Addendum

REGIONS OF CONVERGENCE OF DIRICHLET'S, NEWTON'S AND FACTORIAL SERIES

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The region of convergence of each of the series studied by Obrechkoff is of the kind $\{s \in \mathbb{C} : \Re s > \tau,\ -\infty \le \tau < \infty\}$, i.e., it is either the whole complex plane or a "right" half-plane with respect to a line orthogonal to the real axes. This is, in fact, a consequence of the validity of Abel's type assertions for each of the series in question. The proofs of the last ones are based on a simple "key" lemma which is nothing but a version of a criterion for convergence of series due also to Abel, namely:

Lemma*. *If the series*

$$\sum_{n=0}^{\infty} u_n, \quad u_n \in \mathbb{C}, \quad n = 0, 1, 2, \dots,$$

is convergent and $\{\lambda_n\}_{n=0}^{\infty}$ *is a bounded sequence of complex numbers such that*

$$\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty,$$

then so does the series

$$\sum_{n=0}^{\infty} \lambda_n u_n.$$

P r o o f. Let

$$s_\nu = \sum_{k=0}^{\nu} u_k, \quad \nu = 0, 1, 2, \dots,$$

and suppose first that:

I. $s = \sum_{\nu=0}^{\infty} u_\nu = 0$. Then, the assertion is an immediate consequence of the equality

$$\sum_{\nu=n+1}^{n+p} \lambda_\nu u_\nu = \sum_{\nu=n+1}^{n+p} \lambda_\nu (s_\nu - s_{\nu-1})$$

$$= \lambda_{n+p} s_{n+p} - \lambda_{n+1} s_n + \sum_{\nu=n+1}^{n+p-1} (\lambda_\nu - \lambda_{\nu+1}) s_\nu.$$

II. If $s \neq 0$, then the series

$$\sum_{n=0}^{\infty} \widetilde{u}_n, \quad \widetilde{u}_n = u_n - \frac{s}{2^{n+1}}, \quad n = 0, 1, 2, \ldots,$$

is convergent with sum $\widetilde{s} = 0$. Hence, the series

$$\sum_{n=0}^{\infty} \lambda_n \widetilde{u} = \sum_{n=0}^{\infty} \left(\lambda_n u_n - \frac{\lambda_n s}{2^{n+1}} \right)$$

is convergent and so does the series

$$\sum_{n=0}^{\infty} \lambda_n u_n = \sum_{n=0}^{\infty} \lambda_n \widetilde{u}_n + \sum_{n=0}^{\infty} \frac{\lambda_n s}{2^{n+1}}.$$

Theorem 1*. *If the series*

$$\sum_{n=0}^{\infty} \frac{a_n}{(n+1)^s}, \quad a_n \in \mathbb{C}, \quad n = 0, 1, 2, \ldots,$$

is convergent for $s = s_0 \in \mathbb{C}$, then it is convergent for each $s \in \mathbb{C}$ such that $\Re s > \Re s_0$.

P r o o f. Let now

$$\xi_n(s, s_0) = \frac{1}{(n+1)^{s - s_0}}, \quad n = 0, 1, 2, \ldots,$$

then

$$\frac{a_n}{(n+1)^s} = \xi_n(s, s_0) \frac{a_n}{(n+1)^{s_0}}, \quad n = 0, 1, 2, \ldots.$$

Further,

$$\xi_n(s, s_0) - \xi_{n+1}(s, s_0) = \frac{1}{(n+1)^{s - s_0}} \left\{ 1 - \left(1 - \frac{1}{n+2} \right)^{s - s_0} \right\}$$

$$= \frac{1}{(n+1)^{s - s_0}} \left\{ \frac{s - s_o}{n+2} + O\left(\frac{1}{n^2} \right) \right\}, \quad n \to \infty,$$

hence

$$|\xi_n(s, s_0) - \xi_{n+1}(s, s_0)| = O\left(\frac{1}{n^{\sigma - \sigma_0 + 1}} \right), \quad n \to \infty,$$

where $\sigma = \Re s, \sigma_0 = \Re s_0$.

After replacing s by $-s$ in the series

$$\sum_{n=0}^{\infty} a_n s(s-1)(s-2)\ldots(s-n),$$

it becomes

$$\sum_{n=0}^{\infty}(-1)^{n+1}a_n s(s+1)(s+2)\dots(s+n)$$

and hence, the validity of a statement like **Theorem 1*** for the Newton series is equivalent to the following one:

Theorem 2*. *If the series*

$$\sum_{n=0}^{\infty} a_n s(s+1)(s+2)\dots(s+n)$$

is convergent for $s = s_0 \in G = \mathbb{C} \setminus \{0, -1, -2, \dots\}$*, then it is convergent for each* $s \in G$ *such that* $\Re s < \Re s_0$.

Proof. If

$$A_n(s) = a_n \prod_{\nu=0}^{n}(s+\nu), \quad n = 0, 1, 2, \dots,$$

then

$$A_n(s) = A_n(s_0)\eta_n(s, s_0), \quad n = 0, 1, 2, \dots,$$

where

$$\eta_n(s, s_0) = \prod_{\nu=0}^{n} \frac{s+\nu}{s_0+\nu}, \quad n = 0, 1, 2, \dots.$$

Further,

$$\eta_n(s, s_0) = \frac{\Gamma_n(s, s_0)}{(n+1)^{s_0-s}},$$

where

$$\Gamma_n(s, s_0) = \frac{s}{s_0}\frac{\exp(-\gamma_n s)}{\exp(-\gamma_n s_0)} \prod_{\nu=1}^{n} \frac{\left(1+\dfrac{s}{\nu}\right)\exp\left(-\dfrac{s}{\nu}\right)}{\left(1+\dfrac{s_0}{\nu}\right)\exp\left(-\dfrac{s_0}{\nu}\right)},$$

$$\gamma_n = \log(n+1) - \sum_{\nu=1}^{n}\frac{1}{\nu},$$

and, moreover,

$$\lim_{n\to\infty}\Gamma_n(s, s_0) = \frac{\Gamma(s_0)}{\Gamma(s)}.$$

Then,

$$\eta_n(s, s_0) - \eta_{n+1}(s, s_0)$$

$$= \frac{\Gamma_n(s,s_0)}{(n+1)^{s-s_0}} \left\{ 1 - \left(\frac{n+2}{n+1}\right)^{s-s_0} E_n(s,s_0) \right\},$$

where

$$E_n(s,s_0) = \exp\left\{-(\gamma_{n+1}-\gamma_n)(s-s_0)\right\} \frac{\left(1+\frac{s}{n+1}\right)\exp\left(-\frac{s}{n+1}\right)}{\left(1+\frac{s_0}{n+1}\right)\exp\left(-\frac{s_0}{n+1}\right)}.$$

Further,

$$\left(\frac{n+2}{n+1}\right)^{s-s_0} = \left(1+\frac{1}{n+1}\right)^{s-s_0} = 1 + O\left(\frac{1}{n}\right), \quad n \to \infty,$$

$$\exp\left\{-(\gamma_{n+1}-\gamma_n)(s-s_0)\right\}$$
$$= \exp\left\{-\left[\log\left(1+\frac{1}{n}\right) + \frac{1}{n+1}\right](s-s_0)\right\} = 1 + \left(\frac{1}{n}\right), \quad n \to \infty$$

and

$$\frac{\left(1+\frac{s}{s_0}\right)\exp\left(-\frac{s}{n+1}\right)}{\left(1+\frac{s_0}{n+1}\right)\exp\left(-\frac{s_0}{n+1}\right)} = 1 + \left(\frac{1}{n}\right), \quad n \to \infty.$$

Hence,

$$E_n(s,s_0) = 1 + \left(\frac{1}{n}\right), \quad n \to \infty,$$

i.e.,

$$\eta_n(s,s_0) - \eta_{n+1}(s,s_0) = O\left(\frac{1}{n^{\sigma_0-\sigma+1}}\right), \quad n \to \infty,$$

where $\sigma_0 = \Re s_0$, $\sigma = \Re s$ and the validity of **Theorem 2*** is a consequence of the "key" **Lemma***.

After the translation $s \mapsto s-1$ the factorial series

$$\sum_{n=1}^{\infty} \frac{a_n}{(s+1)(s+2)(s+3)\ldots(s+n)}$$

becomes

$$\sum_{n=0}^{\infty} \frac{a_{n+1}}{s(s+1)(s+2)\ldots(s+n)},$$

i.e., it has to be proved the following theorem.

Theorem 3*. *If the series*

$$\sum_{n=0}^{\infty} \frac{b_n}{s(s+1)(s+2)\dots(s+n)}$$

is convergent for $s = s_0 \in G$, *then the same is true for each* $s \in G$ *such that* $\Re s > \Re s_0$.

P r o o f. But if

$$B_n(s) = \frac{b_n}{s(s+1)(s+2)\dots(s+n)},$$

then

$$B_n(s) = B_n(s_0)\zeta(s, s_0),$$

where

$$\zeta(s, s_0) = \prod_{\nu=0}^{n} \frac{s_0 + \nu}{s + \nu} = \eta(s_0, s)$$

and the assertion is again a consequence of the "key" **Lemma***.

Let the set D of $x \in \mathbb{R}$, such that the Dirichlet series is convergent for $s = x$, be non-empty and let $\tau \in [-\infty, \infty)$ be defined as $\inf D$. If $\tau \in \mathbb{R}$, then it is easy to prove that this series is convergent in the half-plane $H_\tau = \{s \in \mathbb{C} : \Re s > \tau\}$ and diverges in the half-plane $\mathbb{C} \setminus \overline{H}_\tau$. Indeed, if $s_0 \in H_\tau$, then there exists $x_0 \in (\tau, \Re s_0)$ such that the Dirichlet series is convergent for $s = x_0$ and hence by **Theorem 1***, it is convergent for $s = s_0$. On the contrary, the assumption that there is a point s_0 outside $\overline{H}_\tau$ such that this series is convergent at this point contradicts to the definition of τ.

Further, it holds the equality

$$\tau = \limsup_{n\to\infty} \frac{\log |a_0 + a_1 + \cdots + a_n|}{\log(n+1)}$$

which may be called formula of Cauchy-Hadamard for the Dirichlet series. Moreover, it holds also for the series of Newton as well as for the factorial series. This formula can be established as Obrechkoff has done this for the regions of Euler-Knopp summation of the series already mentioned. It may be "guessed" by taking into consideration that the usual convergence of a series is its Euler-Knopp summation when the parameter $k = 0$.