_0^\infty t^{-\alpha-1}e^{-mu+1)t}\sum\binom{m-p}{\mu}\frac{\mu^{-\gamma}}{(m-p+1-\mu)}\frac{z^{m+1-\mu}}{1-z}\,dt$$

$$\le K_2 m^{p+1-\gamma}\int_0^\infty t^{-\alpha-1}e^{-(\mu+1)}\sum_{\mu=1}^{m-p}\binom{m-p}{\mu}\frac{z^{m+1-\mu}}{1-z}\,dt$$

$$\le K_3 m^{p+1-\gamma}\int_0^\infty t^{p-\alpha}e^{-t}(z+e^{-t})^{m-p}\,dt$$

$$= K_3 m^{p+1-\gamma}\int_0^\infty t^{p-\alpha}e^{-t}\left(\frac{q+e^{-t}}{q+1}\right)^{m-p}\,dt.$$

But it was already established that

$$\int_0^\infty t^{p-\alpha}e^{-t}\left(\frac{q+e^{-t}}{q+1}\right)^{m-p}\,dt = O\left(\frac{1}{m^{p-\alpha+1}}\right)$$

and since $\gamma - \alpha = -\beta - \varepsilon - (-\beta - 2\varepsilon) = \varepsilon$, it follows that

$$\sum_{\mu=1}^{m-p} R_\mu B_{m,\mu}(\alpha) = O\left(m^{p+1-\gamma}\frac{1}{m^{p-\alpha+1}}\right) = O\left(\frac{1}{m^\varepsilon}\right) = o(1), \quad m\to\infty.$$

It remains now the behavior of the sum

$$S_m(\alpha) = \sum_{\mu=1}^{m-p} A_{m,\mu}(\alpha)$$

to be studied when $m \to \infty$.

Since $1 - z - e^{-t} = (1 - e^{-t})\dfrac{q}{q+1} \leq \dfrac{2q}{q+1}t,\ 0 \leq t < \infty$ and $1 - z \geq \dfrac{1}{q+1}$, it holds the inequality

$$S_m(\alpha) \leq K_4 \int_0^{\infty} t^{-\alpha} \sum_{\mu=1}^{m-p} |R_\mu| e^{-\mu t} \sum_{\nu=0}^{\mu-1} \binom{m+1}{\nu} \frac{z^{m+1-\nu}}{(1-z)^{\mu-\nu}}\, dt.$$

Further,

$$S_m(\alpha) \leq K_5 \int_0^{\infty} t^{-\alpha} \sum_{\mu=1}^{m-p} \mu^{-\gamma} e^{-\mu t} \sum_{\nu=0}^{\mu-1} \binom{m+1}{\nu} \frac{z^{m+1-\nu}}{(1-z)^{\mu-\nu}}\, dt$$

$$= K_5 \int_0^{\infty} t^{-\alpha} \sum_{\nu=0}^{m-p-1} \binom{m+1}{\nu} z^{m+1-\nu} \sum_{\mu=\nu+1}^{m-p} \frac{e^{-\mu t}}{\mu^\gamma} \frac{dt}{(1-z)^{\mu-\nu}}$$

$$= K_5 \int_0^{\infty} t^{-\alpha} \sum_{\nu=0}^{m-p-1} \binom{m+1}{\nu} e^{-(\nu+1)t} \frac{z^{m+1-\nu}}{1-z} \sum_{\lambda=0}^{m-p-1} \frac{g^\lambda}{(k+1+\lambda)^\gamma}\, dt,$$

where $g = \dfrac{e^{-t}}{1-z}$, $0 < g < 1$.

Since the sequence $\{(\nu+1+\lambda)^{-\gamma}\}_{\lambda=0}^{\infty}$ is decreasing and, moreover,

$$\sum_{\lambda=0}^{n} g^\lambda = \frac{1-g^{n+1}}{1-g} < \frac{1}{1-g} = \frac{1-z}{1-z-e^{-t}} \leq Mt^{-1},$$

$$M = Const., \quad \nu = 1, 2, \dots, \ n = 1, 2, \dots, \quad 0 < t < \infty,$$

by a theorem of Abel,

$$\sum_{\lambda=0}^{m-p-1} \frac{g^\lambda}{(\nu+1+\lambda)^\gamma} \leq \frac{M}{(\nu+1)t}, \quad m \geq p+1, \quad t > 0.$$

Then,

$$S_m(\alpha) \leq K_6 \int_0^{\infty} t^{-\alpha-1} \sum_{\nu=0}^{m-p-1} \binom{m+1}{\nu} (\nu+1)^{-\gamma} z^{m+1-\nu} e^{-(\nu+1)t}\, dt$$

and taking into account the inequality

$$(\nu+1)^{-\gamma} \binom{m+1}{\nu} < m^{p+2-\gamma} \binom{m-p-1}{\nu}, \quad m > p+1,$$

one gets that

$$S_m(\alpha) \le K_6 m^{p+2-\gamma} \int_0^\infty t^{-\alpha-1} e^{-t} \sum_{\nu=0}^{m-p-1} \binom{m-p-1}{\nu} z^{m+1-\nu} e^{-\nu t}\, dt$$

$$\le K_7\, m^{p+2-\gamma} \int_0^\infty t^{-\alpha+p+1} e^{-t} (z+e^{-t})^{m-p-1}\, dt$$

$$= K_7\, m^{p+2-\gamma} \int_0^\infty t^{-\alpha+p+1} e^{-t} \left(\frac{q+e^{-t}}{q+1}\right)^{m-p-1} dt$$

$$= O\left(m^{p+2-\gamma} \frac{1}{m^{p+2-\alpha}}\right) = O\left(\frac{1}{m^{\gamma-\alpha}}\right) = O\left(\frac{1}{m^{\varepsilon}}\right) = o(1), \quad m \to \infty,$$

and thus the first part of the proof is finished.

Let now the series (2.4) be E_k-summable for some $s = -\alpha < 0$, i.e., the series

$$\sum_{n=0}^{\infty} A_n^k(-\alpha), \tag{2.38}$$

$$A_n^k(-\alpha) = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} a_\nu (\nu+1)^\alpha, \quad n = 0, 1, 2 \ldots,$$

is convergent.

Further,

$$A_n^k = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} a_\nu, \quad n = 0, 1, 2 \ldots,$$

then

$$A_n^k = \sum_{\mu=0}^{n} A_\mu^k(-\alpha) \binom{n}{\mu} \left(\frac{q}{q+1}\right)^{n-\mu} \sum_{\nu=0}^{n-\mu} (-1)^\nu \binom{n-\mu}{\nu} (\mu+\nu+1)^{-\alpha}$$

$$= \sum_{\mu=0}^{n} A_\mu^k(-\alpha) \binom{n}{\mu} \left(\frac{q}{q+1}\right)^{n-\mu} \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-(\mu+1)t} (1-e^{-t})^{n-\mu}\, dt.$$

Moreover,

$$\sum_{n=m}^{\nu} A_n^k = \sum_{n=m}^{\nu} \sum_{\mu=0}^{m} A_\mu^k(-\alpha) g_{n,\mu} + \sum_{\mu=m}^{\nu} \sum_{n=\mu}^{\nu} A_\mu^k(-\alpha) g_{n,\mu}(\alpha),$$

where

$$g_{n,\mu}(\alpha) = \frac{1}{\Gamma(\alpha)}\binom{n}{\mu}\int_0^\infty t^{\alpha-1}e^{-(\mu+1)t}z^{n-\mu}\,dt, \quad z = (1-e^{-t})\frac{q}{q+1}$$

and

$$\delta_{\nu,\mu}(\alpha) = \sum_{n=\mu}^{\nu} g_{n,\mu}(\alpha) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-(\mu+1)t}\sum_{\tau=0}^{\nu-\mu}\binom{\mu+\tau}{\mu}z^\tau\,dt, \quad \nu \geq \mu.$$

The following equalities

$$\lim_{\nu\to\infty}\delta_{\nu,\mu}(\alpha) = \delta_\mu(\alpha) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-\mu+1)t}\frac{dt}{(1-z)^{\mu+1}}, \tag{2.39}$$

$$\lim_{\nu\to\infty}\delta_{\nu,\nu}(\alpha) = \lim_{\nu\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-(\nu+1)t}\,dt = 0$$

are immediate consequences of the preceding one.

Let now $I_{m,\nu}(-\alpha)$ be defined as

$$I_{m,\nu}(-\alpha) = \sum_{\mu=m}^{\nu}\sum_{n=\mu}^{\nu} A_\mu^k(-\alpha)g(n,\mu)(\alpha), \quad \nu \geq m,$$

then

$$I_{m,\nu}(-\alpha) = \sum_{\mu=m}^{\nu} A_\mu^k(-\alpha)\delta_{\nu,\mu}(\alpha)$$

$$= -S_{m-1}^k(-\alpha)\delta_{\nu,m}(\alpha) + \sum_{\mu=m}^{\nu-1}(\delta_{\nu,\mu}(\alpha) - \delta_{\nu,\mu+1}(\alpha)), \quad \nu \geq m+1,$$

where

$$S_{m-1}^k(-\alpha) = A_0^k(-\alpha) + A_1^k(-\alpha) + \cdots + A_{m-1}^k(-\alpha).$$

Further, the series

$$\sum_{\mu=m}^{\infty} S_\mu^k(-\alpha)(\delta_\mu(\alpha) - \delta_{\mu+1}(\alpha)) \tag{2.40}$$

is absolutely convergent. Indeed,

$$\delta_\mu(\alpha) - \delta_{\mu+1}(\alpha) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-(\mu+1)t}\,\frac{1-z-e^{-t}}{(1-z)^{\mu+2}}\,dt$$

$$= O\left(\int_0^\infty t^{\alpha-1}e^{-(\mu+1)t}\left(\frac{e^t(q+1)}{q+e^t}\right)^{\mu+1}\frac{1-e^{-t}}{1-z}\,dt\right)$$

$$= O\left(\int_0^\infty t^{\alpha}e^{-(\mu+1)t}\,dt\right) = O\left(\frac{1}{(\mu+1)^{\alpha+1}}\right),$$

and, moreover, the sequence $\{S_\mu^k(-\alpha)\}_{\mu=m}^\infty$ is bounded since the series (2.38) is convergent.

A consequence of the convergence of the series (2.40) as well as of the equalities (2.38) is that if m is fixed, then

$$\lim_{\nu\to\infty} I_{m,\nu}(\alpha) = -S_{m-1}^k(\alpha)\delta_m(\alpha) + \sum_{\mu=m}^{\infty} S_\mu^k(\alpha)(\delta_\mu(\alpha) - \delta_{\mu+1}(\alpha)). \tag{2.41}$$

Indeed, if p is an arbitrary positive integer and $\nu \geq m+p+1$, then

$$\Delta_{m,\nu}(\alpha) = \left|\sum_{\mu=m}^{\nu-1} S_\mu^k(\alpha)(\delta_{\nu,\mu}(\alpha) - \delta_{\nu,\mu+1}(\alpha)) - \sum_{\mu=m}^{\infty} S_\mu^k(\alpha)(\delta_\mu(\alpha) - \delta_\mu(\alpha))\right|$$

$$\leq \left|\sum_{\mu=m+p}^{\nu-1} S_\mu^k(\alpha)(\delta_{\nu,\mu}(\alpha) - \delta_{\nu,\mu+1}(\alpha)) - \sum_{\mu=m+p}^{\infty} S_\mu^k(\alpha)(\delta_\mu(\alpha) - \delta_{\mu+1}(\alpha))\right|,$$

hence

$$\limsup_{\nu\to\infty} \Delta_{m,\nu}(\alpha)$$

$$\leq \left|\sum_{\mu=m}^{m+p-1} S_\mu^k(\alpha)(\delta_{\nu,\mu}(\alpha) - \delta_{\nu,\mu+1}(\alpha)) - \sum_{\mu=m}^{m+p_1} S_\mu^k(\alpha)(\delta_\mu(\alpha) - \delta_{\mu+1}(\alpha))\right|$$

and letting $p \to \infty$, one gets the equality (2.41).

Further, the sum

$$J_{\nu,m}(\alpha) = \sum_{n=m}^{\nu}\sum_{\mu=0}^{m} A_\mu^k(\alpha)g_{n,\mu}(\alpha)$$

is to be studied when $\nu \to \infty$ provided that m is fixed. Since

$$\sum_{n=m}^{\nu} g_{n,\mu}(\alpha) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-(\mu+1)t}\sum_{n=m}^{\nu}\binom{n}{\mu}z^{n-\mu}\,dt$$

and moreover, the series

$$\sum_{\tau=0}^{\infty}\binom{m+\tau}{\mu}z^{\tau}, \quad z=(1-e^{-t})\frac{q}{q+1},$$

is uniformly convergent in the interval $[0,\infty)$, it follows that there exists

$$L_{\mu,m}(\alpha)=\lim_{\nu\to\infty}\sum_{n=m}^{\nu}g_{n,\mu}(\alpha)$$

$$=\int_0^{\infty}t^{\alpha-1}e^{-(\mu+1)t}\sum_{\tau=0}^{\infty}\binom{m+\tau}{\mu}z^{\tau+m-\mu}\,dt,$$

and hence, there also exists

$$J_m(\alpha)=\lim_{\nu\to\infty}J_{\nu,m}(\alpha)=\sum_{\mu=0}^{m}A_{\mu}^{k}(\alpha)L_{\mu,m}(\alpha). \tag{2.42}$$

Let

$$f_{m,\mu}(z)=\sum_{\tau=0}^{\infty}\binom{m+\tau}{\mu}z^{\tau+m-\mu}=\sum_{\lambda=m-\mu}^{\infty}\binom{\lambda+\mu}{\mu}z^{\lambda},$$

then, as it is easily seen,

$$f_{m,\mu}(z)=\frac{1}{\mu!}\frac{d^{\mu}}{dz^{\mu}}\sum_{\lambda=m-\mu}^{\infty}z^{\lambda+\mu}=\frac{1}{\mu!}\frac{d^{\mu}}{dz^{\mu}}(z^{m}(1-z)^{-1})$$

$$=\frac{1}{\mu!}\sum_{p=0}^{\mu}m(m-1)\dots(m-p+1)z^{m-p}(1-z)^{p-\mu-1}$$

$$=\sum_{p=0}^{\mu}D_{m,\mu}^{p}\frac{z^{m-p}}{(1-z)\mu+1-p},$$

where

$$D_{m,\mu}^{p}=\frac{m(m-1)\dots(m-p+1)}{\mu(\mu-1)\dots(\mu-p+1)},$$

and hence,

$$L_{\mu,m}(\alpha)=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-(\mu+1)}D_{p=0}^{\mu}\frac{z^{m-p}}{(1-z)\mu-p+1}\,dt.$$

Since $A^k_\mu(\alpha) = S^k_\mu(\alpha) - S^k_{\mu-1}(\alpha)$, the equality (2.42) yields that

$$J_m(\alpha) = S^k_m(\alpha)L_{m,m}(\alpha) + \sum_{\mu=0}^{m-1} S^k_\mu(\alpha)(L_{\mu,m}(\alpha) - L_{\mu+1,m}(\alpha)) \tag{2.43}$$

$$= S^k_m(\alpha)L_{m,m}(\alpha) + \widetilde{J}_m(\alpha).$$

Further,

$$S^k_m(\alpha)L_{m,m}(\alpha) = \frac{S^k_m(\alpha)}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}e^{-(m+1)t} \sum_{\lambda=0}^{\infty} \binom{m+\lambda}{\lambda} z^\lambda \, dt \tag{2.44}$$

$$= \frac{S^k_m(\alpha)}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}e^{-(m+1)t} \frac{dt}{(1-z)^{m+1}} = O\left(\frac{1}{(m+1)^\alpha}\right), \quad m \to \infty.$$

For $\Gamma(\alpha)(L_{\mu,m}(\alpha) - Ł_{\mu+1,m}(\alpha))$ one gets that

$$\Gamma(\alpha)(L_{\mu,m}(\alpha) - L_{\mu+1,m}(\alpha))$$

$$= \int_0^\infty t^{\alpha-1}e^{-(\mu+1)t} \sum_{p=0}^{\mu} D^p_{m,\mu} \frac{z^{m-p}}{(1-z)^{\mu-p+1}} \, dt$$

$$- \int_0^\infty t^{\alpha-1}e^{-(\mu+2)t} \sum_{p=0}^{\mu+1} D^p_{m,\mu+1} \frac{z^{m-p}}{(1-z)^{\mu-p+1}} \, dt,$$

whence

$$L_{\mu.m}(\alpha) - L_{\mu+1,m}(\alpha)$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}e^{-(\mu+1)t} \frac{1-z-e^{-t}}{1-z} \sum_{p=0}^{\mu} D^p_{m,\mu} \frac{z^{m-p}}{(1-z)^{\mu-p+1}} \, dt$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}e^{-(\mu+1)} \sum_{p=0}^{\mu} (D^p_{m,\mu} - D^p_{m,\mu+1}) \frac{z^{m-p}}{(1-z)^{\mu-p+2}} \, dt$$

$$- \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}e^{-(\mu+2)t} D^{\mu+1}_{m,\mu+1} \frac{z^{m-\mu-1}}{1-z} \, dt = T^{(1)}_{m,\mu}(\alpha) + T^{(2)}_{m,\mu}(\alpha) + T^{(3)}_{m,\mu}(\alpha),$$

and hence,

$$\widetilde{J}_m(\alpha) = \sum_{\mu=0}^{m-1} S^k_\mu(\alpha)T^{(1)}_{m,\mu} + \sum_{\mu=0}^{m-1} S^k_\mu(\alpha)T^{(2)}_{m,\mu}(\alpha) + \sum_{\mu=0}^{m-1} S^k_\mu(\alpha)T^{(3)}_{m,\mu}(\alpha) \tag{2.45}$$

$$= \widetilde{J}_m^{(1)}(\alpha) + \widetilde{J}_m^{(2)}(\alpha) + \widetilde{J}_m^{(3)}(\alpha).$$

Further,

$$\widetilde{J}_m^{(1)}(\alpha)\Gamma(\alpha)$$

$$= \int_0^\infty t^{\alpha-1} \frac{1-z-e^{-t}}{1-z} \sum_{\mu=0}^{m-1} \sum_{p=0}^{\mu} S_\mu^k(\alpha) D_{m,\mu}^p \frac{z^{m-p}}{(1-z)^{\mu-p+1}} e^{-(\mu+1)t}\, dt$$

$$= \int_0^\infty t^{\alpha-1} \frac{1-z-e^{-t}}{1-z} \sum_{p=0}^{m-1} \sum_{\mu=p}^{m-1} S_\mu^k(\alpha) D_{m,\mu}^p \frac{z^{m-p}}{(1-z)^{\mu-p+1}} e^{-(\mu+1)t}\, dt.$$

But

$$\{m(m-1)\dots(m-p+1)\}^{-1} \sum_{\mu=p}^{m-1} D_{m,\mu}^p \frac{e^{-(\mu+1)t}}{(1-z)^{\mu-p+1}}$$

$$= \frac{e^{-(p+1)t}}{1-z} \sum_{\lambda=0}^{m-p-1} \frac{1}{(p+\lambda)(p+\lambda-1)\dots(\lambda+1)} \frac{e^{-\lambda t}}{(1-z)^\lambda},$$

the sequence

$$\left\{ \frac{1}{(p+\lambda)(p+\lambda-1)\dots(\lambda+1)} \right\}_{\lambda=0}^\infty$$

is decreasing and, moreover,

$$\sum_{\lambda=0}^{n} \frac{e^{-\lambda t}}{(1-z)^\lambda} < \frac{1-z}{1-z-e^{-t}}, \quad n = 0, 1, 2, \dots, \quad t \in (0, \infty),$$

a theorem of Abel yields that

$$\sum_{\mu=p}^{m-1} D_{m,\mu} \frac{e^{-(p+1)t}}{(1-z)^{\mu-p+1}} < \binom{m}{p} \frac{e^{-(p+1)t}}{1-z-e^{-t}}, \quad t \in (0, \infty).$$

Then,

$$\widetilde{J}_m^{(1)}(\alpha) = O\left(\int_0^\infty t^{\alpha-1} \frac{e^{-t}}{1-z} \sum_{p=0}^{m-1} \binom{m}{p} z^{m-p} e^{-pt}\, dt \right) \tag{2.46}$$

$$O\left(\int_0^\infty t^{\alpha-1} e^{-t} (z+e^{-t})^m\, dt \right) = \left(\frac{1}{m^\alpha} \right).$$

For $\widetilde{J}_m^{(3)}(\alpha)$ one gets that

$$\begin{aligned}(2.47)\qquad &\widetilde{J}_m^{(3)}(\alpha)\\ &= \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\frac{1}{1-z}\sum_{\mu=0}^{m} S_\mu^k(\alpha)\binom{m}{\mu+1} z^{m-\mu-1}e^{-(\mu+2)t}\,dt\\ &= O\left(\int_0^\infty t^{\alpha-1}e^{-t}\sum_{\mu=1}^{m} z^{m-\mu}e^{-\mu t}\,dt\right) = O\left(\frac{1}{m^\alpha}\right).\end{aligned}$$

It remains $\widetilde{J}_m^{(2)}(\alpha)$ to be estimated. From the inequality

$$D_{m,\mu}^p - D_{m,\mu+1}^p = \frac{p}{(\mu+1)\mu\ldots(\mu-p+1)} > 0,$$

it follows that the sequence $\{D_{m,\mu}^p\}_{\mu=1}^\infty$ is decreasing and since

$$\frac{e^{-t}}{1-z} < 1, \quad 0 < t < \infty,$$

the same holds for the sequence $\left\{\frac{e^{-\mu t}}{(1-z)^\mu}\right\}_{\mu=1}^\infty$, hence,

$$\begin{aligned}\sum_{\mu=p}^{m-1}(D_{m,\mu}^p - D_{m,\mu+1}^p)\frac{e^{-\mu t}}{(1-z)^\mu} &< \frac{e^{-pt}}{(1-z)^p}\sum_{\mu=p}^{m-1}(D_{m,\mu}^p - D_{m,\mu+1}^p)\\ &< \frac{e^{-pt}}{(1-z)^p}D_{m,p}^p = \binom{m}{p}\frac{e^{-pt}}{(1-z)^p}.\end{aligned}$$

Then

$$\begin{aligned}(2.48)\qquad \widetilde{J}_m^{(2)}(\alpha) &= O\left(\int_0^\infty t^{\alpha-1}e^{-t}\frac{1}{(1-z)^2}\sum_{p=0}^{m-1}\binom{m}{p}z^{m-p}e^{-pt}\,dt\right)\\ &= O\left(\int_0^\infty t^{\alpha-1}e^{-t}(z+e^{-t})^m\,dt\right) = O\left(\frac{1}{m^\alpha}\right).\end{aligned}$$

As a consequence of (2.45), (2.46), (2.47) and (2.48) one gets that $\widetilde{J}_m(\alpha) = O\left(\frac{1}{m^\alpha}\right)$ and then from (2.43) and (2.44) it follows that $J_m(\alpha) = O\left(\frac{1}{m^\alpha}\right)$, i.e., the proof of Theorem 11 is finished.

Theorem 12. *Let $\overline{e_k}$ be the abscissa of the $|E_k|$-summation for the series* (2.4). *If $\overline{e}_k \geq 0$, then*

$$\overline{e}_k = \limsup_{n\to\infty} \frac{\log(|A_0^k| + |A_1^k|| + \dots |A_n^k|)}{\log(n+1)}, \tag{2.49}$$

where $\{A_n^k\}_{n=0}^{\infty}$ are given by the equalities (2.3).

The proof proceeds as that of Theorem 5. More precisely, it is based on the representation

$$\sum_{\mu=0}^{m} A_\mu^k(s) = \sum_{\mu=0}^{m} A_\mu^k l_{m,\,u}(s), \tag{2.50}$$

where

$$A_m^k(s) = \frac{1}{(q+1)^{\mu+1}} \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} q^{\mu-\nu} a_\nu (\nu+1)^{-s},$$

$$l_{m,\mu}(s) = \frac{1}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} t^{-(\mu+1)} \sum_{\nu=0}^{m-\mu} \binom{\mu+\nu}{\nu} (1-e^{-t})^\nu \left(\frac{q}{q+1}\right)^\nu dt,$$

provided that $\sigma = \Re s > 0$. Then (2.50) leads to the inequality

$$\sum_{\mu=0}^{m} |A_\mu^k(s)| \leq \frac{1}{|\Gamma(s)|} \sum_{\mu=0}^{m} |A_\mu^k| \int_0^\infty t^{\sigma-1} \left(\frac{q+1}{q+e^t}\right)^{\mu+1} dt$$

$$\leq \frac{1}{|\Gamma(s)|} \sum_{\mu=0}^{m-1} B_\mu \int_0^\infty t^{\sigma-1} \frac{e^t-1}{e^t+q} \left(\frac{q+1}{q+e^t}\right)^{\mu+1} dt$$

$$+ \frac{B_\mu}{|\Gamma(s)|} \int_0^\infty t^{\sigma-1} \left(\frac{q+1}{q+e^t}\right)^{m+1} = I_m^{(1)}(s) + I_m^{(2)}(s),$$

where

$$B_\mu^k = |A_0^k| + |A_1^k| + \dots |A_\mu^k|, \quad m = 0, 1, 2, \dots.$$

Let α be the right-hand side of (2.49) and let $\alpha \geq 0$. Then from (2.49) it follows that for each $\varepsilon > 0$ there exists $K = K(\varepsilon)$ such that

$$B_\mu \leq K(\mu+1)^{\alpha+\varepsilon}, \quad \mu = 0, 1, 2, \dots.$$

Then

$$I_m^{(1)}(s) \le K_1 \int_0^\infty t^{\sigma-1} \frac{(q+e^t)^{\alpha+\varepsilon-1}}{(e^t-1)^{\alpha+\varepsilon}}\, dt$$

and the integral in the right-hand side is convergent provided that $\sigma > \alpha+\varepsilon$.

For the integral $I_m^{(2)}(s)$ one gets that

$$I_m^{(2)}(s) \le K_2 m^{\alpha+\varepsilon} \int_0^\infty t^{\sigma-1} \left(\frac{q+1}{q+e^t}\right)^{m+1} dt$$

$$= K_2 m^{\alpha+\varepsilon} \left\{ \int_0^1 t^{\sigma-1} \left(\frac{q+1}{q+e^t}\right)^{m+1} dt + \int_1^\infty t^{\sigma-1} \left(\frac{q+1}{q+e^t}\right)^{m+1} dt \right\}$$

Since

$$\frac{q+1}{q+e^t} \le \frac{q+1}{q+1+t} \le 1 - \frac{t}{q+2}, \quad 0 \le t \le 1,$$

it holds that

$$\int_0^1 t^{\sigma-1} \left(\frac{q+1}{q+e^t}\right)^{m+1} dt < \int_0^1 t^{\sigma-1} \left(1-\frac{t}{q+2}\right)^{m+1} dt$$

$$= (q+2)^\sigma \int_0^{1/(q+2)} t^{\sigma-1}(1-t)^{m+1}\, dt < (q+2)^\sigma \int_0^1 t^{\sigma-1}(1-t)^{m+1}\, dt$$

$$= (q+2)^\sigma \frac{\Gamma(\sigma)\Gamma(m+2)}{\Gamma(\sigma+m+2)},$$

and hence,

$$m^{\alpha+\varepsilon} \int_0^1 t^{\sigma-1} \left(\frac{q+1}{q+e^t}\right)^{m+1} dt = O(m^{\alpha-\sigma+\varepsilon}) = o(1), \quad m \to \infty.$$

Further,

$$m^{\alpha+\varepsilon} \int_1^\infty t^{\sigma-1} \left(\frac{q+1}{q+e^t}\right)^{m+1} dt \le m^{\alpha+\varepsilon} \left(\frac{q+1}{q+e}\right)^m \int_1^\infty t^{\sigma-1} \frac{q+1}{q+e^t}\, dt$$

and since

$$m^{\alpha+\varepsilon} \left(\frac{q+1}{q+e}\right)^m \int_1^\infty t^{\sigma-1} \frac{q+1}{q+e^t}\, dt = o(1), \quad m \to \infty,$$

it follows that $I_m^{(s)} = o(1)$, $m \to \infty$. Thus it is established that the series (2.4) is $|E_k|$-summable for each s such that $\sigma = \Re s > \alpha$, Let now the series (2.4) be $|E_k|$-summable for $s = \alpha > 0$, i.e., the series (2.6) is absolutely convergent for $s = \alpha$. If α is a positive integer, then the inequality

$$B_m^k(\alpha) \le \sum_{\mu=0}^{m} |A_\mu^k(\alpha)| \sum_{\nu=0}^{m-\mu} \binom{\mu+\nu}{\nu} \left(\frac{q}{q+1}\right)^\nu |\Delta^\nu((\mu+1)^\alpha)|$$

and the fact that $\Delta^\nu((\mu+1)^\alpha) = 0$ when $\nu > \alpha$, leads to the asymptotics of $B_m^k(\alpha)$ when $m \to \infty$, namely that $B_m^k(\alpha) = O(m^\alpha)$.

If α is not an integer and $p = [\alpha] + 1$, i.e., $\alpha < p < p + 1$, then (2.25) and (2.26) yield that

$$B_m^k(\alpha) \le I_m^{(1)}(\alpha) + I_m^{(2)}(\alpha) + I_m^{(3)}(\alpha),$$

where

$$I_m^{(1)}(\alpha) = \sum_{\mu=0}^{m-p+1} |A_\mu^k(\alpha) \sum_{\nu=0}^{p-1} \binom{\mu+\nu}{\nu} \left(\frac{q}{q+1}\right)^\nu |\Delta^\nu(\mu+1)^\alpha|,$$

$$I_m^{(2)}(\alpha) = \sum_{\mu=m-p+2}^{m} |A_\mu^k(\alpha)| \sum_{\nu=0}^{m-\mu} \binom{\mu+\nu}{\nu} \left(\frac{q}{q+1}\right)^\nu |\Delta^\nu(\mu+1)^\alpha|$$

and

$$I_m^{(3)}(\alpha = \sum_{\mu=0}^{m-p+1} |A_\mu^k(\alpha)| \sum_{\nu=p}^{m-\mu} \binom{\mu+\nu}{\nu} \left(\frac{q}{q+1}\right)^\nu |\Delta^\nu(\mu+1)^\alpha|.$$

But

$$I_m^{(3)}(\alpha) = \frac{\left(\frac{q}{q+1}\right)^p}{g(\alpha)} \sum_{\mu=0}^{m-p+1} |A_m^k(\alpha)| \int_0^\infty t^{-\alpha-1} e^{-(\mu+1)t} (1-e^{-t})^p P_{\mu,m}(t)\, dt,$$

$$P_{\mu,m}(t) = \sum_{\nu=0}^{m-\mu-p} \binom{\mu+p+\nu}{\mu} \left(\frac{q}{q+1}\right)^\nu (1-e^{-t})^\nu$$

and since

$$\binom{\mu+p+\nu}{\mu} \le K(\mu+1)^p \binom{\mu+\nu}{\mu}, \quad n = 0, 1, 2 \dots,$$

it follows that

$$I_m^{(3)}(\alpha) \le K_1 \sum_{\mu=0}^{m-p+1} (\mu+1)^p |A_\mu^k(\alpha)| \int_0^\infty t^{-\alpha-1} e^{-(\mu+1)t} (1-e^{-t})^p Q_{\mu,m}(t)\, dt,$$

where

$$Q_{\mu,m}(t) = \sum_{\nu=0}^{m-\mu-p} \binom{\mu+\nu}{\mu} \left(\frac{q}{q+1}\right)^{\nu} (1-e^{-t})^{\nu},$$

and hence,

$$\begin{aligned} I_m^{(3)}(\alpha) &\le K_2 \sum_{\mu=0}^{m} (\mu+1)^p |A_\mu^k(\alpha)| \int_0^\infty t^{-\alpha-1} e^{-(\mu+1)t} (1-e^{-t})^p \, dt \\ &\le K_2 \sum_{\mu=0}^{m} (\mu+1)^p |A_\mu^k(\alpha)| \int_0^\infty t^{p-\alpha-1} e^{-(\mu+1)t} \, dt \\ &= K_2 \sum_{\mu=0}^{m} (\mu+1)^\alpha |A_\mu^k(\alpha)| \int_0^\infty t^{p-\alpha-1} e^{-t} \, dt, \end{aligned}$$

i.e.,

$$\le K_3 m^\alpha \sum_{\mu=0}^{m} |A_\mu^k(\alpha)| \le K_4 m^\alpha.$$

Since

$$\binom{\mu+\nu}{\mu} |\Delta^\nu (\mu+1)^\nu| = O(\mu^{\alpha-\nu}\mu^\nu) = O(\mu^\alpha),$$

it follows that

$$I_m^{(1)}(\alpha) + I_m^{(2)}(\alpha) = O(m^\alpha), \quad m \to \infty,$$

and thus the theorem is proved.

Theorem 13. *If $\overline{e}_k$ is negative, then*

$$\overline{e}_k = \limsup_{n\to\infty} \frac{\log(|A_n| + |A_{n+1}| + \dots)}{\log(n+1)}. \tag{2.51}$$

Let, as in the proof of Theorem 5, $\beta \in (-\infty, 0)$ be the right-hand side of (2.51), $\varepsilon > 0$ be such that $\beta + 2\varepsilon < 0$ and $-\alpha = \beta + 2\varepsilon$, i.e., $\alpha > 0$. Then there exists a positive $K = K(\varepsilon)$ such that

$$|A_n| + |A_{n+1}| + |A_{n+2}| + \dots \le K(n+1)^{\beta+\varepsilon}, \quad n = 0, 1, 2, \dots .$$

Further, let as before, $\sum_{n=0}^\infty A_n^k(-\alpha)$ be the E_k-transform of series (2.4) for $s = -\alpha$ and let α be an integer. Since $\Delta^{n-\mu}(\mu+1)^\alpha = 0$ when $n - \mu > \alpha$,

$$A_n^k(-\alpha) = \left(\frac{q}{q+1}\right)^n \sum_{\mu=n-\alpha}^{n} \left(\frac{q+1}{q}\right)^\mu \binom{n}{\mu} \Delta^{n-\mu}(\mu+1)^\alpha = \sum_{\lambda=0}^{\alpha} J_n^{(\lambda)}(\alpha),$$

where

$$J_n^{(\lambda)}(\alpha) = A_{n-\lambda}^k \left(\frac{q}{q+1}\right)^\lambda \binom{n}{\lambda} \Delta^\lambda (n-\lambda+1)^\alpha.$$

Let $R_n^k = |A_n^k| + |A_{n+1}^k| + \dots$, then, since

$$|J_n^{(\lambda)}(\alpha)| \le K_1 \binom{n}{\lambda} (n-\lambda)^{\alpha-\lambda} |A_{n-\lambda}^k| \le K_2 |A_{n-\lambda}^k| (n-\lambda)^\alpha,$$

it follows that

$$\sum_{n=N}^{N+m} |J_n^{(\lambda)}(\alpha)|$$

$$\le K_2 \left\{ (N-\lambda)^\alpha (R_{N-\lambda}^k - R_{N-\lambda+1}) + \sum_{\nu=1}^{m} (N+\nu-\lambda)^\alpha (R_{n+\nu-\lambda}^k - R_{n+\nu+1-\lambda}^k) \right\}$$

$$= K_2 \left\{ (N-\lambda)^\alpha R_{N-\lambda}^k + \sum_{\nu=1}^{n} R_{N+\nu-\lambda}^k ((N+\nu-\lambda)^\alpha - (N+\nu-1-\lambda)^\alpha) \right\}$$

$$-K_2 R_{N+m+1-\lambda}^k (N+m-\lambda)^\alpha.$$

Since $R_p^k = O(p^{\beta+\varepsilon})$ and $(p+1)^\alpha - p^\alpha = O(p^{\alpha-1}), p \to \infty$, it follows that $\lim_{N\to\infty} (N-\lambda)^\alpha R_{n-\lambda}^k = 0$ and $\lim_{N\to\infty} (N+m-\lambda)^\alpha R_{N+m-\lambda} = 0$, hence

$$\sum_{n=N}^{\infty} |J_n^{(\lambda)}(\alpha)| \le K_2 (N-\lambda)^\alpha R_{N-\lambda}^k + K_3 \sum_{\nu=1}^{\infty} \frac{1}{(N+\nu-\lambda)^{1+\varepsilon}}.$$

That means the series $\sum_{n=1}^{\infty} |J_n^{(\lambda)}(\alpha)|$ is convergent for each $\lambda = 0, 1, 2, \dots, \alpha$ and so does the series $\sum_{n=0}^{\infty} |A_n^k(-\alpha)|$.

Let now α be not an integer. Then,

$$I_m^{(3)}(\alpha) \le L \sum_{\mu=0}^{m} |A_\mu^k| (\mu+1)^\alpha$$

$$\le L_1 \sum_{\mu=0}^{m} (R_\mu^k - R_{\mu+1}^k)(\mu+1)^\alpha \le L_2 \sum_{\mu=0}^{m} R_\mu^k ((\mu+1)^\alpha - \mu^\alpha)$$

$$= L_2 \left\{ R_0^k + \sum_{\mu=1}^{m} R_\mu^k ((\mu+1)^\alpha - \mu^\alpha) - R_m^k (m+1)^\alpha \right\}.$$

But since $R_\mu^k = O(\mu^{\beta+\varepsilon})$ and $(\mu+1)^\alpha - \mu^\alpha = O(\mu^{\alpha-1})$, it follows that

$$I_m^{(3)}(\alpha) = O\left(R_0^k + \sum_{\mu=1}^{m} \frac{1}{\mu^{1+\varepsilon}}\right) = O(1), \ m \to \infty.$$

In a similar way one gets that $I_m^{(1)}(\alpha)$ as well as $I_m^{(2)}(\alpha) = O(1)$ when $m \to \infty$ and, hence, the series (2.4) is $|E_k|$-summable for each s such that $\Re s > \beta$.

Let, conversely, the series (2.4) be $|E_k|$-summable for $s = \alpha < 0$, i.e., the series (2.29) is convergent. Then, taking into consideration that

$$(\mu+\nu+1)^\alpha = \frac{1}{(\mu+\nu+1)^{-\alpha}} = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-(\mu+\nu+1)t}\, dt,$$

one gets that

$$A_n^k = \sum_{\nu=0}^{n} \binom{n}{\nu} \left(\frac{q}{q+1}\right)^{n-\nu} A_\nu^k(-\alpha) \sum_{\mu=0}^{n-\nu} (-1)^\mu \binom{n-\nu}{\mu} (\mu+\nu+1)^\alpha$$

$$= \sum_{\nu=0}^{n} \binom{n}{\nu} \left(\frac{q}{q+1}\right)^{n-\nu} A_n^k(-\alpha) b_{n,\nu}(\alpha),$$

where

$$b_{n,\nu}(\alpha) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-(\nu+1)t} (1-e^{-t})^{n-\nu}\, dt.$$

Hence,

$$\sum_{n=m}^{\lambda} |A_n^k| \leq \sum_{n=m}^{\lambda} \sum_{\nu=0}^{n} \binom{n}{\nu} \left(\frac{q}{q+1}\right)^{n-\nu} |A_\nu^k(-\alpha)| b_{n,\nu}(\alpha)$$

$$< \sum_{n=m}^{\lambda} \sum_{\nu=0}^{m} \binom{n}{\nu} \left(\frac{q}{q+1}\right)^{n-\nu} |A_\nu^k(-\alpha)| b_{n,\nu}(\alpha)$$

$$+ \sum_{n=m}^{\lambda} \sum_{n=\nu}^{\lambda} \binom{n}{\nu} \left(\frac{q}{q+1}\right)^{n-\nu} |A_\nu^k(-\alpha)| b_{n,\nu}(\alpha)$$

$$= J_{m,\lambda}^{(1)}(\alpha) + J_{m,\lambda}^{(2)}(\alpha).$$

But

$$B_{\nu,\lambda}(\alpha) = \sum_{n=\nu}^{\lambda} \binom{n}{\nu} \left(\frac{q}{q+1}\right)^{n-\nu} b_{n,\nu}(\alpha)$$

$$= \frac{1}{\Gamma(-\alpha)} \sum_{n=\nu}^{\lambda} \binom{n}{\nu} \left(\frac{q}{q+1}\right)^{n-\nu} \int_0^\infty t^{-\alpha-1} e^{-(\nu+1)t} (1-e^{-t})^{n-\nu}\, dt$$

$$< \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-(\nu+1)t} \sum_{\tau=0}^{\infty} \binom{\nu+\tau}{\tau} \left(\frac{q}{q+1}\right)^{\tau} (1-e^{-t})^{\tau}\, dt$$

$$= \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-(\nu+1)t} \left(1 - \frac{q}{q+1}(1-e^{-t})\right)^{-\nu-1} dt$$

$$= \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} \left(\frac{q+1}{q+e^t}\right)^{\nu+1} dt$$

whence

$$\limsup_{\lambda\to\infty} B_{\nu,\lambda}(\alpha) \le B_\nu(\alpha),$$

where

$$B_\nu(\alpha) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} \left(\frac{q+1}{q+e^t}\right)^{\nu+1} dt.$$

Since

$$J_{m,\lambda}^{(2)}(\alpha) = \sum_{\nu=m}^{\lambda} |A_\nu^k(-\alpha)| B_{\nu,\lambda}(\alpha)$$

and the series $\sum_{\nu=0}^{\infty} |A_\nu^k(-\alpha)|$ is convergent, it follows that

$$J_m^{(2)}(\alpha) = \lim_{\lambda\to\infty} J_{m,\lambda}^{(2)}(\alpha) \le \sum_{\nu=m}^{\infty} |A_\nu^k(-\alpha)| B_\nu(\alpha).$$

Let $S_\nu^k(-\alpha) = |A_0^k(-\alpha)| + A_1^k(-\alpha) + \cdots + |A_\nu^k(-\alpha)|$, then $|A_\nu^k(-\alpha)| = S_\nu^k(-\alpha) - S_{\nu-1}^k(\alpha), \nu = 1, 2, \ldots$ and hence

$$\sum_{\nu=m}^{\infty} |A_\nu^k(-\alpha)| B_\nu(\alpha) = S_m^k(-\alpha) B_m(\alpha) - \sum_{\nu=m-1}^{\infty} S_\nu^k(\alpha)(B_\nu(\alpha) - B_{\nu+1}(\alpha)).$$

But the sequence $\{B_\nu(\alpha)\}_{\nu=0}^{\infty}$ is decreasing and, moreover,

$$B_m(\alpha) = O\left(\int_0^\infty t^{-\alpha-1} e^{-(m+1)t)}\, dt\right) = O(m^\alpha), \quad m \to \infty,$$

i.e.,

$$\sum_{\nu=m}^{\infty} |A_\nu(-\alpha)| B_\nu(\alpha) = O(m^\alpha), \quad m \to \infty$$

which yields that $J_m^{(2)}(\alpha) = O(m^\alpha),\ m \to \infty$.

It remains

$$J_{m,\lambda}^{(1)}(\alpha) = \sum_{n=m}^{\lambda}\sum_{\nu=0}^{m}\binom{n}{\nu}\left(\frac{q}{q+1}\right)^{n-\nu}|A_\nu^k(-\alpha)| = \sum_{\nu=0}^{m}|A_\nu^k(-\alpha)|G_{m,\lambda}^{(\nu)}(\alpha),$$

where

$$G_{m,\lambda}^{(\nu)}(\alpha) = \sum_{n=m}^{\lambda}\binom{n}{\nu}\left(\frac{q}{q+1}\right)^{n-\nu} b_{n,\nu}(\alpha),$$

to be studied. Since $(1-e^{-t})\frac{q}{q+1} \le \frac{q}{q+1}$, from

$$G_{m,\lambda}^{(\nu)}(\alpha) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty t^{-\alpha-1}e^{-(\nu+1)t}\sum_{n=m}^{\lambda}\binom{n}{\nu}\left(\frac{q}{q+1}\right)^{n-\nu}(1-e^{-t})^{n-\nu}\,dt$$

it follows that there exists

$$G_m^{(\nu)}(\alpha) = \lim_{\lambda\to\infty} G_{m,\lambda}^{(\nu)}(\alpha) = \frac{1}{\Gamma(-\alpha)}\left(\frac{q}{q+1}\right)^{m-\nu}\int_0^\infty t^{-\alpha-1}e^{-(\nu+1)t}(1-e^{-t})^{m-\nu}H_{m,\nu}(t)\,dt,$$

where

$$H_m(t) = \sum_{\tau=0}^{\infty}\binom{m+\tau}{\tau}\left(\frac{q}{q+1}\right)^{\tau}(1-e^{-t})^\tau,\quad 0 \le t < \infty,$$

and hence, there also exists

$$J_m^{(1)}(\alpha) = \lim_{\lambda\to\infty} J_{m,\lambda}^{(1)}(\alpha) = \sum_{\nu=0}^{m}|A_\nu^k(-\alpha)|G_m^{(\nu)}(\alpha).$$

Further, if $z = (1-e^{-t})\frac{q}{q+1}$, then

$$\left(\frac{q}{q+1}\right)^{m-\nu}(1-e^{-t})^{m-\nu}H_m(t) = \sum_{\tau=0}^{\infty}\binom{m+\tau}{\tau}z^{m-\nu+\tau} = \sum_{\tau=m-\nu}^{\infty}\binom{\tau+\nu}{\tau}z^\tau$$

and, as in the proof of Theorem 6, one gets that $J_m^{(1)}(\alpha) = O(m^\alpha),\ m \to \infty$.

Theorem 14. *The inequalities* $e_k \le \overline{e}_k \le e_k + 1$ *hold true.*

The left inequality one is evident. In order to prove the right one, it is sufficient to establish that if the series (2.4) is E_k-summable for some $s = s_0$, then it is $|E_k|$-summable for $s = s_0 + \alpha, \alpha >$. It may be assumed that $s_0 = 0$ and in such a case the change of α by $-\alpha$ immediately gives the representation

$$A_n^k(\alpha)$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{\mu=0}^{n} \binom{n}{\mu} A_\mu^k \int_0^\infty t^{\alpha-1} e(-(\mu+1)t) z^{n-\mu}\, dt\, z = (1-e^{-t}) \frac{q}{q+1}.$$

Since the sequence $\{A_n^k\}_{n=0}^\infty$ tends to zero,

$$|A_n^k(\alpha)| \le K \sum_{\mu=0}^{n} \int_0^\infty t^{\alpha-1} e^{-(\mu+1)t} z^{n-\mu}\, dt$$

$$= K \int_0^\infty t^{\alpha-1} e^{-t} (e^{-t} + z)^n \, dt = K \int_0^\infty t^{\alpha-1} e^{-t} \left(\frac{q+e^{-t}}{q+1} \right)^n dt.$$

But

$$\int_0^\infty t^{\alpha-1} e^{-t} \left(\frac{q+e^{-t}}{q+1} \right)^n dt = O(n^{-\alpha}), \quad n \to \infty,$$

hence the series $\sum_{n=0}^\infty |A_n^k(\alpha)|$ is convergent for each $\alpha > 1$. In fact, a more general theorem is established, namely:

Theorem 15. *If the sequence* $\{|A_n^k(s_0)|\}_{n=0}^\infty$ *is bounded, then the series* (2.4) *is* $|E_k|$*-summable for each* s *such that* $\Re s > \Re s_0 + 1$,

3. Summation of factorial series

The general form of these series is

$$a_0 + \sum_{\nu=1}^{\infty} \frac{a_\nu}{(s+1)(s+2)\dots(s+\nu)}, \tag{3.1}$$

$$s \in \mathbb{C} \setminus \mathbb{Z}^-, \ \mathbb{Z}^- = \{-1, -2, -3, \dots\}.$$

Let

$$\sum_{n=0}^{n} A_n^k(s)$$

be the E_k-transform of the series (3.1), i.e.,

$$A_n^k(s) = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \frac{a_\nu}{(s+1)(s+2)\dots(s+\nu)},$$

$$n = 1, 2, 3, \dots, \quad A_0^k(s) = a_0.$$

A basic result for the E_k-summation of the series (3.1) is the following theorem:

Theorem 16. *Let the series* (3.1) *be E_k-sumable for some $s = s_0$, i.e., the series*

$$\sum_{n=0}^{\infty} A_n^k(s_0)$$

is convergent. Then, it is E_k-summable for each $s \in \mathbb{C} \setminus \mathbb{Z}^-$ such that $\Re s > \Re s_0$ and its E_k-sum is a function $f(s)$ holomorphic in the region $H(s_0; s) = \{s : \Re s > \Re s_0\} \setminus \mathbb{Z}^-$ with possible poles at the points of the set $\mathbb{Z}^-$. Moreover, if $\Re s_0 \geq 0$, then

$$f(s) = \frac{\Gamma(s+1)}{\Gamma(s-s_0)\Gamma(s_0+1)} \sum_{\mu=0}^{\infty} S_\mu^k(s_0) T_\mu(s_0; s), \tag{3.2}$$

$$T_\mu(s_0; s) = \int_0^1 t^{s-s_0} (1-t)^{\mu+s_0} \left(\frac{q+1}{q+1-qt} \right)^{\mu+1} \frac{dt}{q+1-qt},$$

where

$$S_\mu^k(s_0) = \sum_{\nu=0}^{\mu} A_\nu^k(s_0).$$

If $\Re s_0 < 0$, then

$$f(s) = \sum_{\nu=0}^{p-1} \frac{a_\nu}{(s+1)(s+2)\dots(s+\nu)}$$

$$+ \frac{\Gamma(s+1)}{\Gamma(s-s_0)\Gamma(s_0+1)} \sum_{\mu=0}^{\infty} S_{\mu,p}^k(s_0) T_{\mu,p}(s_0; s),$$

$$T_{\mu,p}(s_0; s) = \int_0^1 t^{s-s_0} (1-t)^{\mu+s_0+p} \left(\frac{q+1}{q+1-qt} \right)^{\mu+1} \frac{dt}{q+1-qt},$$

where p is a positive integer greater than $-\Re s_0$,

$$S_{\mu,p}^k(s_0) = \sum_{\nu=0}^{\mu} A_{\nu,p}^k(s_0)$$

and

$$A_{\nu,p}^k(s_0) = \frac{1}{(q+1)^{\nu+1}} \sum_{\tau=0}^{\nu} q^{\nu-\tau} \binom{\nu}{\tau} \frac{a_{\tau+p}}{(s_0+1)(s_0+2)\dots(s_0+\tau+p)}.$$

Let $\Re s > \Re s_0 \geq 0$ and let as before

$$b_0(s) = a_0,\ b_\nu(s) = \frac{a_\nu}{(s+1)(s+2)\dots(s+\nu)}, \quad \nu \geq 1.$$

If

$$\lambda_0(s_0; s) = 1,\ \lambda_\nu(s_0; s) = \frac{(s_0+1)(s_0+2)\dots(s_0+\nu)}{(s+1)(s+2)\dots(s+\nu)}, \quad \nu \geq 1,$$

then the series (3.1) can be written in the form

$$\sum_{\nu=0}^{\infty} b_\nu(s_0)\lambda_\nu(s_0; s). \tag{3.3}$$

If $\sum_{n=0}^{\infty} A_n^k(s)$ is the E_k-transform of the series (3.1), then as before

$$A_n^k(s) = \sum_{\mu=0}^{n} A_\mu^k(s_0) \left(\frac{q}{q+1}\right)^{n-\mu} \binom{n}{\mu} \Delta^{n-\mu}\lambda_\mu(s_0; s). \tag{3.4}$$

But

$$\lambda_{\mu+1}(s_0; s) = \lambda_\mu(s_0; s)\frac{s_0+\mu+1}{s+\mu+1},$$

$$\Delta\lambda_\mu(s_0; s) = \lambda_\mu(s_0; s) - \lambda_{\mu+1}(s_0; s) = \lambda_\mu(s_0; s)(s_0, s)\frac{s-s_0}{s+\mu+1},$$

$$\Delta^2\lambda_\mu(s_0; s) = \Delta\lambda_\mu(s_0; s) - \Delta\lambda_{\mu+1}(s_0; s) = \lambda_\mu(s_0; s)\frac{(s-s_0)(s-s_0+1)}{(s+\mu+1)(s+\mu+2)},$$

. . . .

$$\Delta^\nu\lambda_\mu(s_0; s) = \lambda_\mu(s_0; s)\frac{(s-s_0)(s-s_0+1)\dots(s-s_0+\nu-1)}{(s+\mu+1)(s+\mu+2)\dots(s+\mu+k)}.$$

If $d = s - s_0$, then

$$\Delta^{n-\mu}\lambda_\mu(s_0;s) = \frac{d(d+1)\dots(d+n-\mu-1)(s_0+1)(s_0+2)\dots(s_0+\mu)}{(s+1)(s+2)\dots(s+n)}$$

$$= \frac{\Gamma(d+n-\mu)\Gamma(s_0+\mu+1)\Gamma(s+1)}{\Gamma(d)\Gamma(s_0+1)\Gamma(s+n+1)}$$

and since

$$\Delta^{n-\mu}\lambda_\mu(s_0;s) = \frac{\Gamma(s+1)}{\Gamma(d)\Gamma(s_0+1)}\int_0^1 t^{d+n-\mu-1}(1-t)^{s_0+\mu}\,dt,\ \ \mu = 0,1,2,\dots,n,$$

it follows that

$$A_n^k(s)$$

$$= \frac{\Gamma(s+1)}{\Gamma(s-s_0)\Gamma(s_0+1)}\sum_{\mu=0}^{n} A_\mu^k(s_0)\left(\frac{q}{q+1}\right)^{n-\mu}\binom{n}{\mu}\int_0^1 t^{d+n-\mu-1}(1-t)^{s_0+\mu}\,dt.$$

Further,

$$S_m^k(s) = \sum_{n=0}^{m} A_n^k(s)\frac{\Gamma(s+1)}{\Gamma(s-s_0)\Gamma(s_0+1)}\sum_{\mu=0}^{m} A_\mu^k(s_0) l_{m,\mu}(s_0;s),$$

where

$$l_{m,\mu}(s_0;s) = \sum_{\tau=0}^{m-\mu}\left(\frac{q}{q+1}\right)^{\tau}\binom{\mu+\tau}{\mu}\int_0^1 t^{d+\tau-1}(1-t)^{s_0+\mu}\,dt.$$

If μ is fixed, then

$$l_\mu(s_0;s) = \lim_{m\to\infty} l_{m,\mu}(s_0;s)$$

$$= \sum_{\tau=0}^{\infty}\left(\frac{q}{q+1}\right)^{\tau}\binom{\mu+\tau}{\mu}\int_0^1 t^{d+\tau-1}(1-t)^{s_0+\mu}\,dt$$

$$= \int_0^1 t^{d-1}(1-t)^{s_0+\mu}\left(\frac{q+1}{q+1-qt}qq\right)^{\mu+1}\,dt.$$

For $S_m^k(s)$ it holds that

$$S_m^k(s) = \sum_{\mu=0}^{m-1} S_\mu^k(s_0)h_{m,\mu}(s_0;s) + S_m^k(s_0)l_{m,m}(s_0;s),$$

where

$$h_{m,\mu}(s_0;s) = l_{m,\mu}(s_0;s) - l_{m,\mu+1}(s_0;s),\ \ \mu = 0,1,2,\dots,m-1.$$

It is easily seen that

$$\lim_{m\to\infty} S_m^k(s_0) l_{m,m}(s_0;s) = 0$$

as well as that

$$h_\mu(s_0;s) = \lim_{m\to\infty} h_{m,\mu}(s_0;s) = l_\mu(s_0;s) - l_{\mu+1}(s_0;s)$$

$$= \int_0^1 t^{d-1}(1-t)^{\mu+s_0}\left(\frac{q+1}{q+1-qt}\right)^{\mu+1}\frac{dt}{q+1-qt}.$$

Further, it holds the representation

$$l_{m,\mu}(s_0;s) = \int_0^1 t^{d-1}(1-t)^{\mu+s_0} P_{m,\mu}(\zeta)\,dt, \quad \zeta = \frac{qt}{q+1}, \quad 0 < t < 1,$$

where

$$P_{m,\mu}(\zeta) = \sum_{\nu=0}^{m-\mu}\binom{\mu+\nu}{\mu}\zeta^\nu, \quad \mu = 0,1,2,\dots,m.$$

But

$$P_{m,\mu}(\zeta) = -\sum_{\nu=1}^{\mu}\binom{m+1}{\nu}\frac{\zeta^{m+1-\nu}}{(1-\zeta)^{\mu-\nu+1}}$$

and since

$$h_{m,\mu}(s_0;s) = \int_0^1 t^{d-1}(1-t)^{\mu+s_0}\{P_{m,\mu}(\zeta) - (1-t)P_{m,\mu+1}(\zeta)\}\,dt,$$

it follows that

$$h_{m,\mu}(s_0;s) = -\frac{1}{q+1}\int_0^1 t^{d-1}(1-t)^{\mu+s_0}\sum_{\nu=0}^{\mu}\binom{m+1}{\nu}\frac{\zeta^{m+1-\nu}}{(1-\zeta)^{\mu+2-\nu}}\,dt$$

$$+\binom{m+1}{\mu+1}\int_0^1 t^{d-1}(1-t)^{\mu+1+s_0}\frac{\zeta^{m-\mu}}{1-\zeta}\,dt, \quad \mu = 0,1,2,\dots,m-1.$$

Then, in a way analogous to that already used by treating Dirichlet's series, one can prove that

$$\sum_{\mu=0}^{m-1} |h_{m,\mu}(s_0;s)| = O(1), \quad m \to \infty,$$

for each s such that $\Re s > \Re s_0$ and thus it is established that the sequence $\{S_m^k(s)\}_{m=0}^\infty$ tends to the right-side of the equality (3.2) under the same condition on s.

Since

$$\sum_{\mu=0}^{\infty} \left(\frac{(1-t)(q+1)}{q+1-qt}\right)^{\mu+1} = \frac{q+1-qt}{t}, \quad 0 < t < 1,$$

it follows that

$$\sum_{\mu=0}^{\infty} S_\mu^k(s_0) \int_0^1 t^{s-s_0}(1-t)^{\mu+s_0} \left(\frac{q+1}{q+1-qt}\right)^{\mu+1} \frac{dt}{q+1-qt}$$

$$= F^k(s_0;s) + G^k(s_0;s),$$

where

$$F^k(s_0;s) = S^k(s_0) \int_0^1 t^{s-s_0-1}(1-t)^{s_0} \frac{q+1}{q+1-qt}\, dt,$$

$$S^k(s_0) = \sum_{n=0}^{\infty} A_n^k(s_0),$$

$$G^k(s_0;s) = \sum_{\mu=0}^{\infty} \varepsilon^k(s_0) \int_0^1 t^{s-s_0}(1-t)^{\mu+s_0} \left(\frac{q+1}{q+1-qt}\right)^{\mu+1} \frac{dt}{q+1-qt},$$

$$\varepsilon_\mu^k(s_0) = S_\mu^k(s_0) - S^k(s_0).$$

It is clear that whatever $\delta > 0$ may be, the integral defining $F^k(s_0;s)$ is absolutely uniformly convergent on the closed half-plane $\widetilde{H}_\delta(s_0;s) = \{s : \Re(s-s_0) \geq \delta, s \notin \mathbb{Z}^-\}$, i.e., it defines a function holomorphic in the half-plane $\widetilde{H}(s_0;s) = \{s : \Re(s-s_0) > 0\}$.

Further, if $\varepsilon > 0$, then there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|\varepsilon_\mu^k(s_0)| \leq \varepsilon, \mu \geq N$ and the inequality

$$\left|\sum_{\mu=N}^{\infty} \varepsilon_\mu^k(s_0) \int_0^1 t^{s-s_0}(1-t)^{\mu+s_0} \left(\frac{q+1}{q+1-qt}\right)^{\mu+1} \frac{dt}{q+1-qt}\right|$$

$$\leq \varepsilon \int_0^1 t^{\sigma-\sigma_0}(1-t)^{\sigma_0} \frac{q+1}{q+1-qt}, \quad \sigma = \Re s, \quad \sigma_0 = \Re s_0,$$

yields that the series defining $G^k(s_0, s)$ is uniformly convergent in each of the half-planes $\widetilde{H}_\delta(s_0; s), \delta > 0$, hence, it also defines a function holomorphic in the $\widetilde{H}(s_0; s)$. Therefore, since $\Re s_0 \geq 0$, the function f defined by the equality (3.2) is holomorphic in this half-plane.

Let now $\Re s_0 < 0$, then the series (3.1) can be written as follows

$$\sum_{\nu=0}^{p-1} \frac{a_\nu}{(s+1)\dots(s+\nu)}$$

$$+ \sum_{\nu=p}^{\infty} b_\nu(s_0) \frac{(s_0+1)\dots(s_0+p)}{(s+1)\dots(s+p)} \cdot \frac{(s_0+p+1)\dots(s_0+p+\nu)}{(s+p+1)\dots(s+p+\nu)},$$

where p is a positive integer greater than $-\Re s_0$. If $b_{p+\nu}(s) = c_\nu(s), s_0 + p = u_0, s + p = u$, then the latter series takes the form

$$\sum_{\nu=0}^{p-1} \frac{a_\nu}{(s+1)\dots(s+\nu)}$$

$$+ \frac{(s_0+1)\dots(s_0+p)}{(s+1)\dots(s+p)} \sum_{\nu=0}^{\infty} c_\nu(s_0) \frac{(u_0+1)\dots(u_0+\nu)}{(u+1)\dots(u+\nu)}.$$

The series

$$\sum_{\nu=0}^{\infty} c_\nu(s_0) \frac{(u_0+1)\dots(u_0+\nu)}{(u+1)\dots(u+\nu)} \tag{3.5}$$

is E_k-summable for $u = u_0$ and if

$$A_{n,p}^k(s_0) = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} c_\nu(s_0), \; S_{n,p}^k(s_0) = \sum_{\nu=0}^{n} A_{n,p}^k(s_0),$$

then by the first part of the theorem the series (3.5) is E_k-summable for each u such that $\Re u > \Re u_0$ with sum

$$\frac{\Gamma(u+1)}{\Gamma(u-u_0)\Gamma(u_0+1)} \sum_{\mu=0}^{\infty} S_{\mu,p}^k \int_0^1 t^{u-u_0}(1-t)^{\mu+u_0} \left(\frac{q+1}{q+1-qt}\right)^{\mu+1} \frac{dt}{q+1-qt}.$$

Since
$$\frac{(s_0+1)\dots(s_0+p)}{(s+1)\dots(s+p)} = \frac{\Gamma(s_0+p+1)}{\Gamma(s_0+1)} \cdot \frac{\Gamma(s+1)}{\Gamma(s+p+1)},$$
it follows that the series (3.1) is E_k-summable for each s such that $\Re s > \Re s_0$ with the sum

$$f(s) = \sum_{\nu=0}^{p-1} \frac{a_\nu}{(s+1)\dots(s+\nu)} + \frac{\Gamma(s+1)}{\Gamma(s-s_0)\Gamma(s_0+1)} \sum_{\mu=0}^{\infty} S^k_{\mu,p} T_{\mu,p}(s_0;s),$$

$$T_{\mu.p}(s_0;s) = \int_0^1 t^{s-s_0}(1-t)^{\mu+s_0+p} \left(\frac{q+1}{q+1-qt}\right)^{\mu+1} \frac{dt}{q+t-qt}.$$

A consequence of the theorem just proved is that there exists a number f_k with the property that the series (3.1) is E^k-summable if $\Re s > f_k$ and it is not E^k-summable when $\Re s < f_k$. This number is called abscissa of E_k-summability for the series (3.1). Let $A^k_n = A^k_n(0), n = 0,1,2,\dots,$ then it holds the following theorem:

Theorem 17. *If $f_k \geq 0$, then*

$$f_k = \limsup_{n\to\infty} \frac{\log|A^k_0 + A^k_1 + \dots + A^k_n|}{\log(n+1)}. \tag{3.6}$$

Let α be the right-hand side of (3.6) and let suppose that $\alpha \in [0,\infty)$. If $\varepsilon > 0$, then
$$|A^k_0 + A^k_2 + \dots + A^k_n| = O((n+1)^{\alpha+\varepsilon}), \quad n\to\infty,$$
i.e., there exists $K = K(\varepsilon) > 0$ such that

$$|A_0 + A^k_1 + \dots + A^k_n| \leq K(n+1)^{\alpha+\varepsilon}, \quad n = 0,1,2,\dots.$$

Let further

$$\lambda_\nu(s) = \frac{\nu!}{(s+1)\dots(s+\nu)}, \quad \nu = 0,1,2,\dots,$$

then it holds a representation similar to (3.5), namely

$$A^k_n(s) = \sum_{\mu=0}^{n} A^k_\mu \binom{n}{\mu} \left(\frac{q}{q+1}\right)^{n-\mu} \Delta^{n-\mu}\lambda_\mu(s),$$

and moreover,

$$\Delta^{n-\mu}\lambda_\mu(s) = s\int_0^1 t^{s-1+n-\mu}(1-t)^\mu\,dt,$$

provided that $\Re s > 0$. If $S_m^k = \sum_{\mu=0}^m A_\mu^k$, then

$$S_m^k(s) = \sum_{\mu=0}^m A_\mu^k(s) = \sum_{\mu=0}^m A_\mu^k l_{m,\mu}(s) = s\sum_{\mu=0}^{m-1} S_\mu^k h_{m,\mu}(s) + sS_m^k l_{m,m}(s),$$

$$h_{m,\mu}(s) = l_{m,\mu}(s) - l_{m,\mu+1}(s), \quad \mu = 0,1,2,\dots,m-1,$$

where

$$l_{m,\mu}(s) = \int_0^1 t^{s-1}(1-t)^\mu \sum_{\nu=0}^{m-\mu}\zeta^\nu\,dt, \quad \zeta = \frac{qt}{q+1}.$$

Since $S_m^k = O(m^{\alpha+\varepsilon})$,

$$S_m^k l_{m,m}(s) = O\left(m^{\alpha+\varepsilon}\int_0^1 t^{s-1}(1-t)^m\,dt\right)$$

$$= O\left(m^{\alpha+\varepsilon}\frac{\Gamma(s)\Gamma(m+1)}{\Gamma(s+m+1)}\right) = O(m^{\alpha+\varepsilon}m^{-s}) = o(1), \quad m\to\infty.$$

If μ is fixed, then

$$\lim_{m\to\infty} l_{m,\mu}(s) = l_\mu(s) = \int_0^1 t^{s-1}(1-t)^\mu\left(\frac{q+1}{q+1-qt}\right)^{\mu+1}dt,$$

and hence

$$\lim_{m\to\infty} h_{m,\mu}(s) = h_\mu(s) = \int_0^1 t^s(1-t)^\mu\left(\frac{q+1}{q+1-qt}\right)^{\mu+1}\frac{dt}{q+1-qt}.$$

The series $\sum_{\mu=0}^\infty h_\mu(s)$ is convergent for each $s > \alpha+\varepsilon$. Indeed, if $\mu \geq 1$, then

$$h_\mu(s) = \int_0^1 t^{s-1}(1-t)^{\mu-1}R_\mu(t)\frac{dt}{q+1-qt},$$

where

$$R_\mu(t) = t(1-t)\left(\frac{q+1}{q+1-qt}\right)^{\mu+1}, \quad 0\leq t\leq 1.$$

An elementary calculation gives that the rational function R_μ has its maximum in the interval (0,1) at a point $t_\mu = O(\mu^{-1})$. Moreover, $R_\mu(t_\mu) \sim K\mu^{-1}$, $K = Const.$ and hence,

$$h_\mu(s) = O\left(\mu^{-1}\int_0^1 t^{s-1}(1-t)^{\mu-1}\,dt\right)$$
$$= O\left(\mu^{-1}\frac{\Gamma(s)\Gamma(\mu)}{\Gamma(s+\mu)}\right) = O(\mu^{-s-1}), \quad \mu\to\infty,$$

whence it immediately follows that the series

$$\sum_{\mu=0}^{\infty} S_\mu^k h_\mu(s)$$

is absolutely convergent for each $s > \alpha+\varepsilon$, since it is majorized by the convergent series

$$\sum_{\mu=0}^{\infty}(\mu+1)^{\alpha+\varepsilon}|h_\mu(s)|.$$

Then, in order to establish the E_k-summability of the series (3.1) for $s > \alpha+\varepsilon$, one has to prove that

$$\lim_{m\to\infty}\sum_{\mu=0}^{m-1} S_\mu^k h_{m,\mu}(s) = \sum_{\mu=0}^{\infty} S_\mu^k h_\mu(s).$$

To that end it is sufficient to show that

$$T_m(s) = \sum_{\mu=0}^{m-1}(\mu+1)^{\alpha+\varepsilon}|h_{m,\mu}(s)| = O(1), \quad m\to\infty.$$

But

$$h_{m,\mu}(s) = -\frac{1}{q+1}\int_0^1 t^s(1-t)^\mu \sum_{\nu=0}^{\mu}\frac{\zeta^{m-\nu+1}}{(1-\zeta)^{\mu-\nu+1}}\,dt$$
$$+\binom{m+1}{\mu+1}\int_0^1 t^{s-1}(1-t)^{\mu+1}\frac{\zeta^{m-\mu}}{1-\zeta}\,dt = A_{m,\mu}(s) + B_{m,\mu}(s),$$

and hence

$$T_m(s) \le \sum_{\mu=0}^{m-1}(\mu+1)^{\alpha+\varepsilon}|A_{m,\mu}(s)| + \sum_{\mu=0}^{m-1}(\mu+1)^{\alpha+\varepsilon}|B_{m,\mu}(s)|$$

$$= \mathcal{A}_m(s) + \mathcal{B}_m(s).$$

Further,

$$\mathcal{B}_m(s) \leq (m+1)^{\alpha+\varepsilon} \int_0^1 t^{s-1} \sum_{\mu=0}^{m-1} \binom{m+1}{\mu+1} (1-t)^{\mu+1} \frac{z^{m-\mu}}{1-z}\, dt$$

$$\leq (m+1)^{\alpha+\varepsilon}(q+1) \int_0^1 t^{s-1}(1-t+z)^{m+1}\, dt$$

$$= (m+1)^{\alpha+\varepsilon}(q+1) \int_0^1 t^{s-1} \left(1 - \frac{t}{q+1}\right)^{m+1} dt.$$

The change of the variable t by $\dfrac{t}{m+1}$ yields that

$$\mathcal{B}_m(s) < (m+1)^{\alpha+\varepsilon-s}(q+1) \int_0^{m+1} t^{s-1} \left(1 - \frac{t}{(q+1)(m+1)}\right)^{m+1} dt$$

$$< (m+1)^{\alpha+\varepsilon-s}(q+1) \int_0^{m+1} t^{s-1} e^{-t/(q+1)}\, dt,$$

i.e.,

$$\mathcal{B}_m(s) < (m+1)^{\alpha+\varepsilon-s}(q+1) \int_0^{\infty} t^{s-1} e^{-t/(q+1)}\, dt = o(1), \quad m \to \infty,$$

since $s > \alpha + \varepsilon$.

For $\mathcal{A}_m(s)$ one gets that

$$\mathcal{A}_m(s) \leq \int_0^1 t^s \sum_{\mu=0}^{m-1} (\mu+1)^{\alpha+\varepsilon}(1-t)^\mu \sum_{\nu=0}^{\mu} \binom{m+1}{\nu} \frac{\zeta^{m+1-\nu}}{(1-\zeta)^{\mu+2-\nu}}\, dt$$

$$< \int_0^1 t^s \sum_{\nu=0}^{m+1} \binom{m+1}{\nu} \frac{\zeta^{m+1-\nu} z^\nu}{(1-\zeta)^2} \sum_{\mu=0}^{\infty} (\mu+1)^{\alpha+\varepsilon} \left(\frac{1-t}{1-\zeta}\right)^\mu dt$$

$$\leq K_1 \int_0^1 \frac{t^s}{\left(1 - \dfrac{1-t}{1-\zeta}\right)^{\alpha+\varepsilon+1}} \frac{dt}{(1-z)^2}$$

$$\leq K_2 \int_0^1 t^{s-\alpha-\varepsilon-1}(q+1-qt)^{\alpha+\varepsilon+1} \frac{dt}{(1-\zeta)^2}$$

$$\leq K_3(q+1)^{\alpha+\varepsilon+1}\int_0^1 t^{s-\alpha-\varepsilon-1}\,dt$$

$$= K_3\frac{(q+1)^{\alpha+\varepsilon+1}}{s-\alpha-\varepsilon,} \quad K_j = Const., \quad j=1,2,3,$$

and thus it is established that the sequence $\{\mathcal{A}_m(s)\}_{m=1}^{\infty}$ is bounded.

Let now the series (3.1) be E_k-summable for $s=\alpha>0$, i.e., the series $\sum_{n=0}^{\infty} A_n^k(\alpha)$ is convergent. If

$$a_\nu(\alpha) = \frac{a_\nu}{(\alpha+1)(\alpha+2)\dots(\alpha+\nu)} = \frac{\Gamma(\alpha+1)a_n u}{\Gamma(\nu+\alpha+1)}, \quad \nu = 0,1,2,\dots,$$

$$b_\nu(\alpha) = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+\nu)}{\nu!} = \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\nu+1)}, \quad \nu=0,1,2,\dots,$$

then the series

$$\sum_{\nu=0}^{\infty}\frac{a_\nu}{\nu!}$$

can be written in the form

$$\sum_{\nu=0}^{\infty} a_\nu(\alpha)b_\nu(\alpha),$$

which leads to the representation

$$A_n^k = \sum_{\mu=0}^{n} A_\mu^k(\alpha)\binom{n}{\mu}\left(\frac{q}{q+1}\right)^{n-\mu}\Delta^{n-\mu}b_\mu(\alpha).$$

Further,

$$\Delta^{n-\mu}b_\mu(\alpha)$$

$$= \frac{1}{n!}(-\alpha)(-\alpha+1)\dots(-\alpha+n-\mu-1)(\alpha+1)(\alpha+2)\dots(\alpha+\mu)$$

$$= \frac{\Gamma(-\alpha+n-\mu)\Gamma(\alpha+\mu+1)}{\Gamma(-\alpha)\Gamma(\alpha+1)\Gamma(n+1)}$$

and if $\mu < n-\alpha$, then

$$\Delta^{n-\mu}b_\mu(\alpha) = \frac{1}{\Gamma(-\alpha)\Gamma(\alpha+1)}\int_0^1 t^{-\alpha+n-\mu-1}(1-t)^{\mu+\alpha}\,dt.$$

Let p be a integer such that $\alpha < p \leq \alpha+1$, then

$$A_n^k = K_n(\alpha) + L_n(\alpha),$$

where

$$K_n(\alpha) = \sum_{\mu=0}^{n-p} A_\mu^k(\alpha) \binom{n}{\mu} \left(\frac{q}{q+1}\right)^{n-\mu} \Delta^{n-\mu} b_\mu(\alpha)$$

and

$$L_n(\alpha) = \sum_{\mu=n-p+1}^{n} A_\mu^k(\alpha) \binom{n}{\mu} \left(\frac{q}{q+1}\right)^{n-\mu} b_\mu(\alpha).$$

If

$$\mathcal{K}_m(\alpha) = \sum_{n=p}^{m} K_n(\alpha),$$

then

$$\Gamma(-\alpha)\Gamma(\alpha+1)\mathcal{M}_m(\alpha)$$

$$= \sum_{\mu=0}^{m-p} A_\mu^k(\alpha) \sum_{n=\mu+p}^{m} \binom{n}{\mu} \left(\frac{q}{q+1}\right)^{n-\mu} \int_0^1 t^{-\alpha-1+n-\mu}(1-t)^{\mu+\alpha}\, dt$$

and hence,

$$\mathcal{K}_m(\alpha) = \frac{1}{\Gamma(-\alpha)\Gamma(\alpha+1)} A_\mu^k(\alpha) l_{m,\mu}(\alpha),$$

where

$$l_{m,\mu}(\alpha) = \sum_{\nu=p}^{m-\mu} \binom{\mu+\nu}{\mu} \int_0^1 t^{-\alpha-1}(1-t)^{\mu+\alpha}\zeta^\nu\, dt, \quad \zeta = \frac{qt}{q+1}.$$

Since

$$\sum_{\nu=p}^{m-\mu} \binom{\mu+\mu}{\mu} \zeta^\nu$$

$$= -\sum_{\nu=0}^{\mu} \binom{m+1}{\nu} \frac{\zeta^{m+1-\nu}}{(1-\zeta)^{\mu+1-\nu}} + \sum_{\nu=0}^{\mu} \binom{\mu+p}{\nu} \frac{\zeta^{\mu+p-\nu}}{(1-\zeta)^{\mu+1-\nu}},$$

it follows that $l_{m,\mu}(\alpha) = l_{m,\mu}^{(1)}(\alpha) - l_{m,\mu}^{(2)}(\alpha)$, where

$$l_{m,\mu}^{(1)}(\alpha) = -\sum_{\nu=0}^{\mu} \binom{m+1}{\nu} \int_0^1 t^{-\alpha-1}(1-t)^{\mu+\alpha} \frac{\zeta^{m+1-\nu}}{(1-\zeta)^{\mu+1-\nu}}\, dt,$$

$$l_{m,\mu}^{(2)}(\alpha) = \sum_{\nu=0}^{\mu} \binom{\mu+p}{\nu} \int_0^1 t^{-\alpha-1}(1-t)^{\mu+\alpha} \frac{\zeta^{\mu+p-1}}{(1-\zeta)^{\mu+1-\nu}}\, dt.$$

Then,

$$\mathcal{K}_m(\alpha) = \frac{1}{\Gamma(-\alpha)\Gamma(\alpha+1)} \left(\sum_{\mu=0}^{m-p} A_\mu^k(\alpha) l_{m,\mu}^{(1)}(\alpha) - \sum_{\mu=0}^{m-p} A_\mu^k(\alpha) l_{m,\mu}^{(2)}(\alpha) \right)$$

$$= \mathcal{K}_m^{(1)}(\alpha) - \mathcal{K}_m^{(2)}(\alpha),$$

where

$$\mathcal{K}_m^{(1)}(\alpha) = \frac{1}{\Gamma(-\alpha)\Gamma(\alpha+1)} \left(\sum_{\mu=0}^{m-\mu-1} S_\mu(\alpha) h_{m,\mu}^{(1)}(\alpha) + S_{m-p}(\alpha) l_{m,m-p}^{(1)}(\alpha) \right),$$

$$\mathcal{K}_m^{(2)}(\alpha) = \frac{1}{\Gamma(-\alpha)\Gamma(\alpha+1)} \left(\sum_{\mu=0}^{m-p-1} S_\mu(\alpha) h_{m,\mu}^{(2)}(\alpha) + S_{m-p}(\alpha) l_{m,m-p}^{(2)}(\alpha) \right),$$

$$h_{m,\mu}^{(j)}(\alpha) = l_{m,\mu}^{(j)}(\alpha) - l_{m,\mu+1}^{(j)}(\alpha), \quad j = 1, 2, \quad \mu = 0, 1, 2, \dots, m-p-2\,.$$

Further,

$$S_{m-p}^k(\alpha) l_{m,m-p}^{(1)}(\alpha) + S_{m-p}^k(\alpha) l_{m,m-p}^{(2)(\alpha)} = S_{m-p}^k(\alpha) l_{m,m-p}(\alpha)$$

$$= S_{m-p}^k(\alpha) \binom{m}{p} \left(\frac{q}{q+1} \right)^p \int_0^1 t^{p-\alpha-1} (1-t)^{m-p+\alpha}\, dt$$

$$= O(m^\alpha) \int_0^1 t^{p-\alpha-1} (1-t)^{m-p+\alpha}\, dt$$

$$= O\left(m^p \frac{\Gamma(p-\alpha)\Gamma(m-p+\alpha-1)}{\Gamma(m+1} \right) = O(m^\alpha).$$

Since

$$h_{m,\mu}(\alpha) = -\frac{1}{q+1} \sum_{\nu=0}^{\mu} \binom{m+1}{\nu} \int t^{-\alpha} (1-t)^{\mu+\alpha} \frac{\zeta^{m+1-\nu}}{(1-\zeta)^{\mu+2-\nu}}\, dt$$

$$+ \binom{m+1}{\mu+1} \int_0^1 t^{-\alpha-1} (1-t)^{\mu+\alpha} \frac{\zeta^{m-\mu}}{1-\zeta}\, dt = -U_{m,\mu}(\alpha) + V_{m,\mu}(\alpha),$$

it follows that

$$\sum_{\mu=0}^{m-p-1} S_\mu^k(\alpha) h_{m,\mu}(\alpha) = \mathcal{U}_m(\alpha) + \mathcal{V}_m(\alpha),$$

where
$$\mathcal{U}_m(\alpha) = -\sum_{\mu=0}^{m-p-1} U_{m,\mu}(\alpha), \quad \mathcal{V}_m(\alpha) = \sum_{\mu=0}^{m-p-1} V_{m,\mu}(\alpha).$$

Then,

$$|\mathcal{V}_m(\alpha)| \le K_1 \sum_{\nu=1}^{m-p} \binom{m+1}{\nu} \int_0^1 t^{-\alpha-1}(1-t)^{\nu+\alpha}\zeta^{m+1-\nu}\,dt$$

$$\le K_1 m^{p+1} \sum_{\nu=0}^{m-p} \binom{m-p}{\nu} \int_0^1 t^{-\alpha-1}(1-t)^{\nu+\alpha}\zeta^{m-p-\nu}\zeta^{p+1}\,dt$$

$$\le K_2 m^{p+1} \int_0^1 t^{p-\alpha}(\zeta+1-t)^{m-p}(1-t)^{\alpha}\,dt$$

$$\le K_3 m^{p+1} \int_0^1 t^{p-\alpha}\left(1-\frac{t}{q+1}\right)^{m-p} dt$$

$$= K_3 \frac{m^{p+1}}{(m-p)^{p+1-\alpha}} \int_0^{m-p} t^{p-\alpha}\left(1-\frac{t}{(q+1)(m-p)}\right)^{m-p} dt$$

$$= O\left(m^{\alpha}\int_0^{\infty} t^{p-\alpha}e^{-t/(q+1)}\,dt\right) = O(m^{\alpha}).$$

Further,

$$|\mathcal{U}_m(\alpha)| \le K \int_0^1 t^{-\alpha}(1-t)^{\alpha} \sum_{\mu=0}^{m-p-1}\sum_{\nu=0}^{\mu} \binom{m+1}{\nu} \frac{(1-t)^{\mu}\zeta^{m+1-\nu}}{(1-\zeta)^{\mu+2-\nu}}\,dt$$

$$= K \int_0^1 t^{-\alpha}(1-t)^{\alpha} \sum_{\nu=0}^{m-p-1} \binom{m+1}{\nu} \frac{\zeta^{m+1-\nu}}{(1-\zeta)^{2-\nu}} \sum_{\mu=\nu}^{m-p-1} \left(\frac{1-t}{1-\zeta}\right)^{\mu} dt$$

$$< K \int_0^1 t^{-\alpha}(1-t)^{\alpha} \sum_{\nu=0}^{m-p-1} \binom{m+1}{\nu} \frac{\zeta^{m+1-\nu}}{(1-\zeta)^{1-\nu}(t-\zeta)} \left(\frac{1-t}{1-\zeta}\right)^{\nu} dt$$

$$= K(q+1) \int_0^1 t^{-\alpha-1}(1-t)^{\alpha} \sum_{\nu=0}^{m-p-1} \binom{m+1}{\nu} \zeta^{m+1-\nu}(1-t)^{\nu} \frac{dt}{1-\zeta}$$

$$\le K_1 m^{p+1} \int_0^1 t^{-\alpha-1}\zeta^{p+2}(1-t)^{\alpha} \sum_{\nu=0}^{m-p-1} \binom{m-p-1}{\nu} \zeta^{m-p-1-\nu}(1-t)^{\nu}\,dt$$

$$= K_1 \left(\frac{q}{q+1}\right)^{p+2} m^{p+2} \int_0^1 t^{p+1-\alpha}(1-t)^{\alpha}(\zeta+1-t)^{m-p-1}\, dt$$

$$= O\left(m^{p+\alpha}\frac{1}{m^{p+2-\alpha}}\right) = O(m^{\alpha}),$$

and hence, $\mathcal{K}_m^{(1)}(\alpha) = O(m^{\alpha})$.

It remains $\mathcal{K}_m^{(2)}(\alpha)$ to be estimated when $m \to \infty$. To this end one needs the function

$$G_{p,\mu}(\zeta) = \sum_{\nu=0}^{\mu} \frac{\zeta^{\mu+p-\nu}}{(1-\zeta)^{\mu+1-\nu}}\,.$$

It has the representation

$$G_{p,\mu}(\zeta) = \sum_{\nu=0}^{\mu+p} \binom{\mu+p}{\nu} \frac{\zeta^{\mu+p-\nu}}{(1-\zeta)^{\mu+1-\nu}} - \sum_{\nu=\mu+1}^{\mu+p} \binom{\mu+p}{\nu} \frac{\zeta^{\mu+p-\nu}}{(1-\zeta)^{\mu+1-\nu}}$$

$$= \frac{1}{(1-\zeta)^{\mu+1}} - \sum_{\nu=0}^{p-1} \binom{\mu+p}{p-1-\nu} \zeta^{p-1-\nu}(1-\zeta)^{\nu},$$

whence for the function

$$F_{p,\mu}(\zeta) = G_{p,\mu}(\zeta) - (1-t)G_{p,\mu+1}(\zeta), \quad t = \frac{q+1}{q}\zeta,$$

it follows that

$$F_{p,\mu}(\zeta) = \frac{\zeta}{q(1-\zeta)^{\mu+2}}$$

$$-\sum_{\nu=0}^{p-1} \zeta^{p-1-\nu}(1-\zeta)^{\nu} \left(\binom{\mu+p}{p-1-\nu} - \binom{\mu+1+p}{p-1-\nu} \left(1 - \frac{q+1}{q}\zeta\right)\right).$$

Further, it holds the representation

$$F_{p,\mu}(\zeta) = \Phi_{p,\mu}(\zeta) + \Psi_{p,\mu}(\zeta),$$

where

$$\Phi_{p,\mu}(\zeta) = O\left((\mu+1)^{p-1}\frac{\zeta^p}{(1-\zeta)^{\mu+p+1}}\right), \quad \Psi_{p,\mu}(\zeta) = O((\mu+1)^{p-1}\zeta^p).$$

Then, since

$$h_{\mu}^{(2)}(\alpha) = \int_0^1 t^{-\alpha-1}(1-t)^{\mu+\alpha}\Phi_{p,\mu}(\alpha)\, dt$$

$$+\int_0^1 t^{-\alpha-1}(1-t)^{\mu+\alpha}\Psi_{p,\mu}(\alpha)\,dt = I_{p,\mu}(\alpha)+J_{p,\mu}(\alpha),$$

and

$$|I_{p,\mu}(\alpha)| < K(\mu+1)^{p-1}\int_0^1 t^{p-\alpha-1}(1-t)^{\mu+\alpha}\,dt$$

$$= O\left(\mu^{p-1}\frac{\Gamma(p-\alpha)\Gamma(\mu+\alpha+1)}{\Gamma(p+\mu+1)}\right) = O(\mu^{p-1}\mu^{\alpha-p}) = O(\mu^{\alpha-1}),$$

as well as

$$|J_{p,\mu}(\alpha)| \le K(\mu+1)^{p-1}\int_0^1 t^{p-\alpha-1}(1-t)^{\mu+\alpha}\frac{dt}{(1-\zeta)^{\mu+p+1}}$$

$$\le K_1(\mu+1)^{p-1}\int_0^1 t^{p-\alpha-1}\left(\frac{(q+1)(1-t)}{q+1-qt}\right)^{\mu} dt$$

$$= K_1(\mu+1)^{p-1}\int_0^1 t^{p-\alpha-1}\left(1-\frac{t}{q+1-qt}\right)^{\mu} dt$$

$$< K_1(\mu+1)^{p-1}\int t^{p-\alpha-1}\left(1-\frac{t}{q+1}\right)^{\mu} dt$$

$$= K_1(\mu+1)^{p-1}\frac{1}{\mu^{p-\alpha}}\int_0^{\mu}\left(1-\frac{t}{q+1}\right)^{\mu} dt$$

$$\le K_2\mu^{\alpha-1}\int_0^{\infty} t^{p-\alpha-1}e^{-t/(q+1)}\,dt = O(\mu^{\alpha-1}),$$

it follows that

$$\sum_{\mu=0}^{m}|h_{m,\mu}(\alpha)| = \sum_{\mu=1}^{m+1}O(\mu^{\alpha-1}) = O(m^{\alpha}).$$

Further,

$$\mathcal{L}_m(\alpha) = \sum_{\mu=0}^{m}L_m(\alpha) = \sum_{\nu=0}^{p-1}\left(\frac{q}{q+1}\right)^{\nu}\sum_{\mu=\nu}^{m}A_{\mu-\nu}^{m}\binom{\mu}{\nu}A_{\mu-\nu}^{k}(\alpha)\Delta^{\nu}b_{\mu-\nu}(\alpha).$$

Since

$$\Delta^{\nu}b_{\mu-\nu}(\alpha)$$

$$= \frac{1}{\mu!}(-\alpha)(-\alpha+1)\ldots(-\alpha+\nu-1)(\alpha+1)(\alpha+2)\ldots(\alpha+\mu-\nu),$$

one has that

$$\frac{\nu!}{(-\alpha)(-\alpha+1)\dots(-\alpha+\nu-1)}\sum_{\mu=\nu}^{m}\binom{\mu}{\nu}A^k_{\mu-\nu}(\alpha)\Delta^\nu b_{\mu-\nu}(\alpha)$$

$$\sum_{\mu=\nu}^{m}A^k_{\mu-\nu}(\alpha)\frac{(\alpha+1)(\alpha+2)\dots(\alpha+\mu-\nu)}{(\mu-\nu)!}$$

$$=\sum_{\mu=\nu}^{m}A^k_{\mu-\nu}(\alpha)b_{\mu-\nu}(\alpha)=\sum_{\mu=0}^{m-\nu}A^k_{\mu}(\alpha)b_{\mu}(\alpha)$$

$$=S^k_{m-\nu}(\alpha)b_\mu(\alpha)+\sum_{\mu=0}^{m-\nu-1}S^k_\mu(\alpha)(b_\mu(\alpha)-b_{\mu+1}(\alpha)).$$

Since $b_\mu(\alpha)=O(\mu^\alpha)$ and

$$b_\mu(\alpha)-b_{\mu+1}(\alpha)=\frac{\Gamma(\mu+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\mu+1)}-\frac{\Gamma(\mu+\alpha+2)}{\Gamma(\alpha+1)\Gamma(\mu+2)}$$

$$=\frac{\Gamma(\mu+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\mu+1)}\left(1-\frac{\mu+\alpha+1}{\mu+1}\right)=O(\mu^{\alpha-1}),$$

it follows that

$$\mathcal{L}_m(\alpha)=O(m^\alpha)+\sum_{\mu=1}^{m-\nu}O(\mu^{\alpha-1})=O(\mu^\alpha)$$

and thus the proof of Theorem 17 is completed.

Theorem 18. *If $f_k<0$, then*

$$f_k=\limsup_{n\to\infty}\frac{\log|A_n+A_{n+1}+\dots|}{\log(n+1)}.$$

Let

$$|A_n+A_{n+1}+\dots|\le K(n+1)^\alpha,$$

where $\alpha<0$ and $n>N=N(\alpha)\in\mathbb{N}$.

For the terms $\{A^k_n(s)\}_{n=0}^\infty$ of the E_k-transform of the series (3.1) for $s>\alpha$ it holds the representation

$$A^k_n(s)=c_n(s)\sum_{\mu=0}^{n}A_n\left(\frac{q}{q+1}\right)^{n-\mu}b_{n-\mu}(s),$$

where

$$c_n(s) = (b_n(s))^{-1} = \frac{\Gamma(s+1)\Gamma(n+1)}{\Gamma(n+s+1)}, \quad g = \frac{q}{q+1}.$$

Then

$$S_m^k(s) = \sum_{n=0}^{m} A_n^k(s) = \sum_{\mu=0}^{m} T_{m.\mu}(s),$$

where

$$T_{m,\mu}(s) = \sum_{\nu=0}^{m-\mu} c_{\mu+\nu}(s) \left(\frac{q}{q+1}\right)^{\nu} b_\nu(s-1).$$

The validity of equality

$$\lim_{m\to\infty} T_{m,\mu}(s) = T_m(s) = \sum_{\nu=0}^{\infty} c_{\mu+\nu}(s) \left(\frac{q}{q+1}\right)^{\nu} b_\nu(s-1)$$

for $\mu = 0, 1, 2, \ldots$ is a consequence of that fact that the series on its right-hand side is convergent for each fixed μ.

If

$$R_\mu = A_\mu^k + A_{\mu+1}^k + \ldots,$$

then

$$S_m^k(s) = \sum_{\mu=0}^{m} (R_\mu - R_{\mu+1}) T_{m,\mu}(s)$$

$$= \sum_{\mu=1}^{m} R_m (T_{m,\mu}(s) - T_{m,\mu-1}(s)) + R_0 T_{m,0}(s) - R_{m+1} T_{m,m}(s),$$

where

$$T_{m,0}(s) = \sum_{\nu=0}^{m} c_\nu(s) \left(\frac{q}{q+1}\right)^{\nu} b_\nu(s-1), \; T_{m,m}(s) = c_m(s).$$

Since $R_{m+1} = O(m^\alpha)$, $c_m(s) = O(m^{-s})$ when $m \to \infty$ and moreover $\alpha - s < 0$, it follows that $\lim_{m\to\infty} R_{m+1} T_{m,m}(s) = 0$.

Further,

$$T_{m,\mu}(s) - T_{m,\mu-1}(s)$$

$$= \sum_{\nu=0}^{m-\mu} (c_{\mu+\nu}(s) - c_{\mu+\nu-1}(s)) \left(\frac{q}{q+1}\right)^{\nu} b_\nu(s-1)$$

$$-c_m(s) \left(\frac{q}{q+1}\right)^{m-\mu+1} b_{m-\mu+1}(s-1) = T_{m,\mu}^{(1)}(s) - T_{m,\mu}^{(2)}(s),$$

whence

$$I_m(s) = \sum_{\mu-1}^{m-1} R_\mu(T_{m,\mu}(s) - T_{m,\mu-1}(s))$$

$$= \sum_{\mu=1}^{m-1} R_\mu T^{(1)}_{m,\mu}(s) - \sum_{\mu=1}^{m-1} R_m T^{(2)}_{m,\mu}(s)$$

$$= I^{(1)}_m(s) - I^{(2)}_m(s).$$

Further,

$$|I^{(2)}_m(s)| \le \sum_{\mu=1}^{[m/2]} |R_\mu||T^{(1)}_{m,\mu}(s)| + \sum_{\mu=[m/2]+1}^{m-1} |R_\mu||T^{(2)}_{m,\mu}(s)|,$$

$$\sum_{\mu=1}^{[m/2]} |R_\mu||T^{(1)}_{m,\mu}(s)| \le K_1 m^{-s} \sum_{\mu=1}^{[m/2]} \mu^\alpha \left(\frac{q}{q+1}\right)^{m-\mu+1} |b_{m-\mu+1}(s)|$$

and since $\dfrac{q}{q+1} < 1$, it follows that

$$\sum_{\mu=1}^{[m/2]} |R_\mu||T^{(1)(s)}_{m,\mu}| \le K_1 m^{-s} \sum_{\mu=1}^{[m/2]} \mu^\alpha \left(\frac{q}{q+1}\right)^{m-\mu+1} |b_{m-\mu+1}(s-1)|$$

$$\le K_2 m^{-s} \left(\sqrt{\frac{q}{q+1}}\right)^{m+1-[m/2]} \sum_{\mu=1}^{m-1} \left(\sqrt{\frac{q}{q+1}}\right)^{m-\mu+1} |b_{m-\mu+1}(s-1)| = o(1),$$

as well as

$$\sum_{\mu=[m/2]+1}^{m-1} |R_\mu||T^{(2)}_{m,\,u}(s)| \le K m^{-s} \left(\frac{m}{2}\right)^\alpha \sum_{\mu=1}^{m-1} \frac{1}{(m-\mu)^{1-s}} = o(1),$$

i.e., $I^{(2)}_m(s) = o(1),\ m \to \infty$.

Further,

$$\lim_{m\to\infty} I^{(1)}_m(s) = \sum_{\mu=1}^{\infty} R_\mu(T^{(1)}_{m,\mu}(s) - T^{(2)}_{m,\mu}(s)). \tag{3.7}$$

Indeed,

$$c_{\mu+\nu}(s) - c_{\mu+\nu-1}(s) = c_{\mu+\nu-1}(s)\frac{s}{\mu+\nu+s},$$

whence

$$|T^{(1)}_{m,\mu}(s)| \leq K \sum_{\nu=0}^{m-\mu} \frac{1}{(\mu-\nu)^{s+1}} \left(\frac{q}{q+1}\right)^{\nu} |b_\nu(s-1)|.$$

If $s+1>0$, then

$$|T_{m,\mu}(s)| \leq K_1 \mu^{-s-1} \sum_{\nu=0}^{m-\mu} \left(\frac{q}{q+1}\right)^{\nu} |b_\nu(s-1)| \leq K_2 \mu^{-s-1}.$$

Let $s+1 \leq 0$, then since $\mu+\nu \leq \mu(\nu+1)$, it follows that

$$|T^{(1)}_{m,\mu}(s)| \leq K_3 \mu^{-s-1} \sum_{\nu=0}^{m-\mu} (\nu+1)^{-s-1} \left(\frac{q}{q+1}\right)^{\nu} |b_\nu(s-1)| \leq K_4 \mu^{-s-1}$$

and hence,

$$|T^{(1)}_{m,\mu}(s)| = O\left(\frac{1}{\mu^{s-\alpha+1}}\right), \quad \mu \to \infty.$$

In the same way one gets that the series

$$\sum_{\mu=1}^{\infty} R_\mu T^{(2)}_{m,\mu}(s)$$

is majorized by an absolutely convergent series and thus the equality (3.7) is established.

Let now the series (3.1) be E_k-summable for $s=\alpha<0$. Then,

$$A^k_n = \sum_{\mu=0}^{n} \binom{n}{\mu} A^k_\mu(\alpha) \left(\frac{q}{q+1}\right)^{n-\mu} \Delta^{n-\mu} b_\mu(\alpha)$$

$$= c(\alpha) \sum_{\mu=0}^{n} A^k_\mu(\alpha) \left(\frac{q}{q+1}\right)^{n-\mu} b_{n-\mu}(-\alpha-1) b_\mu(\alpha), \; c(\alpha) = \frac{1}{\Gamma(-\alpha)\Gamma(\alpha+1)},$$

and hence,

$J_{m,N}(\alpha)$

$$= \sum_{n=m}^{N} A^k_n = c(\alpha) \sum_{\mu=0}^{n} \sum_{n=\mu}^{N} A^k_n(\alpha) \left(\frac{q}{q+1}\right)^{n-\mu} b_{n-\mu}(-\alpha-1) b_\mu(\alpha)$$

$$+c(\alpha)\sum_{\mu=m}^{N}\sum_{n=\mu}^{N}A_{\mu}^{k}(\alpha)\left(\frac{q}{q+1}\right)^{n-\mu}b_{n-\mu}(-\alpha-1)b_{\mu}(\alpha)$$

$$=J_{m,N}^{(1)}(\alpha)+J_{m,N}^{(2)}(\alpha).$$

Evidently,

$$\lim_{N\to\infty}J_{m,N}^{(1)}(\alpha)=J_{m}^{(1)}(\alpha)$$

$$=c(\alpha)\sum_{\mu=0}^{m}A_{\mu}^{k}(\alpha)b_{\mu}(\alpha)\sum_{n=\mu}^{\infty}\left(\frac{q}{q+1}\right)^{n-\mu}b_{n-\mu}(-\alpha-1).$$

Further, it is easily seen that the series

$$\sum_{\mu=0}^{\infty}A_{\mu}^{k}(\alpha)b_{\mu}(\alpha)$$

is convergent. Indeed,

$$\sum_{\mu=m}^{N}A_{\mu}^{k}(\alpha)b_{\mu}(\alpha)$$

$$=\sum_{\mu=m+1}^{N}R_{\mu}(\alpha)(b_{\mu}(\alpha)-b_{\mu+1}(\alpha))-R_{N+1}b_{N}(\alpha)+R_{m}(\alpha)b_{m}(\alpha),$$

where

$$R_{n}(\alpha)=A_{n}^{k}(\alpha)+A_{n+1}^{k}(\alpha)+\dots,$$

whence

$$\left|\sum_{\mu=m}^{N}A_{\mu}^{k}(\alpha)b_{\mu}(\alpha)\right|\le K\left(\sum_{\mu=m+1}^{N}\frac{1}{\mu^{1-\alpha}}+N^{\alpha}+m^{\alpha}\right)<\varepsilon$$

provided that $m>M=M(\varepsilon)\in\mathbb{N}$. Then, by letting $n\to\infty$, one gets that

$$\left|\sum_{\mu=m}^{\infty}A_{m}^{k}(\alpha)b_{\mu}(\alpha)\right|=\left(\sum_{\mu=m+1}^{\infty}\frac{1}{\mu^{1-\alpha}+m^{\alpha}}\right)=O(m^{\alpha}),\quad m\to\infty.$$

As before one makes sure that there exists

$$\lim_{N\to\infty}J_{m,N}(\alpha)=J_{m}(\alpha)=\sum_{\mu=m}^{\infty}A_{\mu}^{k}(\alpha)b_{\mu}(\alpha)\sum_{n=\mu}^{\infty}\left(\frac{q}{q+1}\right)^{n-\mu}b_{n-\mu}(-\alpha-1)$$

$$= \left(1 - \frac{q}{q+1}\right)^{\alpha} \sum_{\mu=m} A_\mu^k(\alpha) b_\mu(\alpha) = O(m^\alpha), \quad m \to \infty.$$

Further,

$$\sum_{n=m}^{\infty} \left(\frac{q}{q+1}\right)^{n-\mu} b_{n-\mu}(-\alpha-1)$$

$$\leq \left(\sqrt{\frac{q}{q+1}}\right)^{m-\mu} \sum_{n=m}^{\infty} \left(\sqrt{\frac{q}{q+1}}\right)^{n-\mu} b_{n-\mu}(-\alpha-1) \leq K \left(\sqrt{\frac{q}{q+1}}\right)^{m-\mu},$$

hence

$$|J_m^{(1)}(\alpha)| \leq K_1 \sum_{\mu=0}^{m} \left(\sqrt{\frac{q}{q+1}}\right)^{m-\mu} |b_\mu(\alpha)|.$$

Further, since $\frac{q}{q+1} < 1$, it follows that

$$\sum_{\mu=0}^{[m/2]} \left(\sqrt{\frac{q}{q+1}}\right)^{m-\mu} |b_\mu(\alpha)| \leq \left(\sqrt{\frac{q}{q+1}}\right)^{m-[m/2]} \sum_{\mu=0}^{[m/2]} \leq K_2 m^\alpha$$

as well as

$$\sum_{\mu=[m/2]+1}^{m} \left(\sqrt{\frac{q}{q+1}}\right)^{m-\mu} |b_\mu(\alpha)| \leq K_3 \sum_{\mu=[m/2]+1}^{m} \left(\sqrt{\frac{q}{q+1}}\right)^{m-\mu} (\mu+1)^\alpha$$

$$< K_3([m/2]+1)^\alpha \sum_{\mu=0}^{m} \left(\sqrt{\frac{q}{q+1}}\right)^{m-\mu} < K_4 m^\alpha,$$

so that it is established that $J_m^{(1)}(\alpha) = O(m^\alpha), m \to \infty$, and thus Theorem 18 is proved.

Theorem 19. *If the series* (3.1) *is* $|E_k|$*-summable for* $s = s_0$*, then it is* $|E_k|$*-summable for each* s *such that* $\Re s > \Re s_0$*.*

It has to be proved that the absolute convergence of the series $\sum_{n=0}^{\infty} A_n^k(s_0)$ implies the same for the series $\sum_{n=0}^{\infty} A_n^k(s)$ provided that $\Re s > \Re s_0$.

From (3.5) it follows that

$$|A_n^k(s)| \leq \sum_{\mu=0}^{n} |A_\mu^k(s_0)| \binom{n}{\mu} \left(\frac{q}{q+1}\right)^{n-\mu} |\Delta^{n-\mu} \lambda_\mu(s_0; s)|.$$

If $\Re s > \Re s_0$, then

$$|\Delta^{n-\mu}\lambda_\mu(s_0; s)|$$

$$\leq \left|\frac{\Gamma(s+1)}{\Gamma(s-s_0)\Gamma(s_0+1)}\right| \int_0^1 t^{\sigma-\sigma_0+n-\mu-1}(1-t)^{\sigma_0+\mu}\, dt,$$

$$\sigma = \Re s,\ \sigma_0 = \Re s_0.$$

If

$$U_n^k(s) = |A_0^k(s)| + |A_1^k(s)| + \cdots + |A_m^k(s)|,$$

then

$$U_m^k(s) \leq \left|\frac{\Gamma(s+1)}{\Gamma(s-s_0)\Gamma(s_0+1)}\right| \sum_{\mu=0}^{m} |A_\mu^k(s_0)| L_{m,\mu}(s),$$

where

$$L_{m,\mu}(s) = \sum_{\nu=0}^{m-\mu} \left(\frac{q}{q+1}\right)^\nu \binom{\nu+\mu}{\mu} \int_0^1 t^{\sigma-\sigma_0+\nu-1}(1-t)^{\sigma_0+\mu}\, dt.$$

But

$$L_{m,\mu}(s) \leq L_m(s) = \sum_{\nu=0}^{\infty} \left(\frac{q}{q+1}\right)^\nu \binom{\nu+\mu}{\mu} \int_0^1 t^{\sigma-\sigma_0+\nu-1}(1-t)^{\sigma_0+\mu}\, dt$$

$$= \int_0^1 t^{\sigma-\sigma_0-1}(1-t)^{\sigma_0+\mu} \left(\frac{q+1}{q+1-qt}\right)^{\mu+1} dt,$$

hence

$$U_m^k(s) \leq \frac{q}{q+1} \sum_{\mu=0}^{m} |A_\mu^k(s_0)| l_\mu(s)$$

$$= \frac{q}{q+1} \sum_{\mu=0}^{m-1} U_\mu^k(s_0)(L_\mu(s) - L_{\mu+1}) + \frac{q}{q+1} U_m^k(s_0) L_m(s).$$

Since

$$L_\mu(s) - L_{\mu+1}(s)$$

$$= \int_0^1 t^{\sigma-\sigma_0}(1-t)^{\sigma_0+\mu} \left(\frac{q+1}{q+1-qt}\right)^{\mu+1} \frac{dt}{q+1-qt} > 0,$$

$$\mu = 0, 1, 2. \ldots, m-1,$$

and $U^k_\mu(s_0) \le U_0 < \infty$, $\mu = 0, 1, 2, \dots$, it follows that

$$U^k_m(s) \le \frac{qU_0}{q+1}\sum_{\mu=0}^{m-1}(L_\mu(s) - L_{\mu+1}(s)) + \frac{qU_0}{q+1}L_0(s)$$

$$= \frac{qU_0}{q+1}\int_0^1 t^{\sigma-\sigma_0-1}(1-t)^{\sigma_0}\frac{q+1}{q+1-qt}\,dt \le qU_0\int_0^1 t^{\sigma-\sigma_0-1}(1-t)^{\sigma_0}\,dt$$

$$= qU_0\frac{\Gamma(\sigma-\sigma_0)\Gamma(\sigma_0+1)}{\Gamma(\sigma+1)}, \quad m = 0, 1, 2, \dots.$$

If $\Re s < 0$, then it has to be taken into attention that by an author's theorem the subscripts of the terms of $|E_k|$-summable series can be both increased and decreased without changing its summability.

Theorem 20. *It holds the inequality* $\overline{f}_k - f_k \le 1$.

Indeed, since $|A^k_n(s_0)| \le A(s_0) < \infty, n = 0, 1, 2, \dots$, it follows that

$$|A^k_n(s)| \le A(s_0)\sum_{\mu=0}^{n}\binom{n}{\mu}\left(\frac{q}{q+1}\right)^{n-\mu}|\Delta^{n-\mu}\lambda_\mu(s_0; s)|$$

and if $\Re s > \Re s_0 \ge 0$, then

$$|A^k_n(s)| \le \frac{qA(s_0)}{q+1}\sum_{\mu=0}^{n}\binom{n}{\mu}\left(\frac{q}{q+1}\right)^{n-\mu}\int_0^1 t^{\sigma-\sigma_0+n-\mu-1}(1-t)^{\sigma_0+\mu}\,dt$$

$$= \frac{qA(s_0)}{q+1}\int_0^1 t^{\sigma-\sigma_0-1}(1-t)^{\sigma_0}\left(\frac{qt}{q+1} + 1 - t\right)^n dt$$

$$\le \frac{qA(s_0)}{q+1}\int_0^1 t^{\sigma-\sigma_0-1}\left(1 - \frac{t}{q+1}\right)^n dt.$$

Substituting t for t/n, one gets that

$$|A^k_n(s)| \le \frac{qA(s_0)}{q+1}n^{-(\sigma-\sigma_0)}\int_0^n t^{-(\sigma-\sigma_0-1)}\left(1 - \frac{t}{(q+1)n}\right)^n dt$$

$$\le \frac{qA(s_0)}{q+1}n^{-(\sigma-\sigma_0)}\int_0^\infty t^{\sigma-\sigma_0-1}e^{-t/(q+1)}\,dt.$$

If $\sigma - \sigma_0 > 1$, then the series $\sum_{n=0}^{\infty} A^k_n(s)$ is absolutely convergent and thus Theorem 20 is proved since the case $\Re s_0 < 0$ can immediately be reduced to the previous one.

Theorem 21. *If* $\overline{f}_k < 0$, *then*

$$\overline{f}_k = \limsup_{n\to\infty} \frac{\log(|A_n^k| + |A_{n+1}^k| + \dots)}{\log(n+1)}. \tag{3.8}$$

The proof proceeds as that of Theorem 12. Let

$$|A_n^k| + |A_{n+1}^k| + \cdots \le K(n+1)^\alpha, \quad \alpha < 0,\ n > N = N(\alpha) \in \mathbb{N}$$

and $s \in (\alpha, 0)$. Then

$$U_m(s) \le \sum_{\mu=0}^{m} |A_\mu^k(s)| T_\mu(s),\ T_\mu(s) = \sum_{\nu=0}^{\infty} |c_{\mu+\nu}(s)| \left(\frac{q}{q+1}\right)^\nu b_\nu(s-1).$$

Since $\mu + \nu + 1 \le (\mu+1)(\nu+1)$, it follows that

$$T_\mu(s) \le K \sum_{\nu=0}^{\infty} (\mu+\nu+1)^{-s} \left(\frac{q}{q+1}\right)^\nu (\nu+1)^{s-1}$$

$$\le K(\mu+1)^{-s} \sum_{\nu=0}^{\infty} (\tau+1)^{-1} \left(\frac{q}{q+1}\right)^\nu \le (\mu+1)^{-s}$$

hence

$$U_\mu(s) \le K_1 \sum_{\mu=0}^{m} |A_m u^k| (\mu+1)^{-s}.$$

If, as before,

$$R_n = \sum_{\mu=n}^{\infty} |A_n^k|,$$

then

$$\sum_{\mu=0}^{m} |A_\mu^k| (\mu+1)^{-s}$$

$$= \sum_{\mu=1}^{m-1} R_\mu((\mu+1)^{-s} - \mu^{-s}) + R_0 - R_{m+1}(m+1)^{-s}$$

$$\le K_2 \sum_{mu=1}^{m-1} \mu^\alpha((\mu+1)^{-s} - \mu^{-s}) + R_0$$

$$\leq K_3 \sum_{\mu=1}^{m-1} \mu^\alpha \mu^{-s-1} + R_0 < K_3 \sum_{\mu=1}^{\infty} \frac{1}{\mu^{s-\alpha+1}} + R_0.$$

Let now the series (3.1) be $|E_k|$-summable for $s = \alpha < 0$. Then in a way already used one obtains that

$$\sum_{n=m}^{\infty} |A_n^k| \leq J_m^{(1)}(\alpha) + J_m^{(2)}(\alpha),$$

where

$$J_m^{(1)}(\alpha) = |c(\alpha)| \sum_{\mu=0}^{m} |A_\mu^k(\alpha) \sum_{n=m}^{\infty} \left(\frac{q}{q+1}\right)^{n-\mu} b_{n-\mu}(-\alpha-1)$$

and

$$J_m^{(2)}(\alpha) = |c(\alpha)| \sum_{\mu=m}^{\infty} |A_\mu^k(\alpha)| \sum_{\nu=0}^{\infty} \left(\frac{q}{q+1}\right)^{\nu} b_\nu(-\alpha-1).$$

Further, if

$$R_\mu(\alpha) = \sum_{\nu=m}^{\infty} |A_\nu^k(\alpha)|,$$

then

$$J_m^{(2)}(\alpha) \leq K \sum_{\mu=m}^{\infty} |A_\mu^k(\alpha)| b_m u(\alpha)$$

$$\leq K_1 m^\alpha \sum_{\mu=m}^{\infty} (R_\mu(\alpha) - R_{\mu+1}(\alpha)) = K_1 m^\alpha R_m(\alpha) = O(m^\alpha), \quad m \to \infty.$$

The same estimate holds also for $J_m^{(1)}(\alpha)$ and, hence,

$$\sum_{n=m}^{\infty} |A_n^k| = O(m^\alpha), \quad m \to \infty.$$

It is at hand now a second prof of Theorem 16 which gives an unified representation of the function $f(s)$ in the region $\{s : \Re s < \Re s_0\}$ provided that the series (3.1) is E^k-summable for $s = s_0$. More precisely:

Theorem 22. *If the series* (3.1) *is* E_k*-summable for* $s = s_0$*, then it is* E_k*-summable for each* s *such that* $\Re s > \Re s_0$ *with sum*

$$f(s) = \Gamma(s+1) \sum_{\mu=0}^{\infty} A_\mu^k b_\mu(s_0) \sum_{\nu=0}^{\infty} \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+\nu+s+1)} g^\nu b_\nu(s-s_0-1), \quad g = \frac{q}{q+1}.$$

If $\Re s > \Re s_0$, then it holds the representation

$$A_n^k(s) = \sum_{\mu=0}^{n} A_\mu^k \binom{n}{\mu} g^{n-\mu} \Delta^{n-\mu} \lambda(s_0; s),$$

where

$$\Delta^{n-\mu}\lambda_\mu(s_0; s) = \frac{\Gamma(s - s_0 + n - \mu)\Gamma(s_0 + \mu + 1)\Gamma(s + 1)}{\Gamma(n - \mu + 1)\Gamma(s - s_0)\Gamma(\mu + 1)\Gamma(s_0 + 1)\Gamma(s + n + 1)}$$
$$= c_n(s) b_{n-\mu}(s - s_0 - 1) b_\mu(s_0)$$

and hence

$$A_n^k(s) = c_n(s) \sum_{\mu=0}^{n} A_\mu^k g^{n-\mu} b_{n-\mu}(s - s_0 - 1) b_\mu(s_0).$$

Then,

$$S_m^k(s) \sum_{n=0}^{m} A_n^k(s) = \sum_{\mu=0}^{m} A_\mu^k b_\mu(s_0) V_{m,\mu}(s_0; s)$$

and

$$V_{m,\mu}(s_0; s) = \sum_{\nu=0}^{m-\mu} c_{\mu+\nu}(s) g^\nu b_\nu(s - s_0 - 1).$$

If μ is fixed, then

$$\lim_{m\to\infty} V_{m,\mu}(s_0; s) = V_\mu(s_0; s) = \sum_{\nu=0}^{\infty} g^\nu b_\nu(s - s_0 - 1),$$

since, as it is easily seen, the series in the right-hand side of the last equality is convergent, which follows from the estimate $c_n(s) = O(n^{-\sigma}), n \to \infty, \sigma = \Re s$. Then,

$$S_m^k(s) = \sum_{\mu=0}^{m-1} S_\mu^k(b_\mu(s_0)V_{m,\mu}(s_0; s) - b_{\mu+1}(s_0)V_{m,\mu+1}(s_0; s)) + S_m^k b_m(s_0)V_{m,m}(s_0; s),$$

$$S_m^k b_m(s_0) V_{m,m}(s_0 : s) = O(m^{\sigma_0} c_m(s)) = O(m^{\sigma-\sigma_0}) = o(1), \quad m \to \infty$$

and

$$b_\mu(s_0)V_{m,\mu}(s_0; s) - b_{\mu+1}(s_0)V_{m,\mu+1}(s_0; s)$$
$$= \sum_{\nu=0}^{m-\mu-1} (b_\mu(s_0)c_{\mu+\nu}(s) - b_{\mu+1}(s_0)c_{\mu+\nu+1}(s)) g^\nu b_\nu(s - s_0 - 1).$$

Further, the behaviour of the sum

$$\mathcal{C}_m(s) = |c_m(s)| \sum_{\mu=0}^{m-1} g^{n-\mu} |b_{n-\mu}(s - s_0 - 1)$$

has to be studied when $m \to \infty$.

If $\sigma_0 > 1$, then

$$\mathcal{C}_m(s) \leq k m^{-\sigma} \sum_{\mu=0}^{m} b_{m-\mu}(\sigma - \sigma_0) b_\mu(\sigma_0)$$

$$= K m^{-\sigma} b_m(\sigma) = O(1), \quad m \in \mathbb{N}.$$

If $\sigma_0 < -1$ and $p > -\sigma$ is a positive integer, then

$$\mathcal{C}_m(s) = |c_m(s)| \sum_{\mu=0}^{p-1} g^{m-\mu} |b_{m-\mu}(s - s_0 - 1) b_\mu(s_0)|$$

$$+ |c_m(s)| \sum_{\mu=p}^{m-1} g^{m-\mu} |b_{m-\mu}(s - s_0 - 1) b_\mu(s_0)| = \mathcal{C}_m^{(1)}(s) + \mathcal{C}_m^{(2)}(s).$$

Since $g < 1$, it is quite evident that

$$\mathcal{C}_m^{(1)}(s) = O(m^{-\sigma} g^m m^{\sigma - \sigma_0 - 1}) = o(1), \quad m \to \infty.$$

If $\mu \geq p$, then $b_\mu(\sigma)$ has the sign of $(-1)^p$ and therefore

$$\mathcal{C}_m^{(2)}(s) \leq K m^{-\alpha} \sum_{\mu=0}^{m} (-1)^p g^{m-mu} b_{n-\mu}(\sigma - \sigma_0 - 1) b_\mu(\sigma_0).$$

Further, if $\mu > (g - \sigma_0)/(1-g)$ and $j \in ((q - \sigma_0)/(1-g), \mu]$ is a positive integer, then

$$\sum_{\mu=j}^{m} (-1)^p g^{m-\mu} b_{m-\mu}(\sigma - \sigma_0 - 1) b_\mu(\sigma_0) < (-1)^p b_m(\sigma_0) \sum_{\mu=j}^{m} b_{m-\mu}(\sigma - \sigma_0 - 1)$$

$$< (-1)^p b_m(\sigma_0) \sum_{\mu=j}^{m} b_{m-\mu}(\sigma - \sigma_0 - 1) = (-1)^p b_m(\sigma - \sigma_0) b_m(\sigma_0)$$

and hence

$$\mathcal{C}_m^{(2)} \le Km^{-\sigma}\left(b_m(\sigma-\sigma_0)b_m(\sigma_0)+\sum_{\mu=p}^{j}(-1)^p g^{m-\mu}b_{m-\mu}(\sigma-\sigma_0-1)b_\mu(\sigma_0)\right)$$

$$= O(m^{-\sigma}m^{\sigma_0}m^{\sigma-\sigma_0})+O(m^{-\sigma}m^{\sigma-\sigma_0-1}g^m)=O(1).$$

If $-\sigma_0$ is a positive integer, then $b_m(\sigma_0)=0$ when $m>-\sigma_0$. But $\sigma\neq -m$ so that $b_m(\sigma)=O(m^{\sigma_0})$. If one takes $\sigma_0+\eta, \eta>0$, instead of σ_0, then $b_m(\sigma_0)=O(m^{\sigma_0+\eta})$, provided that η is sufficiently small.

It remains to show that the sequence

$$T_m(s)=\sum_{\mu=0}^{m-1}|b_\mu(s)V_{m,mu}(s)-b_{\mu+1}(s)V_{m,\mu+1}(s)|$$

is bounded.

Since

$$b_\mu(\sigma_0)C_{\mu+\nu}(s)-b_{\mu+1}(\sigma_0)c_{\mu+\nu+1}(s)$$
$$=b_\mu(s_0)c_{\mu+\nu}(s)\left(\frac{s}{\mu+\nu+s+1}-\frac{s_0}{\mu+1}\frac{\mu+\nu+1}{\mu+\nu+s+1}\right),$$
$$(b_\mu(s_0)c_{\mu+\nu}(s)-b_{\mu+1}(s_0)c_{\mu+\nu+1}(s))g^\nu b_\nu(s-s_0-1)$$
$$=O(b_\mu(s_0)c_{\mu+\nu}(s)(\mu+1)^{-1})=O((\mu+1)^{-\sigma-\sigma_0-1}(\nu+1)^{\sigma-\sigma_0}g^\nu),$$

it follows that

$$J_m(s)=\sum_{\nu=0}^{m-\mu-1}(b_\mu(s_0)c_{\mu+\nu}(s)-b_{\mu+1}(s_0)c^{\mu+\nu+1}(s))g^\nu b_\nu(s-s_0-1)$$

$$=\frac{1}{(\mu+1)^{\sigma-\sigma_0-1}}\sum_{\nu=0}^{m}O((\nu+1)^{\sigma-\sigma_0-1}g^\nu)=O\left(\frac{1}{(\mu+1)^{\sigma-\sigma_0+1}}\right)$$

and hence,

$$T_m(s)=\sum_{\mu=0}^{m}O\left(\frac{1}{(\mu+1)^{\sigma-\sigma_0+1}}\right)=O(1).$$

If $u,v\ge 0$ and $\lambda\in(0,1]$, then it holds the inequality

$$(u+v)^\lambda \le u^\lambda+v^\lambda. \tag{3.9}$$

Indeed, it is equivalent to the inequality $(1+t)^\lambda \leq 1+t^\lambda, t \geq 0$. If $\omega(t) = (1+t)^\lambda - 1 - t^\lambda$, then $\omega'(t) = \lambda((1+t)^{-\lambda+1} - t^{-\lambda+1}) < 0, t > 0$, i.e., $\omega(t) < \omega(0) = 0$.

Let now $0 < \sigma \leq 1$, then

$$(\mu+\nu)^{-\sigma} \leq \mu^{-\sigma} + \nu^{-\sigma}.$$

Hence

$$(b_\mu(s_0)c_{\mu+\nu}(s) - b_{\mu+1}(s_0)c_{\mu+\nu+1}(s))g^\nu b_\nu(s-s_0-1)$$

$$= O((\mu+1)^{\sigma-\sigma_0-1}(\nu+1)^{\sigma-\sigma_0-1}g^\nu + O(\nu^{-\sigma}(\mu+1)^{\sigma_0-1}(\nu+1)^{\sigma-\sigma_0-1},$$

$$J_m(s) = \sum_{\nu=0}^{m-\mu-1} O((\mu+1)^{-\sigma-\sigma_0-1}(\nu+1)^{\sigma-\sigma_0-1}g^\nu)$$

$$+ \sum_{\nu=0}^{m-\mu-1} O((\mu+1)^{\sigma_0-1}(\nu+1)^{-\sigma_0-1}g^\nu)$$

$$\leq \frac{K}{(\mu+1)^{\sigma-\sigma_0+1}} \sum_{\nu=0}^{m-\mu-1} (\nu+1)^{\sigma-\sigma_0-1}g^\nu$$

$$+ \frac{K_1}{(\mu+1)^{-\sigma_0+1}} \sum_{\nu=0}^{m-\mu_1} (\nu+1)^{-\sigma_0-1}g^\nu$$

$$= O\left(\frac{1}{(\mu+1)^{\sigma-\sigma_0+1}}\right) + O\left(\frac{1}{(\mu+1)^{-\sigma_0+1}}\right)$$

and since $\sigma_0 < \sigma < 0$, it follows that

$$T_m(s) = \sum_{\mu=0}^{m-1} |J_\mu(s)|$$

$$= \sum_{\mu=0}^{m-1} O\left(\frac{1}{(\mu+1)^{\sigma-\sigma_0+1}}\right) + \sum_{\mu=0}^{m-1} O\left(\frac{1}{(\mu+1)^{-\sigma_0+1}}\right) = O(1).$$

It remains the case $\alpha < -1$ to be considered. Let for the sake of convenience denote $\alpha = -\sigma, \beta = -\sigma_0$, i.e., $\alpha, \beta > 1$ and $\alpha < \beta$. Then,

$$|J_m(s)| \leq K(\mu+1)^{-\beta-1} \sum_{\nu=0}^{m-\mu-1} (\mu+\nu)^\alpha (\nu+1)^{-\alpha+\beta+1} g^\nu$$

and

$$T_m(s) \leq K \sum_{\mu=0}^{m-1} (\mu+1)^{-\beta-1} \sum_{\nu=0}^{m-\mu-1} (\mu+\nu)^{\alpha}(\nu+1)^{-\alpha+\beta+1}$$
$$= K \sum_{\nu=0}^{m-1} h_{m,\nu}(\nu+1)^{-\alpha+\beta+1},$$

where

$$h_{m,\nu} = \sum_{\mu=0}^{m-\nu-1} \frac{(\mu+\nu)^{\alpha}}{(\mu+1)^{\beta+1}}.$$

Since $\alpha < \beta + 1$, each of the functions

$$\frac{(t+\nu)^{\alpha}}{(t+1)^{\beta+1}}, \quad 0 < t < \infty, \quad \nu = 0, 1, 2 \ldots,$$

is decreasing and, hence,

$$h_{m,0} = \sum_{\mu=0}^{m-1} \frac{\mu^{\alpha}}{(\mu+1)^{\beta+1}} < \sum_{\mu=0}^{\infty} \frac{1}{(\mu+1)^{\beta-\alpha+1}},$$

i.e., $h_{m,0} = O(1)$. It remains to estimate the sum

$$H_m = \sum_{\nu=0}^{m-1} h_{m,\nu}(\nu+1)^{\beta-\alpha-1} g^{\nu}.$$

First,

$$h_{m,\nu} \leq \int_0^{m-\nu-1} \frac{(t+\nu)^{\alpha}}{(t+1)^{\beta+1}}\, dt, \quad n = 1, 2, \ldots, m-1.$$

Further, since $\alpha > 1, (t+\nu)^{\alpha} = \nu^{\alpha} + \alpha t(\nu + \theta t)^{\alpha-1}, 0 < \theta < 1$, whence $(t+\nu)^{\alpha} \leq \nu^{\alpha} + t(\nu+t)$ and, hence,

$$h_{m,\nu} < \int_0^{m-\nu-1} \frac{t(t+\nu)^{\alpha-1}}{(t+1)^{\beta+1}}\, dt + \nu^{\alpha} \int_0^{m-\nu-1} \frac{dt}{(t+1)^{\beta+1}}$$
$$< \int_0^{m-\nu-1} \frac{(t+\nu)^{\alpha-1}}{(t+1)^{\beta}}\, dt + \nu^{\alpha} \int_0^{\infty} \frac{dt}{(t+1)^{\beta+1}}\, dt$$
$$= O(\nu^{\alpha}) + \int_0^{m-\nu-1} \frac{(t+\nu)^{\alpha-1}}{(t+1)^{\beta}}\, dt.$$

If $\alpha = p + \eta$, $p \in \mathbb{N}$, $0 < \eta < 1$, then in the same way one gets that

$$h_{m,\nu} \le K\nu^{\alpha} + \int_0^{m-\nu-1} \frac{(t+\nu)^{\eta}}{(t+1)^{\beta-p+1}}\,dt,$$

and if $\eta = 0$, then $h_{m,\nu} = O(\nu^{\alpha})$.

If $0 < \eta < 1$, then the inequality $(\nu + t)^{\eta} \le \nu^{\eta} + t^{\eta}$ yields hat

$$h_{m\nu} \le K_1\nu^{\alpha} + \int_0^{m-\nu-1} \frac{t^{\eta}+\nu^{\eta}}{(t+1)^{\beta-p+1}}$$

$$\le K_1\nu^{\alpha} + \nu^{\eta}\int_0^{\infty} \frac{dt}{(t+1)^{\beta-p+1}}\,dt + \int_0^{\infty} \frac{dt}{(t+1)^{\beta-\alpha+1}} < K_2\nu^{\alpha}$$

and, hence,

$$H_m \le K_3 \sum_{\nu=1}^{m-1} (\nu+1)^{\beta-\alpha-1}\nu^{\alpha}g^{\nu}$$

$$< K_3 \sum_{\nu=1}^{\infty} (\nu+1)^{\beta-\alpha-1}\nu^{\alpha}g^{\nu} = O(1).$$

The case when σ_0 is negative can be treated as before after slight variation of σ_0.

An interesting consequence of Theorem 22 is the following statement:

Theorem 23. *Let the series* (3.1) *be E_k-summable for $s = s_0$. Then it holds the representation*

$$f(s) = s\int_0^1 (1-t)^{s-1}\varphi(t)\frac{dt}{(1-gt)^{s-s_0}}, \tag{3.10}$$

where

$$\varphi(t) = \sum_{\mu=0}^{\infty} A_{\mu}^{k} b_{\mu}(s_0)t^{\mu}, \quad 0 \le t < 1, \tag{3.11}$$

provided $\Re s > 0$ and $\Re s > \Re s_0$.

The formal change of order of summation and integration really leads to representation (3.10). Indeed,

$$f(s) = s\sum_{\mu=0}^{\infty} A_{\mu}^{k} b_{\mu}(s_0) \sum_{\nu=0}^{\infty} \int_0^1 t^{\mu+\nu}(1-t)^{s-1} b_{\nu}(s-s_0-1)g^{\nu}\,dt$$

$$= s\int_0^1 \sum_{\mu=0}^{\infty} A_\mu^k b_\mu(s_0)t^\mu(1-t)^{s-1}\frac{dt}{(1-gt)^{s-s_0}}$$

$$= s\int_0^1 (1-t)^{s-1}\varphi(t)\frac{dt}{(1-gt)^{s-s_0}}.$$

If it is assumed that $S_{-1}^k = 0, \Re s_0 \geq 0$ and $0 \leq t < 1$, then

$$\varphi(t) = \sum_{\mu=0}^{\infty}(S_\mu^k - S_{\mu-1}^k)b_\mu(s_0)t^\mu = \sum_{\mu=0}^{\infty} S_\mu^k(b_\mu(s_0) - b_{\mu+1}(s_0))t^\mu$$

$$= (1-t)\sum_{\mu=0}^{\infty} S_\mu^k b_\mu(s_0)t^\mu - s_0\sum_{\mu=0}^{\infty} S_\mu^k(b_\mu(s_0) - b_{\mu+1}(s_0))\frac{t^{\mu+1}}{\mu+1} = \varphi_1(t) - \varphi_2(t).$$

But $|S_\mu^k| \leq K$, $\mu = 0, 1, 2\dots$ and hence

$$|\varphi_1(t)| \leq K(1-t)\sum_{\mu=0}^{\infty} b_\mu(\sigma_0)t^\mu = \frac{K}{(1-t)^{\sigma_0}},$$

$$|\varphi_2(t)| \leq K|s_0|\sum_{\mu=0}^{\infty}\frac{t^\mu}{\mu+1} \leq K|s_0|\sum_{\mu=0}^{\infty} b_\mu(\sigma_0) = \frac{K|s_0|}{(1-t)^{\sigma_0}},$$

and since $1 - gt \geq \dfrac{1}{q+1}$, $0 \leq t \leq 1$, it follows that

$$\int_0^1 |(1-t|)^{s-1}\varphi(t)|\frac{dt}{|(1-gt)^{s-s_0}|} < L\int_0^1 (1-t)^{\sigma-\sigma_0-1}\,dt,$$

i.e., the integral in (3.10) is absolutely convergent. The same holds if $\sigma_0 < 0$, since then $|\varphi(t)| \leq M < \infty, 0 < t < 1$.

If the real variable t in (3.11) is replaced by $z \in \mathbb{C}$, then $\varphi(z)$ becomes a holomorphic function in the unit disk. Moreover, there exists a constant K such that

$$|\varphi(z)| \leq \frac{K}{(1-|z|)^{\sigma_0}}, \quad |z| < 1,$$

i.e., the function φ is of a finite order in the unit disk in a sense of Hadamard.

For the integral representation (3.10) it can be given a form not involving s_0. To that end another kind of the factorial series is used, namely:

$$\sum_{\nu=0}^{\infty}\frac{\nu! b_\nu}{(s+1)(s+2\dots(s+\nu)}, \quad s \neq -1, -2, \dots. \tag{3.12}$$

Let $\sum_{n=0}^{\infty} B_n^k(s_0)$ be the E_k-transform of the series (3.12) for $s = s_0$, i.e.,

$$B_n^k(s_0) = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} \frac{\nu!\, b_\nu}{(s_0+1)(s_0+2)\dots(s_0+\nu)}.$$

Since

$$b_n(s_0)\binom{n}{\nu}\frac{\nu!\Gamma(s_0+1)}{\Gamma(s_0+\nu+1)} = \frac{\Gamma(s_0+n+1)}{n!\Gamma(s_0+1)}\frac{n!}{\nu!(n-\nu)!}\frac{\nu!\Gamma(s_0+1)}{\Gamma(s_0+\nu+1)}$$

$$= \frac{\Gamma(s_0+n+1)}{\Gamma(n-\nu+1)\Gamma(s_0+\nu+1)}, \quad \nu = 0, 1, 2, \dots,$$

it follows that

$$\varphi(t) = \sum_{n=0}^{\infty} B_n^k(s_0) b_n(s_0) t^n$$

$$= \sum_{\nu=0}^{\infty} \frac{b_\nu}{\Gamma(s_0+\nu+1)} \sum_{n=\nu}^{\infty} \frac{t^n}{(q+1)^{n+1} q^{n-\nu}} \frac{\Gamma(s_0+n+1)}{\Gamma(n-\nu+1)}$$

$$= \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(s_0+n+1)} \sum_{\nu=0}^{\infty} \frac{t^{\nu+n}}{(q+1)^{\nu+n+1}}\, q^\nu\, \frac{\Gamma(\nu+n+s_0+1)}{\Gamma(\nu+1)}$$

$$= \sum_{n=0}^{\infty} \frac{b_n t^n}{(t+1)^{n+1}} \sum_{\nu=0}^{\infty} b_\nu (s_0+\nu)(gt)^\nu$$

$$= \frac{1}{(1-gt)^{s_0}} h(t), \;\; h(t) = \sum_{\nu=0}^{\infty} \frac{b_\nu t^\nu}{(q+1-qt)^{\nu+1}}.$$

The change of order of summations in the two-multiple series defining the function φ is of legal ground, if it is absolutely convergent and it follows now the proof that this is really the fact. Let

$$a_\nu(s_0) = \frac{\nu! b_\nu}{(s_0+1)(s_0+2)\dots(s_0+\nu)}, \quad \nu = 0, 1, 2, \dots,$$

then it is supposed that the series $\sum_{\nu=0}^{\infty} A_\nu^k(s_0)$ is E_k-summable, i.e., if

$$A_n^k(s_0) = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} a_\nu(s_0), \quad n = 0, 1, 2, \dots,$$

then the series $\sum_{n=0}^{\infty} A_n^k(s_0)$ is convergent. Therefore, the sequence $\{A_n^k(s_0)\}_{n=0}^{\infty}$ is bounded, i.e., $|A_n^k(s_0)| \leq A$, $n = 0, 1, 2, \ldots$. Since

$$a_n(s_0) = (q+1)q^n \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{n}{\nu} \left(\frac{q+1}{q}\right)^{\nu} A_\nu^k(s_0),$$

it follows that

$$|a_n(s_0)| \leq A(q+1)q^n \sum_{\nu=0}^{n} \binom{n}{\nu} \left(\frac{q+1}{q}\right)^{\nu}$$
$$= A(q+1)(2q+1)^n, \ \ n = 0, 1, 2, \ldots.$$

Hence, if $|t| < \dfrac{1}{2q+1}$, then the two-multiple series

$$\sum_{n=0}^{\infty} b_n(s_0) \frac{t^n}{(q+1)^{n+1}} \sum_{\nu=0}^{n} q^{n-\nu} \binom{n}{\nu} a_\nu(s_0)$$

is absolutely convergent. Moreover, it is also established that $h(t) = (1 - gt)^{s_0}\varphi(t)$, $0 < t < 1$. But $\varphi(z)$ is holomorphic when $|z| < 1$ and since $0 < g < 1$, the function $h(z)$ is also holomorphic when $|z| < 1$. Then, the representation (3.10) becomes

$$f(s) = s \int_0^1 (1-t)^{s-1} h(t) \frac{dt}{(1-gt)^s}.$$

Let $\dfrac{z}{q+1-qz} = w$, i.e., $z = \dfrac{(q+1)w}{1+qw}$, and

$$\psi(w) = \sum_{n=0}^{\infty} b_n w^n, \quad w = u + iv,$$

then

$$f(s) = \int_0^1 (1-u)^{s-1} \psi(u)\, du.$$

If $(q+1)|w| < |1+qw|$, i.e., $\left| w - \dfrac{q}{2q+1} \right| < \dfrac{q+1}{2q+1}$, then the function $\psi(w)$ is holomorphic in the disk $U(q)$ with center at he point $\dfrac{q}{2q+1}$ and radius $\dfrac{q+1}{2q+1}$. Moreover, this function is also of a finite order in a sense of Hadamard in $U(q)$ so that it is proved the following interesting theorem:

Theorem 24. *Let the series* (3.1) *have a finite abscissa* f_k *of* E_k*-summability. Then, for each* s*, such that* $\Re s > 0$ *and* $\Re s > f_k$*, it hods the representation*

$$f(s) = s \int_0^1 (1-u)^{s-1} \psi(u)\, du, \tag{3.13}$$

where the function

$$\psi(w) = \sum_{n=0}^{\infty} b_n w^n$$

is holomorphic in the disk $U(q)$ *and moreover, it is of finite Hadamard's order in this disk.*

The inverse of the latter theorem is also true. More precisely, let the function $f(s)$ be defined by the equality (3.13), where

$$\psi(w) = \sum_{n=0}^{\infty} b_n w^n$$

is holomorphic in the disk $U(q)$ and of finite Hadamard's order there. Then the factorial series (3.1) has finite abscissa f_k of E_k-summability, i.e., $f_k < \infty$.

Let $u = \dfrac{t}{q+1-qt}$, then

$$f(s) = \int_0^1 (1-t)^{s-1} \tilde{h}(t)\, dt,$$

where

$$\tilde{h}(t) \frac{1}{(1-gt)^{s-s_0}} \sum_{n=0}^{\infty} \tilde{B}_n(s_0; s) t^n,$$

and

$$B_n(s_0; s) = \sum_{\nu=0}^{n} A_\nu^k b_\nu(s_0) b_{n-\nu}(s-s_0-1) g^\nu.$$

The function $\tilde{h}(z)$ is of finite Hadamard's order δ in the disk $U(q)$ and if $\Re s = \sigma > \delta$ and $\sigma > 0$, then by a well-known theorem the series

$$\int_0^1 (1-t)^{s-1} \sum_{n=0}^{n} \widetilde{B}_n(s_o; s) t^n\, dt = \sum_{n=0}^{\infty} \frac{\Gamma(s)\Gamma(n+1)}{\Gamma(n+s+1)} \widetilde{B}_n(s_0; s) = \sum_{n=0}^{\infty} A_n^k(s)$$

is convergent which means that the series (3.1) is E_k-summable.

4. Summation of Newton's series by Euler's transform

The series of Newton are

$$a_0 + \sum_{\nu=1}^{\infty} a_\nu (s-1)(s-2)\dots(s-\nu). \tag{4.1}$$

It is evident that they are convergent for $s = 1, 2, 3, \dots$ so that these values of s can be avoided. A basic theorem for the Euler summation of these series is:

Theorem 25. *If the series (4.1) is E_k-summable for $s = s_0$, then it is E_k-summable for each s with $\Re s > \Re s_0$. Moreover, for the holomorphic function $f(s)$, defined by its E_k-sum for $\Re s > \Re s_0$, it holds the representation*

$$\begin{aligned} f(s) \\ = \frac{\Gamma(1-s_0)}{\Gamma(s-s_0)\Gamma(1-s)} \sum_{\mu=0}^{\infty} S_\mu(s_0) \int_0^1 \frac{t^{s-s_0}(1-t)^{\mu-s}}{q+1-qt} \left(\frac{q+1}{q+1-qt}\right)^{\mu+1} dt, \end{aligned} \tag{4.2}$$

where

$$S_\mu^k(s_0) = \sum_{n=0}^{\mu} A_n^k(s_0),$$

$$A_n^k(s_0) = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} a_\nu (s_0-1)\dots(s_0-\nu),$$

provided that $\Re s > 0$, and $\Re s > \Re s_0$, and

$$\begin{aligned} f(s) = \sum_{\nu=0}^{p} a_\nu (s-1)(s-2)\dots(s-\nu) \\ + \frac{\Gamma(1-s_0)}{\Gamma(s-s_0)\Gamma(1-s)} \sum_{\mu=0}^{\infty} S_{\mu,p}(s_0) \int_0^1 \frac{t^{s-s_0}(1-t)^{\mu+p-s}}{q+1-qt} \left(\frac{q+1}{q+1-qt}\right)^{\mu+1} dt, \end{aligned} \tag{4.3}$$

where

$$S_{\mu,p}^k(s_0) = \sum_{n=0}^{\mu} A_{n,p}^k(s_0),$$

$$A_{n,p}^k(s_0) = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} \binom{n}{\nu} q^{n-\nu} a_{\nu+p} (s_0-1)\dots(s_0-\nu)$$

when $\Re s > 0$, where p is an integer, greater than $\Re s$.

Indeed, if $a_0(s_0) = 1, a_\nu(s_0) = a_\nu(s_0-1)(s_0-2)\dots(s_0-\nu),\ \nu = 1,2,3,\dots,$ then the series (4.1) becomes

$$a_0(s_0) + \sum_{\nu=1}^{\infty} a_\nu(s_0)\,\frac{(s-1)(s-2)\dots(s-\nu)}{(s_0-1)(s_0-2)\dots(s_0-\nu)}.$$

Let, as before,

$$\lambda_0(s_0;s) = 1,\ \lambda_\nu(s_0;s) = \frac{(s-1)(s-2)\dots(s-\nu)}{(s_0-1)(s_0-2)\dots(s_0-\nu)},\quad \nu = 1,2,3,\dots,$$

then

$$A_n^k(s) = \frac{1}{(q+1)^{n+1}}\sum_{\nu=0}^{n}\binom{n}{\nu}q^{n-\nu}a_\nu(s_0)\lambda_\nu(s_0;s)$$

$$= (q+1)^{-n}\sum_{\nu=0}^{n}\binom{n}{\nu}q^{n-\nu}\lambda_\nu(s_0;s)\sum_{\mu_0}^{\nu}(-1)^\mu q^\nu(-1)^\nu\binom{\nu}{\mu}\left(\frac{q+1}{q}\right)^\mu A_\mu^k(s_0)$$

$$= (q+1)^{-n}\sum_{\mu=0}^{n}A_\mu^k(s_0)\left(\frac{q+1}{q}\right)^\mu(-1)^\mu\sum_{\nu=0}^{\mu}(-1)^\nu\lambda_\nu(s_0;s)q^\nu\binom{n}{\nu}\binom{\nu}{\mu}q^{n-\nu}$$

$$= (q+1)^{-n}\sum_{\mu=0}^{n}A_\mu^k(s_0)\left(\frac{q+1}{q}\right)^\mu\binom{n}{\mu}\sum_{\nu=\mu}^{n}q^\nu(-1)^{\mu+\nu}\lambda_\nu(s_0;s)\binom{n-\mu}{\nu-\mu}q^{n-\nu}$$

$$= \sum_{\mu=0}^{n}A_\mu^k(s_0)\left(\frac{q}{q+1}\right)^{n-\mu}\binom{n}{\mu}\sum_{\nu=0}^{n-\mu}(-1)^\nu\binom{n-\mu}{\nu}\lambda_{\mu+\nu}(s_0;s)$$

$$= \sum_{\mu=0}^{n}A_\mu^k(s_0)\binom{n}{\mu}\Delta^{n-\mu}\lambda_\mu(s_0;s).$$

Further, by induction one easily obtains that

$$\Delta^\nu\lambda_\mu(s_0;s)$$

$$= \lambda_\mu(s_0;s)\frac{s_0-s(s_0-s-1)\dots(s_0-s-\nu+1)}{(s_0-\mu-1)(s_0-\mu-2)\dots(s_0-\mu-\nu)},\quad \nu = 0,1,2,\dots$$

and if $s - s_0 = d$, then

$$\Delta^{n-\mu}\lambda_\mu(s_0;s)$$

$$= \frac{d(d+1)\dots(d+n-\mu)(1-s)(2-s)\dots(\mu-s)}{(1-s_0)(2-s_0)\dots(n-s_0)}, \quad \mu = 0,1,2,\dots,n.$$

Since s is not a positive integer, it follows that

$$\Delta^{n-\mu}\lambda_\mu(s_0;s) = \frac{\Gamma(d+n-\mu)\Gamma(\mu-s+1)\Gamma(1-s_0)}{\Gamma(d)\Gamma(n-s_0+1)\Gamma(1-s)}, \quad \mu = 0,1,2,\dots,n$$

and if $\Re s < 0$, then

$$\Delta^{n-\mu}\lambda_\mu(s_0;s) = c(s_0;s)\int_0^1 t^{d-+n-\mu-1}(1-t)^{\mu-s}\,dt, \quad \mu = 0,1,2,\dots,n,$$

where

$$c(s_0;s) = \frac{\Gamma(1-s_0)}{\Gamma(s-s_0)\Gamma(1-s)}.$$

In this way one comes to the representation

$$S_m^k(s) = \sum_{n=0}^{m} A_n^k(s) = c(s_0;s)\sum_{\mu=0}^{m} A_\mu^k(s_0) l_{m,\mu}(s_0;s), \tag{4.4}$$

where

$$l_{m,\mu}(s_0;s) = \sum_{n=\mu}^{m}\left(\frac{q}{q+1}\right)^{n-\mu}\binom{n}{\mu}\int_0^1 t^{d+n-\mu-1}(1-t)^{\mu-s}\,dt$$

$$= \int_0^1 t^{d-1}(1-t)^{\mu-s}\sum_{\nu=0}^{m-\mu}\binom{\mu+\nu}{\mu}\left(\frac{qt}{q+1}\right)^{\nu} dt.$$

If μ is fixed, then

$$\lim_{m\to\infty} l_{m,\mu}(s_0;s) = l_m(s_0;s)$$

$$= \int_0^1 t^{d-1}(1-t)^{\mu-s}\sum_{\nu=0}^{\infty}\binom{\mu+\nu}{\nu}\left(\frac{qt}{q+1}\right)^{\nu} dt$$

$$= \int_0^1 t^{d-1}(1-t)^{\mu-s}\left(\frac{q+1}{q+1-qt}\right)^{\mu+1} dt.$$

If

$$h_{m,\mu}(s_0;s) = l_{m,\mu}(s_0;s) - l_{m,\mu+1}(s_0;s), \quad \mu = 0,1,2,\dots,m-1,$$

then (4.4) yields that

$$S_m^k(s) = c(s_0;s)\sum_{\mu=0}^{m-1} S_\mu^k(s_0)h_{m,\mu}(s_0;s) + c(s_0;s)S_m^k(s_0)l_{m,m}(s_0;s).$$

Moreover,

$$\lim_{m\to\infty} h_{m,\mu}(s_0;s)$$
$$= h_\mu(s_0;s) = \int_0^1 t^d(1-t)^{\mu-s}\left(\frac{q+1}{q+1-qt}\right)^{\mu+1}\frac{dt}{q+1-qt}.$$

Since

$$l_{m,m}(s_0;s) = \int_0^1 t^{d-1}(1-t)^{m-s}\,dt = \frac{\Gamma(d)\Gamma(m-s+1)}{\Gamma(d+m-s+1)}\,,$$

and $\Re d = \Re s - \Re s_0 > 0$, it follows that $\lim_{m\to\infty} l_{m,m}(s_0;s) = 0$ and, hence, $\lim_{m\to\infty} S_m^k(s_0)l_{m,m}(s_0;s) = 0$.

It remains to study the behaviour of the expression

$$\widetilde{S}_m(s_0;s) = \sum_{\mu=0}^{m-1} S_\mu^k(s_0)h_{m,\mu}(s_0;s)$$

when $m\to\infty$.

In order to justify the application of the Toeplitz–Schur theorem, one has to prove that the sequence

$$T_m(s) = \sum_{\mu=0}^{m-1} |h_{m,\mu}(s_0;s)|,\quad m = 1,2,3,\ldots$$

is bounded for each s such that $\Re s < 0$ and $\Re s > \Re s_0$.

For the polynomial

$$P_{m,\mu}(z) = \sum_{\nu=0}^{m-\mu}\binom{\mu+\nu}{\mu}z^\nu$$

it was obtained that

$$P_{m,\mu}(z) = -\sum_{\nu=0}^{\mu}\binom{m+1}{\nu}\frac{z^{m-\nu+1}}{(1-z)^{\mu-\nu+1}},$$

which gives that

$$l_{m,\mu}(s_0;s) = \int_0^1 t^{s-s_0-1}(1-t)^{\mu-s} P_{m,\mu}(z)\,dt, \quad z = \frac{qt}{q+1}.$$

By means of the substitution $1-t = \exp(-u)$ the last representation becomes similar to that used by studying the E_k-summation of the Dirichlet series so that it may be followed the way already known, but there is also a direct approach. Namely, for $h_{m,\mu}(s_0;s)$ it holds that

$$\begin{aligned} h_{m,\mu}(s_0;s) &= \int_0 t^{s-s_0-1}(1-t)^{\mu-s} P_{m,\mu}(z)\,dt \\ &\quad - \int_0^1 t^{s-s_0-1}(1-t)^{\mu+1-s} P_{m,\mu+1}(z)\,dt \\ &= -\int_0^1 \frac{t^{s-s_0}}{q+1}(1-t)^{\mu-s} \sum_{\nu=0}^{\mu} \binom{m+1}{\nu} \frac{z^{m+1-\nu}}{(1-z)}\,dt \\ &\quad + \binom{m+1}{\mu+1} \int_0^1 t^{s-s_0-1}(1-t)^{\mu+1-s} \frac{z^{m-\nu}}{1-z}\,dt \\ &= A_{m,\mu}(s_0;s) + B_{m,\mu}(s_0;s). \end{aligned}$$

If $\sigma = \Re s$ and $\delta = \Re s - \Re s_0$, then

$$\begin{aligned} \sum_{\mu=0}^{m-1} |h_{m,\mu}(s_0;s)| &\le \sum_{\mu=0}^{m-1} |A_{m,\mu}(s_0;s)| + \sum_{\mu=0}^{m-1} |B_{m,\mu}(s_0;s)| \\ &= \mathcal{A}_m(s_0;s) + \mathcal{B}_m(s_0;s). \end{aligned}$$

Further, since

$$1 - z = 1 - \frac{qt}{q+1} \ge \frac{1}{q+1}, \ \ 1 - t + z = 1 - \frac{t}{q+1} \le 1, \quad 0 \le t \le 1,$$

it follows that

$$\begin{aligned} \mathcal{B}_m(s_0;s) &\le \sum_{\mu=0}^{m-1} \binom{m+1}{\mu+1} \int_0^1 t^{\delta-1}(1-t)^{\mu+1-\sigma} \frac{z^{m-\mu}}{1-z}\,dt \\ &< \sum_{\mu=0}^{m+1} \binom{m+1}{\mu} \int_0^1 t^{\delta-1}(1-t)^{\mu-\sigma} \frac{z^{m+1-\mu}}{1-z}\,dt \end{aligned}$$

$$= \int_0^1 t^{\delta-1}(1-t)^{-\sigma}(1-t+z)^{m+1}\frac{dt}{1-z} = O(1),$$

as well as

$$\mathcal{A}_m(s_0;s) \leq \frac{1}{q+1}\int_0^1 t^\delta \sum_{\mu=0}^{m-1}(1-t)^{\mu-\sigma}\sum_{\nu=0}^{\mu}\binom{m+1}{\nu}\frac{z^{m+1-\nu}}{(1-z)^{\mu+2-\nu}}\,dt$$

$$= \frac{1}{q+1}\int_0^1 t^\delta \sum_{\nu=0}^{m-1}\sum_{\mu=\nu}^{m-1} z^{m+1-\nu}\frac{(1-t)^{\nu-\sigma}}{(1-z)^2}\sum_{\tau=0}^{m-1-\nu}\left(\frac{1-t}{1-z}\right)^\tau dt$$

$$= \frac{1}{q+1}\int_0^1 t^\delta \sum_{\nu=0}^{m-1}\binom{m+1}{\nu}z^{m+1-\nu}\frac{(1-t)\nu-\sigma}{(1-z)^2}\sum_{\tau=0}^{m-1-\nu}\left(\frac{1-t}{1-z}\right)^\tau dt$$

$$< \int_0^1 t^{\delta-1}\sum_{\nu=0}^{m-1}\binom{m+1}{\nu}z^{m+1-\nu}\frac{(1-t)^{\nu-\sigma}}{1-z}\,dt$$

$$< (q+1)\int_0^1 t^{\delta-1}(1-t)^{-\sigma}(1-t+z)^{m+1}\,dt = O(1).$$

Then, the Toeplitz theorem yields that there exists $\lim_{m\to\infty} S_m^k(s)$ provided that $\Re s < 0$ and $\Re s > \Re s_0$. Moreover, the E_k-sum $f(s)$ of the series (4.1) is the holomorphic function defined by the equality (4.2) in the region $\{s : \Re s < 0, \Re s > \Re s_0\}$.

It remains to study the case $\Re s \geq 0, \Re s > \Re s_0$ and let $p > s$ be an integer. The Euler summation has the property that if a series is E_k-summable, then the same is true for the series obtained from it after increasing or decreasing the subscripts of its terms with one and the same number. In particular, if the series (4.1) is E_k-summable for $s = s_0$, then the same holds for the series

$$\sum_{\nu=p}^{\infty} b_\nu(s_0)\frac{(s-1)(s-2)\dots(s-\nu)}{(s_0-1)(s_0-2)\dots(s_0-\nu)}.$$

This series can be written as follows

$$\frac{(s-1)(s-2)\dots(s-p)}{(s_0-1)(s_0-2)\dots(s_0-p)}\sum_{\nu=p}^{\infty} b_\nu(s_0)\frac{(s-p-1)(s-p-2)\dots(s-\nu)}{(s_0-p-1)(s_0-p-2)\dots(s_0-\nu)},$$

and if

$$a_p(s_0;s) = \frac{(s-1)(s-2)\dots(s-p)}{(s_0-1)(s_0-2)\dots(s_0-p)},$$

then it takes the form

$$a_p(s_0; s) \sum_{\nu=0}^{\infty} b_{\nu+p}(s_0) \frac{(s-p-1)(s-p-2)\dots(s-p-\nu)}{(s_0-p-1)(s_0-p-2)\dots(s_0-p-\nu)}. \tag{4.5}$$

If $S^k(s)$ is the E_k-sum of the series (4.1) and $S^k_p(s)$ that of (4.5), then

$$S^k(s) = \sum_{\nu=0}^{p-1} a_\nu (s-1)(s-2)\dots(s-\nu) + S^k_p(s).$$

Since $\Re s - p = \Re(s-p) < 0$, the case under consideration reduces to the previous one. That means the E_k-sum of the series (4.5) in the region $\{s; \Re s > 0, \Re s > \Re s_0\}$ is

$$G_p(s_0; s) \int_0^1 t^{s-s_0}(1-t)^{-s_0+p} \frac{q+1}{(q+1-qt)^2} \sum_{\mu=0}^{\infty} S^k_p(s) \left(\frac{(q+1)(1-t)}{q+1-qt} \right)^{\mu} dt$$

where

$$G_p(s_0; s) = \frac{\Gamma(1-s_0+p) a_p(s_0; s)}{\Gamma(s-s_0)\Gamma 1 - s_0 + p)}.$$

But since

$$a_p(s_0; s) = \frac{\Gamma(p-s+1)\Gamma(1-s)}{\Gamma(1-s_0)\Gamma(p-s_0+1)},$$

it follows that, in fact, the series (4.5) is E_k-summable for each s such that $\Re s > 0$ and $\Re s > \Re s_0$ with sum given by (4.4).

From Theorem 21, it follows that there exists a number n_k such that the series (4.1) is E_k-summable for each s such that $\Re s > n_k$ and loses this property when $\Re s < n_k$. This n_k may be also $-\infty$ as well as ∞; it is called abscissa of E_k-summability of the series (4.1).

Theorem 26. *If $n_k \in [0, \infty)$, then*

$$n_k = \limsup_{m \to \infty} \frac{\log |A^k_0 + A^k_1 + \dots + A^k_m|}{\log m},$$

where

$$A^k_n = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n} (-1)^\nu \nu! a_\nu, \quad n = 0, 1, 2, \dots.$$

If

$$A_n^k(s) = \frac{1}{(q+1)^{n+1}} \sum_{\nu=0} \binom{n}{\nu} q^{n-\nu} a_\nu (s-1)(s-2)\dots(s-\nu),$$

then

$$A_n^k(s) = \sum_{\mu=0}^{n} A_\mu^k \left(\frac{q}{q+1}\right)^{n-\mu} \binom{n}{\mu} \Delta^{n-\mu} \lambda_\mu(s),$$

where

$$\lambda_\mu(s) = \frac{(-1)^\mu}{\mu!(s-1)(s-2)\dots(s-\mu)}.$$

Since

$$\binom{n}{\mu} \Delta^{n-\mu} \lambda_\mu(s) = \frac{\Gamma(n+1)}{\Gamma(\mu+1)\Gamma(n-\mu+1)} \frac{\Gamma(s+n-\mu)\Gamma(\mu-s+1)}{\Gamma(s)\Gamma(1-s)}$$

$$= c(s) b_{n-\mu}(s-1) b_\mu(s), \quad c(s) = \frac{1}{\Gamma(s)\Gamma(1-s)},$$

it follows that

$$A_n^k(s) = c(s) \sum_{\mu=0}^{n} A_\mu^k g^{n-\mu} b_{n-\mu}(s-1) b_\mu(-s)$$

$$= c(s) \sum_{\mu=0}^{n} A_{n-\mu}^k g^\mu b_\mu(s-1) b_{n-\mu}(-s), \quad g = \frac{q}{q+1},$$

whence

$$S_m^k(s) = c(s) \sum_{n=0}^{m} \sum_{\mu=0}^{n} A_{n-\mu}^k g^\mu b_\mu(s-1) b_{n-\mu}(-s)$$

$$= c(s) \sum_{\mu=0}^{m} g^\mu b_\mu(s-1) \sum_{n=\mu}^{m} A_{n-\mu} k b_{n-\mu}(-s)$$

$$= c(s) \sum_{\mu=0}^{m} g^\mu b_\mu(s-1) u_{m-\mu}(s),$$

where

$$u_{m-\mu}(s) = \sum_{\nu=0}^{m-\mu} A_\nu^k b_\nu(-s).$$

The series

$$\sum_{\nu=0}^{\infty} A_u^k b_\nu(-s) \tag{4.6}$$

is convergent. Indeed, if $m > n$, then

$$\sum_{\nu=n}^{m} A_\nu^k b_\nu(-s) = \sum_{\nu=n}^{m} S_\nu^k (b_\nu(-s) - b_{\nu+1}(-s))$$

$$+S_m^k b_m(-s) - S_{n-1}^k b_n(-s) = \mathcal{A}_{n,m}(s) + \mathcal{B}_{n,m}(s).$$

If $\varepsilon > 0$ and s is real and greater than $\alpha + \varepsilon$, then

$$\mathcal{B}_{m,n}(s) = O(m^{\alpha+\varepsilon} m^{-s}) + O(n^{\alpha+\varepsilon} n^{-s}) = o(1), \quad m, n \to \infty.$$

Since

$$b_{\nu+1}(-s) = b_\nu(-s) \frac{\nu - s + 1}{\nu + 1},$$

it follows that if $\nu < s - 1$, then $b_\nu(-s)$ and $b_{\nu+1}(-s)$ have one and the same sign and $|b_{\nu+1}(-s)| < |b_\nu(-s)|$. Let $n > s$, then since $|S_\nu^k| \leq K\nu^{\alpha+\varepsilon}$ when ν is sufficiently large, it follows that

$$|\mathcal{A}_{n,m}(s)| \leq K \sum_{nu=n}^{m-1} \nu^{\alpha+\varepsilon} (|b_\nu(-s)| - |b_{\nu+1}(-s)|)$$

$$\leq Kn^{\alpha+\varepsilon} |b_n(-s)| + K \sum_{\nu=n+1}^{m-1} |b_\nu(-s)| (\nu^{\alpha+\varepsilon} - (\nu-1)^{\alpha+\varepsilon}) + K(m-1)^{\alpha+\varepsilon} |b_{m-1}(-s)|$$

$$= O(n^{\alpha+\varepsilon} n^{-s}) + \sum_{\nu=n+1}^{m-1} O(\nu^{\alpha+\varepsilon-1}) \nu^{-s}) + O(m^{\alpha+\varepsilon} m^{-s})$$

$$= O(n^{\alpha+\varepsilon-s}) + \sum_{\nu=n+1}^{m-1} O(\nu^{-(s-\alpha-\varepsilon+1)}) = o(1), \quad m, n \to \infty.$$

Hence, the series (4.6) is convergent and if $A(s)$ is its sum, then

$$\lim_{n\to\infty} u_n(s) = \lim_{n\to\infty} \sum_{\nu=0}^{n} A_\nu^k(s) = A(s).$$

It is easily seen now that

$$\lim_{m\to\infty} S_m^k(s) = c(s)A(s)(1-g)^{-s}.$$

If $\varepsilon > 0$, then there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{\mu=N+1}^{\infty} g^\mu b_\mu(s-1) < \varepsilon.$$

Then, by fixed N it follows that

$$\lim_{m\to\infty} \sum_{\mu=0}^{N} g^\mu b_\mu(s-1) u_{m-\mu}(s) = A(s) \sum_{\mu=0}^{N} g^\mu b_\mu(s-1).$$

Further, if $m > N$, then

$$\left| \sum_{\mu=N+1}^{m} g^\mu b_\mu(s-1) u_{m-\mu}(s) \right| < K \sum_{\mu=N+1}^{m} g^\mu b_\mu(s-1) < K\varepsilon,$$

and hence,

$$\limsup_{m\to\infty} \left| S_m^k(s) - c(s)A(s) \sum_{\mu=0}^{\infty} g^\mu b_\mu(s-1) \right| < 2K|c(s)|\varepsilon.$$

That means the series (4.1) is E_k-summable for each $s > \alpha + \varepsilon$, hence, for each s such that $\Re s > \alpha$.

Conversely, let the series (4.1) be E_k-summable for $s = \alpha > 0$, i.e., the series

$$\sum_{n=0}^{\infty} A_n^k(s)$$

is convergent. Then, since

$$b_{n-\mu}(-\alpha-1) = \frac{\Gamma(-\alpha+n-\mu)}{\Gamma(-\alpha)}$$

it follows that

$$A_n^k = \sum_{\mu=0}^{n} \binom{n}{\mu} A_\mu^k(\alpha) g^{n-\mu} \frac{\mu!(-\alpha)(-\alpha+1)\dots(-\alpha+n-\mu-1)}{(1-\alpha)(2-\alpha)\dots(n-\alpha)},$$

$$c_n(\alpha)\sum_{\mu=0}^{n} A_\mu^k(\alpha)g^{n-\mu}b_{n-\mu}(-\alpha-1) = c_n(\alpha)\sum_{\mu=0}^{n} A_{n-\mu}^k(\alpha)g^{\mu}b_\mu(-\alpha-1),$$

where

$$c_n(\alpha) = \frac{n!}{(1-\alpha)(2-\alpha)\dots(n-\alpha)}.$$

Then,

$$S_m^k = \sum_{n=0}^{m} A_n^k = \sum_{n=0}^{m}\sum_{\mu=0}^{n} c_n(\alpha)A_{n-\mu}(\alpha)g^\mu b_\mu(-\alpha-1)$$

$$\sum_{\mu=0}^{n} g^\mu b_\mu(-\alpha-1)L_{m,\mu}(\alpha),$$

where

$$L_{m,\mu}(\alpha)\sum_{n=\mu}^{m} c_n(\alpha)A_{n-\mu^k}(\alpha) = \sum_{\nu=0}^{m-\mu} c_{\mu+\nu}(\alpha)A_\nu^k(\alpha).$$

If

$$S_n^k(\alpha) = \sum_{\nu=0}^{n} A_\nu^k(\alpha),$$

then

$$L_{m,\mu}(\alpha) = \sum_{\nu=0}^{m-\mu-1} S_\nu^k(\alpha)(c_{\mu+\nu}(\alpha) - c_{\mu+\nu+1}) + c_m(\alpha)S_{n-\mu}^k(\alpha).$$

Since

$$c_{n+1}(\alpha) = c_n(\alpha)\frac{n+1}{n+1-\alpha},$$

all the $c_n(\alpha)$ have one and the same sign when $n > \alpha$ and, moreover, $|c_{n+1}(\alpha)| > |c_n(\alpha)|$.

Further,

$$c_n(\alpha) = \frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma(n-\alpha+1)} = O(n^\alpha), \quad n\to\infty,$$

and $S_m^k(\alpha) = O(1)$, $m\to\infty$, give that

$$L_{m,\mu}(\alpha) = \sum_{\nu=0}^{[\alpha]} S_\nu^k(\alpha)(c_{\mu+\nu}(\alpha) - c_{\mu+\nu+1}(\alpha))$$

$$+\sum_{\nu=[\alpha]+1}^{m-\mu-1} S_\nu^k(\alpha)(c_{\mu+\nu}(\alpha) - c_{\mu+\nu+1}(\alpha)) + O(m^\alpha)$$

$$= \sum_{\nu=0}^{[\alpha]} O(\mu+\nu)^\alpha + \sum_{\nu=[\alpha]+1}^{m-\mu-1} O(|c_{\mu+\nu}(\alpha)| - |c_{\mu+\nu+1}(\alpha)|) + O(m^\alpha)$$

$$= O(\mu^\alpha) + O(m^\alpha) + O(m^\alpha).$$

But since $g < 1$, it follows that

$$|S_m^k| \leq K m^\alpha \sum_{\mu=0}^{m} g^\mu |b_\mu(-\alpha-1)| = O(m^\alpha), \; m \to \infty$$

and thus Theorem 22 is proved.

Theorem 27. *If* $-\infty < n_k < 0$, *then*

$$n_k = \limsup_{n\to\infty} \frac{\log|A_{nk} + A_{n+1}^k + \dots|}{\log n}.$$

Let $\alpha < 0$ be the right-hand side of the last equality, then

$$|R_n^k| = |A_n^k + A_{n+1}^+ \dots| < K n^{\alpha+\varepsilon},$$

where $\varepsilon > 0$ and $n > N = N(\varepsilon)$ and the series

$$\sum_{\nu=0}^{\infty} A_\nu^k b_\nu(-s) \tag{4.7}$$

is convergent provided that $s > \alpha$. Indeed, let $\varepsilon > 0$ be such that $s > \alpha + \varepsilon$ and $m > n > N(\varepsilon)$, then

$$\sum_{\nu=n}^{m} A_\nu b_\nu(-s)$$

$$= R_n^k b_n(-s) + \sum_{\mu=n+1}^{m-1} R_m^k (b_\mu(-s) - b_{\mu-1}(-s)) - R_{m+1}^k b_m(-s)$$

$$= O(n^{\alpha+\varepsilon-s}) + \sum_{\mu=n+1}^{m} O\left(\frac{1}{\mu^{s-\alpha-\varepsilon+1}}\right) + O(m^{\alpha+\varepsilon-s}),$$

and the last estimates ensure the convergence of the series (4.7). Then in the same way already used, it can be proved that $\lim_{m\to\infty} S_m^k(s)$ really exists which means that the series (4.1) is E_k-summable for each $s > \alpha$.

Let now the series (4.1) be E_k-summable for $s = \alpha < 0$. Then,

$$A_n^k = c_n(\alpha) \sum_{\mu=0}^{n} A_{n-\mu}^k(\alpha) g^\mu b_\mu(-\alpha - 1),\ c_n(\alpha) = O(n^\alpha), \quad n \to \infty.$$

For the sum

$$S_{m,N}^k = \sum_{n=m}^{N} A_n^k, \quad N > m,$$

it holds the representation

$$S_{m,N}^k = \sum_{\mu=0}^{m} g^\mu b_\mu(-\alpha - 1) \sum_{n=\mu}^{N} c_n(\alpha) A_{n-\mu}^k(\alpha)$$

$$\sum_{\mu=m}^{N} g^\mu b_\mu(-\alpha - 1) \sum_{n=\mu}^{N} c_n(\alpha) A_{n-\mu}^k(\alpha) = S_{m,N}^{(1)}(\alpha) + S_{m,N}^{(2)}(\alpha).$$

But

$$\sum_{n=m}^{n} c_n(\alpha) A_{n-\mu}^k(\alpha) = \sum_{n=m}^{N} c_n(\alpha)(S_{n-\mu}^k(\alpha) - S_{n-\mu-1}^k(\alpha))$$

$$= -c_m(\alpha) S_{m-\mu-1}^k(\alpha) + c_N(\alpha) S_{N-\mu}^k(\alpha) + \sum_{n=m}^{N-1} S_{n-\mu}^k(\alpha)(c_\mu(\alpha) - c_{\mu+1}(\alpha))$$

$$= O(m^\alpha) + O(N^\alpha) + \sum_{n=m}^{N-1} O(\mu^{\alpha-1})$$

$$= O(m^\alpha) + O(m^(\alpha)) + O(m^\alpha) = O(m^\alpha).$$

Since $g < 1$, it follows that

$$S_{m,N}^{(1)}(\alpha) = O(m^\alpha) \sum_{\mu=0}^{m} g^\mu |b_\mu(-\alpha - 1)| = O(m^\alpha).$$

Further,

$$|S_{m,N}^{(2)}(\alpha)| \leq K \sum_{\mu=m}^{N} g^\mu |b_\mu(-\alpha - 1)| \mu^\alpha$$

$$< K(\sqrt{g})^m \sum_{\mu=m}^{\infty} (\sqrt{g})^\mu \mu^\alpha |b_\mu(-\alpha - 1)| < K_1(\sqrt{g})^m < K_2 m^\alpha,$$

and hence,

$$S_{m,N}(\alpha) = O(m^\alpha), \quad N > m,$$

which yields that

$$|A_m^k + A_{m+1} + \dots| = O(m^\alpha)$$

and thus the proof of Theorem 27 is finished.

Another proof of the basic Theorem 25 leads to the following statement:

Theorem 28. *Let the series* (4.1) *be E_k-summable for $s = s_0$ and let $\sum_{n=0}^\infty A_n^k(s_0)$ be the E_k-transformed series. Then this series is E_k-summable for each s such that $\Re s > \Re s_0$ with sum*

$$\frac{f(s)}{\Gamma(s-s_0)\Gamma(1-s_0)} \tag{4.8}$$

$$= \sum_{\mu=0}^\infty A_\mu^k(s_0) b_\mu(-s) \sum_{\nu=0}^\infty \frac{\Gamma(\mu+\nu+1)}{\Gamma(\mu+\nu+1-s_0)} g^\nu b_\nu(-s_0 = 1).$$

This proof is based on the already known representation

$$A_n^k(s) = \sum_{\mu=0}^n A_\mu^k(s_0) \binom{n}{\mu} g^{n-\mu} \Delta^{n-\mu} \lambda_\mu(s_0; s),$$

as well as on the equality

$$\binom{n}{\mu} \Delta^{n-\mu} \lambda_\mu(s_0; s) = c_n(s_0; s) b_\mu(-s) b_{n-\mu}(s - s_0 - 1),$$

where

$$c_n(s_0; s) = \Gamma(s-s_0)\Gamma(1-s_0) \frac{\Gamma(n+1)}{\Gamma(n+1-s_0)}.$$

Then,

$$S_m^k(s) = \sum_{n=0}^m c_n(s_0; s) \sum_{\mu=0}^n A_\mu^k(s_0) g^{n-\mu} b_\mu(-s) b_{n-\mu}(s - s_0 - 1)$$

$$= \sum_{\mu=0}^m A_\mu^k(s_0) b_m u(-s) u_{m,\mu(s_0;s)},$$

where

$$u_{m,\mu}(s_0; s) = \sum_{n=\mu}^{m} c_n(s_0; s) g^{n-\mu} b_{n-\mu}(s_- s_0 - 1)$$

$$= \sum_{\nu=0}^{m-\mu} c_{\mu+\nu}(s_0; s) g^{\nu} b_{\nu}(s - s_0 - 1).$$

If μ is fixed, then

$$\lim_{m\to\infty} u_{m,\mu}(s_0; s) = u_{\mu}(s_0; s) \sum_{\nu=0}^{\infty} c_{\mu+\nu}(s_0; s) b_{\nu}(s - s_0 - 1),$$

where the series is convergent. Then,

$$S_m^k(s) = \sum_{\mu=0}^{m-1} S_{\mu}^k(s_0)(b_{\mu}(-s) u_{m,\mu}(s_0; s) - b_{\mu+1}(-s) u_{m,\mu+1}(s_0; s))$$

$$+ S_m^k(s_0) b_m(-s) u_{m,m}(s_0; s) = \sum_{\mu=0}^{m-1} S_{\mu}^k(s_0) U_{m,\mu}(s_0; s)$$

$$+ S_m^k(s_0) b_m(-s) u_{m,m}(s_0; s).$$

But if $\sigma = \Re s > \sigma_0 = \Re s_0$, then

$$S_m^k(s_0) b_m(-s) u_{m,m}(s_0; s) = O(m^{-\sigma} m^{\sigma_0}) = o(1), \quad m \to \infty,$$

and it remains to consider the sum

$$\mathcal{S}_m^k(s) = \sum_{\mu=0}^{m-1} S_{\mu}^k(s_0) U_{m,\mu}(s_0; s).$$

If μ is fixed, then

$$\lim_{m\to\infty} U_{m,\mu}(s_0; s) = U_{\mu}(s_0; s) = b_{\mu}(-s) u_{\mu}(s_0; s) - b_{\mu+1}(-s) u_{\mu+1}(s_0; s),$$

and in order to show that there exists $\lim_{m\to\infty} \mathcal{S}_m^k(s)$ it has to be proved that the sequence

$$\mathcal{U}_m(s_0; s) = \sum_{\mu=0}^{m-1} |U_{m,\mu}(s_0; s)|, \quad m = 1, 2, 3, \ldots$$

is bounded for $\Re s > \Re s_0$. The proof is based on the inequality

$$\sum_{\mu=0}^{m-1} |U_{m,\mu}(s_0;s)| \le \sum_{\mu=0}^{m-1} |U^{(1)}_{m,\mu}(s_0;s)| + \sum_{\mu=0}^{m-1} |U^{(2)}_{m,\mu}(s_0;s)|$$

$$= L_m(s_0;s) + T_m(s_0;s),$$

where

$$U^{(1)}_{m,\mu}(s_0;s)$$

$$= \sum_{\nu=0}^{m-\mu-1} (b_\mu(-s)c_{\mu+\nu}(s_0;s) - b_{\nu+1}(-s)c_{\mu+\nu+1}(s_0;s))g^\nu b_\nu(s-s_0-1),$$

and

$$U^{(2)}_{m,\mu}(s_0;s) = c_m(s_0;s)g^{m-\mu}b_{m-\mu}(s-s_0-1)b_\nu(-s).$$

Let p be an integer greater than $-\sigma$, then the sign of $b_\mu(-\sigma)$ is $(-1)^p$ for each $\mu > p$ and, as before, one gets that

$$T_m(s_0;s) \le |c_m(s_0;s)| \sum_{\mu=0}^{p-1} g^{m-\mu}|b_{m-\mu}(s-s_0-1)||b_\mu(-s)|$$

$$+|c_m(s_0;s)| \sum_{\mu=p}^{m-1} g^{m-\mu}|b_{m-\mu}(s-s_0-1)||b_\mu(-s)|$$

$$= T^{(1)}_m(s_0;s) + T^{(2)}_m(s_0;s),$$

where

$$T^{(1)}_m(s_0;s) = O(m^{\sigma_0}g^m m^{\sigma-\sigma_0-1}) = o(1), \quad T^{(2)}_m(s_0;s) \le Km^{\sigma_0}.$$

Further,

$$b_\mu(-s)c_{\mu+\nu}(s_0;s) - b_{\mu+1}(-s)c_{\mu+\nu+1}(s_0;s)$$

$$= b_\mu(-s)c_{\mu+\nu}(s_0;s)\left(1 - \frac{\mu-s+1}{\mu+1}\frac{\mu+\nu+1}{\mu+\nu+1-s_0}\right)$$

$$= -b_\mu(-s)c_{\mu+\nu}\left(\frac{s_0}{\mu+\nu+1-s_0} - \frac{s}{\mu+1}\frac{\mu+\nu+1}{\mu+\nu+1-s_0}\right)$$

$$= O((\mu+1)^{-s-1}(\mu+\nu+1)^{\sigma_0}).$$

If $\sigma_0 \leq 0$, then

$$|U_m^{(1)(s_0;s)}| \leq K(\mu+1)^{-\sigma-1} \sum_{\nu=0}^{m-\mu-1} (\mu+\nu+1)^{\sigma} g^{\nu} b_{\nu}(\sigma-\sigma_0-2)$$

$$\leq K(\mu+1)^{-\sigma+\sigma_0-1} \sum_{\nu=0}^{\infty} g^{\nu} b_{\nu}(\sigma-\sigma_0-1) \leq \frac{K_1}{(\mu+1)^{\sigma-\sigma_0-1}},$$

whence

$$L_m(s_0;s) \leq K_1 \sum_{\mu=0}^{m-1} \frac{1}{(\mu+1)^{\sigma-\sigma_0+1}}.$$

Since $(\mu+\nu+1)^{\sigma_0} \leq (\mu+1)^{\sigma_0} + \nu^{\sigma_0}$ when $0 < \sigma_0 \leq 1$, it follows that

$$|U_m^{(1)}(s_0;s)|$$

$$\leq K_2(\mu+1)^{-\sigma+\sigma_0-1} \sum_{\nu=0}^{m-\mu-1} g^{\nu} b_{\nu}(\sigma-\sigma_0-1)$$

$$+K_2(\mu+1)^{-\sigma-1} \sum_{\nu=0}^{m-\mu-1} g^{\nu} b_{\nu}(\sigma-\sigma_0-1)$$

$$\leq K_2 \left(\frac{1}{(\mu+1)^{\sigma-\sigma_0-1}} + \frac{1}{(\mu+1)^{\sigma+1}} \right),$$

and hence,

$$L_m(s_0;s) \leq K_3 \sum_{\mu=0}^{m-1} \frac{1}{(\mu+1)^{\sigma-\sigma_0+1}} + K_4 \sum_{\mu=0}^{m-1} \frac{1}{(\mu+1)^{\sigma+1}} = O(1).$$

It remains the case when $\sigma > \sigma_0 > 1$. Then,

$$L_m(s_0;s)$$

$$\leq K_5 \sum_{\mu=0}^{m-1} (\mu+1)^{-\sigma-1} \sum_{\nu=0}^{m-\mu-1} (\mu+\nu)^{\sigma_0} g^{\nu} b_{\nu}(\sigma-\sigma_0-1) = K_6 \sum_{\nu=0}^{m-1} h_{\nu}(\sigma_0;\sigma),$$

where

$$h_{\nu}(\sigma_0;\sigma) = \sum_{\mu=0}^{m-\mu-1} \frac{(\mu+1)^{\sigma_0}}{(\mu+1)^{\sigma+1}},$$

and since it was already proved that $h_\nu(\sigma_0; s) = O(\nu^{\sigma_0}), \nu \to \infty$, it follows that

$$L_m(s_0; s) \le K_7 \sum_{\nu=0}^{m-1} \nu^{\sigma_0} g^\sigma b_\nu(\sigma - \sigma_0 - 1) = O(1).$$

Theorem 29. *If the series* (4.1) *is E_k-summable for $s = s_0$, then it is $|E_k|$-summable for each s such that $\Re s > \Re s_0$.*

By assumption, the series

$$\sum_{n=0}^{\infty} A_n^k(s_0),\ A_n^k(s_0) = \frac{1}{(q+1)^{n+1}} \sum_{\mu=0}^{n} \binom{n}{\mu} q^{n-\mu} a_\mu (s_0 - 1)(s_0 - 2)\dots(s_0 - \mu)$$

is absolutely convergent.

If $\Re s < 0$, then since

$$|A_n^k(s)| \le \sum_{\mu=0}^{n} |A_\mu^k(s_0)| \binom{n}{\mu} \left(\frac{q}{q+1}\right)^{n-\mu} |\Delta^{n-\mu} \lambda_\mu(s_0; s)|$$

and

$$|\Delta^{n-\mu} \lambda_\mu(s_0; s)| \le |c(s_0; s)| \int_0^1 t^{\delta+n-\mu-1}(1-t)^{\mu-\sigma}\, dt,$$

$$\delta = \Re(s - s_0),\ \sigma = \Re s,$$

it follows that

$$U_m(s_0; s) = |c(s_0; s)| \sum_{n=0}^{m} |A_n^k(s)| \le |c(s_0; s)| \sum_{\mu=0}^{m} |A_\mu^k(s_0)| L_{m,\mu}(s_0; s),$$

where

$$L_{m,\mu}(s_0; s) = \int_0^1 t^{\delta-1}(1-t)^{\mu-\sigma} \sum_{\nu=0}^{n\mu-\mu} \binom{\mu+\nu}{\mu} \left(\frac{qt}{q+1}\right)^\nu dt.$$

But

$$L_{m,\mu}(s_0; s) \le L_\mu(s_0; s) = \int_0^1 t^{\delta-1}(1-t)^{\mu-\sigma} \sum_{\nu=0}^{\infty} \frac{\mu+\nu}{\mu} \left(\frac{qt}{q+1}\right)^\nu dt$$

$$= \int_0^1 t^{\delta-1}(1-t)^{\mu-\sigma} \left(\frac{q+1}{q+1-qt}\right)^{\mu+1} dt,$$

and hence,

$$U_m(s_0;s) \leq |c(s_0;s)| \sum_{\mu=0}^{m} A_\mu^k(s_0)|L_\mu(s_0;s)$$

$$= |c(s_0;s)| \sum_{\mu=0}^{m-1} \mathcal{A}_m^k(s_0)(L_\mu(s_0;s) - L_{\mu+1}(s_0;s))$$

$$+|c(s_0;s)|\mathcal{A}_m^k(s_0)L_m(s_0;s),$$

where

$$\mathcal{A}_\mu^k(s_0) = |A_0^k(s_0)| + |A_1^k(s_0)| + \cdots + |A_\mu^k(s_0)|.$$

But

$$\mathcal{A}_\mu^k(s_0) \leq \mathcal{A}(s_0), \quad \mu = 0, 1, 2, \ldots,$$

$$L_\mu(s_0;s) < \int_0^1 t^{\delta-1}(1-t)^{-\sigma} \left(\frac{(q+1)(1-t)}{q+1-qt}\right)^\mu \frac{q+1}{q+1-qt}\, dt$$

$$\leq \int t^{\delta-1}(1-t)^{-\sigma} \frac{q+1}{q+1-qt}\, dt = \mathcal{L}(s_0;s), \quad \mu = 0, 1, 2 \ldots,$$

$$L_\mu(s_0;s) - L_{\mu+1}(s_0;s)$$

$$= \int_0^1 t^{\delta-1}(1-t)^{\mu-\sigma} \left(\frac{q+1}{q+1-qt}\right)^{\mu+1} \frac{dt}{q+1-qt} > 0, \quad \mu = 0, 1, 2, \ldots,$$

hence,

$$U_m(s_0;s) < |c(s_0;s)|\mathcal{A}(s_0) \sum_{\mu=0}^{m-1} (L_\mu(s_0;s) - L_{\mu+1}(s_0;s))$$

$$+|c(s_0;s)|\mathcal{A}(s_0)L_0(s_0;s) = |c(s_0;s)|\mathcal{A}(s_0)L_0(s_0;s), \quad m = 0, 1, 2 \ldots,$$

which means that the series

$$\sum_{n=0}^{\infty} |A_n^k(s)|$$

is convergent.

As in the case of Dirichlet and of factorial series, a consequence of Theorem 29 is the existence of abscissa $\overline{n}_k$ of $|E_k|$-summability of the series (4.1) Moreover, if

$$A_n^k = \frac{1}{(q+1)^{n+1}} \sum_{\mu=0}^{n} \binom{n}{\mu} q^{n-\mu} a_\mu, \quad n = 0, 1, 2, \ldots,$$

then statements hold, whose formulation as well as their proofs are completely analogous to those for the Dirichlet and the factorial series, namely:

Theorem 30. *If* $\overline{n}_k \geq 0$, *then*

$$\overline{n}_k = \limsup_{n\to\infty} \frac{\log(|A_0^k| + |A_1^k| + \cdots + |A_n^k|)}{\log n}.$$

Theorem 31. *If* $\overline{n}_k < 0$, *then*

$$\overline{n}_k = \limsup_{n\to\infty} \frac{\log(|A_n^k| + |A_{n+1}^k| + \dots)}{\log n}.$$

In the proofs of some theorems it was used the function

$$g(\alpha) = \int_0^\infty \left(e^{-t} - 1 + t - \frac{t^2}{2!} + \cdots + (-1)^{p-1}\frac{t^{p-1}}{(p-1)!}\right) t^{-\alpha-1}\, dt,$$

where $\alpha \geq 0$ and $p = [\alpha] + 1$. But, in fact, it can be expressed by means of the Euler Gamma-function. To that end the Taylor formula with reminder term in an integral form is needed, namely:

$$f(a+h) = f(a) + f'(a)h + f''(a)2!h^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}h^{n-1}$$

$$+\frac{1}{(n-1)!}\int_0^h (h-t)^{n-1} f^{(n)}(a+t)\, dt.$$

In particular,

$$e^{-t} - 1 + t - \frac{t^2}{2!} + \cdots + (-1)^{p-1}\frac{t^{p-1}}{(p-1)!} = \frac{1}{(p-1)!}\int_0^t (t-u)^{p-1}e^{-u}\, du,$$

and hence,

$$g(\alpha) = \frac{1}{(p-1)!}\int_0^\infty t^{-\alpha-1}\, dt \int_0^t (t-u)^{p-1}e^{-u}\, du$$

$$= \frac{1}{(p-1)!}\int_0^\infty e^{-u}\, du \int_u^\infty t^{-\alpha-1}(t-u)^{p-1}\, dt$$

$$= \frac{1}{(p-1)!}\int_0^\infty e^{-u}u^{p-\alpha-1}\, du \int_0^1 v^{\alpha-p}(1-v)^{p-1}\, dv$$

$$= \frac{1}{(p-1)!}\frac{\Gamma(p-\alpha)\Gamma(\alpha-p+1)\Gamma(p)}{\Gamma(\alpha+1)} = \frac{\Gamma(p-\alpha)\Gamma(1-p+\alpha)}{\Gamma(\alpha+1)}.$$

5. Summation of factorial series by Borel's method

It is at hand now to apply the Borel integral method for summation to factorial series of the kind

$$\sum_{n=0}^{\infty} \frac{n!a_n}{s(s+1)(s+2)\dots(s+n)}, \quad s \neq 0, -1, -2, \dots . \tag{5.1}$$

Let first $\Re s > 0$, then for the function

$$\Phi_s(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{s(s+1)(s+2)\dots(s+n)} \tag{5.2}$$

one gets that

$$\Phi_s(x) = \sum_{n=0}^{\infty} \frac{a_n \Gamma(s)\Gamma(n+1)}{\Gamma(s+n+1)} \frac{x^n}{n!} \tag{5.3}$$

$$= \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \int_0^1 t^n (1-t)^{s-1}\, dt = \int_0^1 (1-t)^{s-1} \varphi_0(xt)\, dt,$$

where

$$\varphi_0(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.$$

For the series (5.1) is said that it is B-summable, if $\Phi_s(x)$ is an entire function and, moreover, the integral

$$\int_0^{\infty} e^{-x} |\Phi^s(x)|\, dx$$

is convergent. In such a case φ_0 is also an entire function. Indeed, from (5.2) it follows that for each $R > 0$ there exists a positive integer $N = N(R)$ such that

$$|a_n| \leq K \left| \frac{\Gamma(s+n+1)}{\Gamma(s)} \right| R^{-n}, \quad n > N,$$

and hence,

$$\frac{|a_n x^n|}{n!} \leq k \left| \Gamma(s+n+1) \right| \left(\frac{|x|}{R} \right)^n \leq K_1 n^{\sigma} \rho^n,$$

provided that $n > N$, $|x| \leq \rho R$, $0 < \rho < 1$, $\sigma = \Re s$.

Further, the integral representation (5.3) yields that

$$\Phi_s(x) = \int_0^x \left(1 - \frac{t}{x}\right)^{s-1} \varphi_0(t) \frac{dt}{x} = x^{-s} G_s(x), \tag{5.4}$$

$$G_s(x) = \int_0^x (x-t)^{s-1} \varphi_0(t)\, dt.$$

Let the series (5.1) be B-summable for some s with $\Re s > 0$. If

$$h_s(x) = \int_0^x e^{-t} \Phi_s(t)\, dt,$$

then $\lim_{x\to\infty} h_s(x)$ exists. If $\Re\delta > 0$, then (5.4) and the equality

$$G_{s+\delta}(x) = \int_0^x (x-t)^{s+\delta-1} \varphi_0(t)\, dt$$

yield that

$$G_{s+\delta}(x) = c(s,\delta) \int_0^x (x-t)^{\delta-1} G_s(t)\, dt,\ \ c(s,\delta) = \frac{\Gamma(s+\delta)}{\Gamma(s)\Gamma(\delta)}.$$

Further, $\Phi_s(x) = e^x h_s'(x), G_s(x) = x^s \Phi_s(x) = x^s e^x h_s'(x)$ and

$$G_{s+\delta} = c(s,\delta) \int_0^x (x-t)^{\delta-1} t^s e^t h_s'(t)\, dt$$

$$= c(s,\delta) \int_0^1 (1-t)^{\delta-1} t^s e^{xt} h_s'(xt)\, dt.$$

We need now to study the integral

$$F(s,\delta;x) = \int_0^x e^{-t} \Phi_{s+\delta}(t)\, dt$$

$$= c(s,\delta) \int_0^x e^{-t}\, dt \int_0^1 (1-u)^{\delta-1} u^s e^{tu} h_s'(tu)\, du$$

$$= c(s,\delta) \int_0^1 (1-u)^{\delta-1} u^s\, du \int_0^x e^{-t(1-u)} h_s'(tu)\, dt.$$

But

$$\int_0^x e^{-t(1-u)} h_s'(tu)\, dt$$

$$= \frac{1}{u} e^{-t(1-u)} h_s(tu) \Big|_0^x + \frac{1-u}{u} \int_0^x h_s(tu) e^{-t(1-u)}\, dt$$

$$\frac{1}{u} e^{-x(1-u)} h_s(xu) + \frac{1-u}{u} \int_0^x h_s(tu) e^{-t(1-u)}\, dt,$$

and hence,

$$F(s,\delta;x) = c(s,\delta) \int_0^1 u^{s-1}(1-u)^{\delta-1} e^{-x(1-u)} h_s(xu)\, du$$

$$+ c(s,\delta) \int_0^1 u^{s-1}(1-u)^{\delta}\, du \int_0^x h_s(tu) e^{-t(1-u)}\, dt$$

$$= U(s,\delta;x) + V(s,\delta;x).$$

The function h_s is bounded, i.e., $|h_s(x) \le M < \infty$, $x \in (0,\infty)$. If $\varepsilon > 0$, then there is $\eta = \eta(\varepsilon) \in (0,1)$, such that

$$M|c(s,\delta)| \int_\eta^1 u^{\sigma-1}(1-u)^{d-1}\, du < \varepsilon,$$

where $\sigma = \Re s, d = \Re\delta$.

Further,

$$U(s,\delta;x) = c(s,\delta) \int_0^\eta u^{s-1}(1-u)^{\delta-1} e^{-x(1-u)} h_s(xu)\, du$$

$$+ c(s,\delta) \int_\eta^1 u^{s-1}(1-u)^{\delta-1} e^{-x(1-u)} h_s(xu)\, du.$$

But

$$|U_2(s,\delta;x)| \le M|c(s,\delta)| \int_\eta^1 u^{\sigma-1}(1-u)^{d-1}\, du < \varepsilon,$$

and from the inequality

$$|U_1(s,\delta;x) \le M|c(s,\delta)| e^{-x(1-\eta)} \int_0^1 u^{\sigma-1}(1-u)^{d-1}\, du,$$

it follows that $\lim_{x\to\infty} U_1(s,\delta;x) = 0$, i.e., $\lim_{x\to\infty} U(s,\delta;x) = 0$.

The integral

$$\int_0^1 u^{s-1}(1-u)^{\delta}\, du \int_0^\infty h_s(tu) e(-t(1-u))\, dt$$

is absolutely convergent since it is majorized by the convergent integral

$$\int_0^1 u^{\sigma-1}(1-u)^d\,du \int_0^\infty e^{-t(1-u)}\,dt = \int_0^1 u^{\sigma-1}(1-u)^{d-1}\,du.$$

Hence, there exists

$$\lim_{x\to\infty} F(s,\delta;x) = c(s,\delta)\int_0^1 u^{s-1}(1-u)^{\delta-1}\,du \int_0^\infty h_s(tu)e^{-t(1-u)}\,dt,$$

and thus it is established that the series (5.1) is B-summable for each $s+\delta$ such that $\Re\delta > 0$.

Let now $\Re s \leq 0, s \neq 0, -1, -2, \dots,$ m be a positive integer such that $\Re(s+m) > 0$, $\psi_{m,s}(x)$ be the function defined by the series

$$\psi_{m,s}(x) = \sum_{n=m}^\infty \frac{a_n x^n}{s(s+1)\dots(s+n)} = \sum_{n=m}^\infty \frac{a_n\Gamma(s)\Gamma(n+1)}{\Gamma(s+n+1)}\frac{x^n}{n!}$$

and let

$$\Phi_{m,s}(x) = \sum_{n=0} m-1 \frac{a_n x^n}{s(s+1)\dots(s+n)} + \psi_{m,s}(x),$$

then

$$\int_0^\infty e^{-x}\Phi_{m,s}(x)\,dx = \sum_{n=0}^{m-1}\frac{n!a_n}{s(s+1)\dots(s+n)} + \int_0^\infty e^{-x}\psi_{m,s}(x)\,dx,$$

which means that the B-summability of the series (5.1) is equivalent to the convergence of the integral

$$\int_0^\infty e^{-x}\psi_{m,s}(x)\,dx.$$

Further,

$$\psi_{m,s}(x) = \frac{1}{s(s+1)\dots(s+m-1)}\sum_{n=m}^\infty \frac{a_n\Gamma(s+m)\Gamma(n-m+1)}{\Gamma(s+n+1)}\frac{x^n}{(n-m)!}$$

$$= \frac{1}{s(s+1)\dots(s+m-1)}\sum_{n=m}^\infty \frac{a_n x^n}{(n-m)!}\int_0^1 t^{n-m}(1-t)^{s+m-1}\,dt$$

$$= \frac{x^m}{s(s+1)\dots(s+m-1)}\int_0^1 (1-t)^{s+m-1}\varphi_m(xt)\,dt,$$

where

$$\varphi_m(x) = \sum_{n=0}^{\infty} \frac{a_{n+m}x^n}{n!},$$

i.e.,

$$\psi_{m,s}(x) = \frac{x^{-s}}{z(s+1)\dots(s+m-1)} \int_0^x (x-u)^{s+m-1}\varphi_m(u)\,du$$
$$= \frac{x^{-s}}{s(s+1)\dots(s+m-1)} G_{m,s}(x),$$

where

$$G_{m,s}(x) = \int_0^x (x-u)^{s+m-1}\varphi_m(u)\,du,\ G_{0,s}(u) = G_s(u).$$

Then, substituting $s+\delta, \Re\delta > 0$ for s, the above equality becomes

$$G_{m,s+\delta}(x) = c(s,\delta) \int_0^x (x-u)^{\delta-1} G_{m,s}(u)\,du.$$

If the series (5.1) is B-summable, then

$$\lim_{x\to\infty} H_{m,,s}(x) = \lim_{x\to\infty} \int_0^x e^{-t}\psi_{m,s}(t)\,dt$$

exists and by following a way completely analogous to that just used, one gets that there exists

$$\lim_{x\to\infty} \int_0^x e^{-t}\psi_{m,s+\delta}(t)\,dt$$
$$= c(s,\delta) \int_0^1 u^{s-1}(1-u)^{\delta}\,du \int_0^{\infty} H_{m,s+\delta}e^{-t(1-u)}\,dt.$$

So, it is proved the following theorem:

Theorem 32. *Let the series* (5.1) *be B-summable for $s = s_0$. Then it is B-summable for each s such that $\Re s > \Re s_0$ and its B-sum $f(s)$ is a meromorphic function in the half-plane $\{s : \Re s > \Re s_0\}$ with possible simple poles at the points* $0, -1, -2, \dots$. *Moreover, if $\Re s_0 > 0$, then*

$$f(s) = \frac{\Gamma(s)}{\Gamma(s_0)\Gamma(s-s_0)} \int_0^1 u^{s_0-1}(1-u)^{s-s_0}\,du \int_0^{\infty} h_{s_0}(tu)e^{-t(1-u)}\,dt,$$

and if $\Re s_0 < 0$, *then*

$$f(s) = \sum_{n=0}^{m-1} \frac{n!a_n}{s(s+1)\dots(s+n)}$$

$$+\frac{\Gamma(s)}{\Gamma(s_0)\Gamma(s-s_0)} \int_0^1 u^{s_0-1}(1-u)^{s-s_0}\,du \int_0^\infty H_{m,s_0}(ut)e^{-t(1-u)}\,dt,$$

where m is a positive integer such that $\Re(s_0+m) > 0$.

The series (5.1) is absolutely B-summable, shortly $|B|$-summable, if the integral

$$\int_0^\infty e^{-x}|\Phi_s(x)|\,dx$$

is convergent. Let this series be convergent for s with $\Re s > 0$ and let

$$\omega(x) = \int_0^\omega e^{-t}|\Phi_s(xt)|\,dt, \quad x \in (0,\infty).$$

Then, since

$$|\Phi_{s+\delta}(x)| \le |c(s,\delta)|x^{-\sigma-d} \int_0^x (x-u)^{d-1}u^\delta|\Phi_s(u)|\,du,$$

where $d = \Re\delta > 0$ and $\sigma = \Re s$, it follows that

$$|\Phi_{s+\delta}(x)| \le |c(s,\delta)|x^{-\sigma-d} \int_0^1 u^\sigma(1-u)^{d-1}\,du \int_0^x e^{-t(1-u)}\omega'(tu)\,dt.$$

Further, an integration by parts yields that

$$\int_0^x e^{-t}|\Phi_{s+\delta}(t)|\,dt \le |c(s,\delta)| \int_0^1 u^{\sigma-1}(1-u)^{d-1}e^{-x(1-u)}\omega'(xu)\,du$$

$$+|c(s,\delta)| \int_0^1 u^{\sigma-1}(1-u)^d\,du \int_0^x \omega(tu)e^{-t(1-u)}\,dt$$

$$\le M|c(s,\delta)| \left\{ \int_0^1 u^{\sigma-1}(1-u)^{d-1}\,du + \int_0^1 u^{\sigma-1}(1-u)^d\,du \int_0^x e^{-t(1-u)}\,dt \right\}$$

whence

$$\int_0^x e^{-t}|\Phi_{s+\delta}(t)|\,dt$$

$$\le M|c(s,\delta) \left\{ \frac{\Gamma(\sigma)\Gamma(d)}{\Gamma(\sigma+d)} + \int_0^1 u^{\sigma-1}(1-u)^d\,du \int_0^\infty e^{-t(1-u)}\,dt \right\},$$

i.e.

$$\int_0^x e^{-t}|\Phi_{s+\delta}(t)|\,dt \le 2M|c(s,\delta)|\frac{\Gamma(\sigma)\Gamma(d)}{\Gamma(\sigma+d)}, \quad x\in(0,\infty),$$

and thus the following theorem is established:

Theorem 33. *If the series* (5.1) *is* $|B|$*-summable for* $s=s_0$*, then it is* $|B|$*-summable for each* s *such that* $\Re s > \Re s_0$*.*

REFERENCES

[1] K. Knopp. Über das Eulersche Sumierungsverfahren. *Math. Z.* **15** (1922), 125–156.

[2] E. Borel. Leçons sur les séries divergentes, II Ed., Paris, 1928.

[3] J. M. Whittaker. The absolutely summability of a Fourier series. *Proc. Edingburgh Math. Soc.*, **2** (1930/31), 1–5.

[4] M. Fékéte. On the absolute summability of an infinite series. *Proc. Edinburgh Math. Soc.* **3** (1933), 132–134.