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MAXIMUM PRINCIPLE FOR WEAKLY COUPLED LINEAR NON-COOPERATIVE SYSTEMS

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In this article is considered the validity of the maximum principle for non-cooperative linear parabolic systems. Unlike the cooperative ones, when cooperativeness is a sufficient condition for maximum principle, simple examples show that for non-cooperative systems there is no general validity of the maximum principle. In this work are given some conditions for global on time variable validity of the maximum principle, as well as some local on time sufficient conditions for it.

1. Introduction

In the theory of the differential equations the maximum principle takes its place as convenient tool in studying uniqueness, stability and solvability of elliptic and parabolic equations and systems. Since 1927, when E.Hopf introduce the idea of a maximum principle in [12], many authors studied maximum and comparison principles. The results cover elliptic and parabolic equations, cooperative systems and some cases of non-cooperative elliptic systems. Nevertheless, the validity of maximum principle for non-cooperative parabolic systems is still open problem due to the lack of a proper tools to study maximum principle in this case. For instance, in the case of cooperative elliptic systems one may use the theory of positive operators on positive cone and obtain spectral properties of the cooperative operator (see [18]) since the inverse operator of the cooperative system

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is positive in a weak sense. Furthermore, a remarkable study on the spectral properties of a cooperative fully-coupled elliptic system is published in [6], where existence of a principal eigenvalue with strongly positive principal eigenfunction is proved for a bounded, open and connected domain, with no requirement of the regularity of the boundary. This result is fundamental for the proof of the comparison principle for weak solutions of non-cooperative systems in [3]. Unfortunately, considering non-cooperative parabolic systems one can not rely on such useful spectral properties.

In this paper is studied the validity of the maximum principle for non-cooperative linear systems of parabolic PDE in a bounded domain.

Let $\Omega \subset R^n$ be bounded domain with at least C^2 smooth boundary $\partial\Omega$. We denote $Q = \Omega \times (0,T)$ and the parabolic boundary of Q by $\Gamma = (\partial\Omega \times [0,T]) \cup \{\Omega \times \{t=0\}\}$. We use the standard order in R^N, i.e. $u < v$, $u, v \in R^N$ if $u^k < v^k$ for every $k = 1, \dots, .N$.

In this article are considered weakly-coupled linear systems of uniformly parabolic equations in Q

$$Pu = f \tag{1}$$

or component-wise

$$u_t^k - \sum_{i,j=1}^{n} D_j \left(a_k^{ij}(x,t) D_i u^k \right) + \sum_{i=1}^{n} b_k^i(x,t) D_i u^k + c_k(x,t) u^k + \sum_{k \neq l=1}^{N} m_{lk}(x,t) u^l = f^k(x,t)$$

$k = 1, \dots, N$, with boundary conditions on Γ

$$u^k(x,t) = g^k(x,t) \tag{2}$$

Note that for the sake of simplicity in notations we suppose $m_{kk} = 0$ for all $k = 1, \dots, N$.

System (1) is uniformly parabolic one, i.e. there are constants $\lambda, \Lambda > 0$ such that

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_k^{ij}(x,t) \xi_i \xi_j \leq \Lambda |\xi|^2 \tag{3}$$

for every k and any $\xi = (\xi_1, \dots, \xi_n) \in R^n$.

The right-hand side of (1) is supposed bounded function, i.e. $|f^l(x,t)| \le C$ in $\overline{Q}$ for every $l = 1, \dots, N$, where $C > 0$ is a constant. Coefficients c_k and m_{lk} in (1) are supposed continuous in $\overline{Q}$, $a_k^{ij}(x,t) \in C^{1+\alpha}(Q) \cap C(\overline{Q})$ and $b_k^i(x,t) \in C^1(Q) \cap C(\overline{Q})$. We presume in addition that for every $k = 1, \dots, N$

$$\left\{ \sum_{i=1}^{n} \left(\sum_{j=1}^{n} D_j a_k^{ij} + b_k^i(x) \right)^2, |c_k| \right\} \le b \tag{4}$$

holds, where $b > 0$ is a constant.

For sake of completeness we recall some well-known definitions. In the first one the terminology follows [11]

Definition 1. *System* (1) *is cooperative one if* $m_{lk} \le 0$ *for* $l \ne k$.

The second one defines is the classical comparison principle:

Definition 2. *Let* $\underline{u}$ *and* $\overline{u}$ *are sub- and super-solutions of* (1), *i.e.* $P\underline{u} \le f$, $P\overline{u} \ge f$. *Comparison principle holds for system (1) if* $\underline{u} \le \overline{u}$ *on* Γ *yields* $\underline{u} \le \overline{u}$ *in* $\overline{Q}$.

Note that in the case of linear systems comparison principle and maximum principle are equivalent terms. Indeed, let $\underline{u}$ and $\overline{u}$ be sub- and super-solutions of (1) and denote $v = \overline{u} - \underline{u}$. Suppose maximum principle hold for system (1). Then $v = \overline{u} - \underline{u} \ge 0$ on Γ and by maximum principle $v = \overline{u} - \underline{u} \ge 0$ in $\overline{Q}$, i.e. comparison principle holds for P as well. If comparison principle holds for P, one can consider $\underline{u} \equiv 0$ and therefore maximum principle holds as well.

Definition 3. *Solution of* (1), (2) *(sub- or super-solution) in a weak sense is a vector-function* $u(x,t) \in \left(C^2(Q) \cap C(\overline{Q})\right)^N$, *such that for every non-negative test function* $\eta \in \left(W_c^{1,1}(Q) \cap C(\overline{Q})\right)^N$ *equality (inequalities)*

$$\int_\Omega u^k(x,t).\eta^k(x,t)dx + \int_{Q_t} \left(-u^k\eta_t^k + \sum_{i,j=1}^{n} a_k^{ij}(x,t) D_i u^k \eta_{x_j}^k \right) dxdt +$$

$$+ \int_{Q_t} \left(\sum_{i=1}^{n} b_k^i(x,t) D_i u^k \eta^k + c_k(x,t) u^k \eta^k \right) dxdt +$$

$$+\int_{Q_t}\left(\sum_{l=1}^{N} m_{lk}(x,t)u^l\eta^k - f^k\eta^k\right)dxdt = 0\,(\leq 0, \geq 0)$$

holds for every $k = 1,\ldots,N$, *and every* $0 \leq t \leq T, Q_t = \Omega \times (0,t)$.

Most results for positiveness of the classical solutions, or validity of the comparison principle on classical sense, are obtained for cooperative systems (see [1, 2, 7, 8, 9, M, 17, 19] for optimal control problems, [10] for diffraction-diffusion systems arising in medicine, [13] for stabilized non-linear system type "heat transfer", in [4] for general quasi-linear cooperative reaction-diffusion systems and many others). One can summarize all this results as "cooperativeness is sufficient condition for the validity of the comparison principle for parabolic systems" (see for instance [4]). On the other hand, the following simple example shows that comparison principle is not a feature of every parabolic system.

Example 1. Let $Q = (0,\pi) \times (0,T)$. Consider the problem

$$\left|\begin{array}{l} u^1_t - u^1_{xx} - u^1 + u^2 = 0 \\ u^2_t - u^2_{xx} = 0 \end{array}\right. \quad \text{in } Q$$

with initial and boundary conditions $u^1(0,t) = u^2(0,t) = u^1(\pi,t) = u^2(\pi,t) = 0$ for $t \in [0,T]$, $u^1(x,0) = u^2(x,0) = 0$ for $x \in [0,\pi]$.

This system is non-cooperative, since $m_{12}(x,t) = 1 > 0$. One solution of this system is the trivial one $v^1 = v^2 = 0$, which is a sub-solution as well. One super-solution is $w^1 = -t.\sin x$, $w^2 = \sin x$. Since $-t.\sin x = w^1 < v^1 = 0$ in Q there is no the comparison principle for the above system.

So the very reasonable question arise: is there comparison principle for some non-cooperative system of parabolic equations? In fact,one of the main results in this paper – Theorem 1 shows the strong correlation between the global on time comparison principle for linear non-cooperative parabolic systems and the existence of positive solution of the L^2-adjoint operator of the system. Furthermore, comparison principle holds as well if the first eigenvalue and the corresponding first eigenfunction are positive. Note that unlike cooperative elliptic systems, which first eigenfunction is positive, in the parabolic case this is not always true. For instance, Theorem 3 gives some conditions such that comparison principle does not hold for system (1), and therefore its first eigenfunction is not positive one.

In the last chapter of this article is applied another approach to the problem of validity of the comparison principle. In Theorem 2 are given some conditions on the coefficients of the system such that in a small neighbourhood of some point t_0 comparison principle holds. The result is useful to investigate the maximal interval (o, t_m) in which the comparison principle holds for system (1). The result is based on the validity of the comparison principle for the elliptic system $Pu(t_0, x) = f$.

Some of the results in this paper are published in [5].

2. Comparison principle for non-cooperative systems and the first eigenvalue

The strong connection between the validity of the comparison principle and the first eigenvalue of the operator is well-known feature of elliptic equations and systems. The main result obtained in this section is the validity of global on t comparison principle for parabolic systems.

Theorem 1. *Let P^* be L^2-adjoint operator of P. Comparison principle holds for system* (1), (2) *if*

(*i*) *there is a positive solution of $P^*v = F(x,t)$ for some $F(x,t) > 0$*

or

(*ii*) *if the first eigenvalue λ of P^* is positive one and the corresponding eigenfunction u^* is positive one as well.*

Note that for parabolic systems the first eigenfunction may not be positive one, unlike the case of elliptic operators.

Proof. Let $\underline{u}$ and $\overline{u}$ be sub- and super-solutions of (1). Denote $w = \overline{u} - \underline{u}$.

1. Let $F(x,t) > 0$ and there is a positive solution of $P^*v = F(x,t)$. If we suppose that there is no comparison principle for P, then $w_- = \min(w, 0) \neq 0$. Let $Q^- = \text{supp}\{w_- \leq 0\}$. Then $0 \leq (Pw^-, v) = (w^-, P^*v) = (w^-, F) \leq 0$. Therefore $w^- \equiv 0$ and $w = \overline{u} - \underline{u} > 0$.

2. Let the first eigenvalue λ of P^* is positive one and the corresponding eigenfunction u^* is positive one as well. Suppose $w_- \leq 0$. Then u^* is suitable test-function and $0 \leq (Pw^-, u^*) = (w^-, P^*u^*) = (w^-, \lambda u^*) \leq 0$. Therefore $w^- \equiv 0$ and $w = \overline{u} - \underline{u} > 0$. □

3. Another approach the comparison principle for non-cooperative systems

In this section we use the conditions for validity of the comparison principle for non-cooperative elliptic systems, obtained in [3]. The idea of transferring that results to the case of parabolic equations is very simple – if we fix t variable we reduce the parabolic system to the elliptic one and we can apply the conditions for the validity of comparison principle for elliptic systems.

Let denote by $L_M u$ the operator

(5)
$$L_M u = -\sum_{i,j=1}^{n} D_j\left(a_k^{ij}(x,t)D_i u^k\right) + \sum_{i=1}^{n} b_k^i(x,t)D_i u^k + c_k(x,t)u^k + \sum_{k\neq l=1}^{N} m_{lk}(x,t)u^l,$$

where $k = 1,\ldots,N$. Let denote by $L_{M^-}u$ the cooperative part of (5), i.e. the operator

$$-\sum_{i,j=1}^{n} D_j\left(a_k^{ij}(x,t)D_i u^k\right) + \sum_{i=1}^{n} b_k^i(x,t)D_i u^k + c_k(x,t)u^k + \sum_{l=1}^{N} m_{lk}^-(x,t)u^l,$$

where $k = 1,\ldots,N$ and $m_{lk}^-(x,t) = \min\{m_{lk}(x,t),0\}$. Let $L_{M^-}^*$ be the L^2-adjoint operator of L_{M^-}.

Theorem 2. *Let* (1) *be a weakly coupled, uniformly parabolic system and the cooperative part of* $L_{M^-}^*$ *is fully coupled. Then the comparison principle holds for of system* (1) *if for every* $t \in [0,T]$ *there is* $x_0(t) \in \Omega$ *such that*

$$\left(\lambda(t) + \sum_{k=1}^{N} m_{kj}^+(x_0,t)\right) > 0 \text{ for } j = 1,\ldots,N \tag{6}$$

where $\lambda(t)$ *is the principal eigenvalue of the operator* $L_{M^-}(t)$ *in* Ω.

Theorem 2 is formulated for fully-coupled cooperative part of the system (1) and the result is based on Theorem (3) in [3]. If the system is competitive one, we can employ Theorem (4) in the same paper. One can formulate a weaker statement instead of Theorem (1) – substitute the principal eigenvalue of the operator L_{M^-} in Ω λ one with λ_k – the principal eigenvalue of the operator

$$-\sum_{i,j=1}^{n} D_j\left(a_k^{ij}(x,t)D_i u^k\right) + \sum_{i=1}^{n} b_k^i(x,t)D_i u^k + c_k(x,t)u^k.$$

P r o o f. Let $\underline{u}$ and $\overline{u}$ be sub- and super-solutions of (1). Then $w = \overline{u} - \underline{u}$ is a super-solition of (1) and $Pw \geq 0$. In other words in Q we have

$$L_M w \geq -w_t \tag{7}$$

Suppose there is no comparison principle for P, i.e. $\min\{w(x,t)\} = w(x_0,t_0) < 0$. If the point $(x_0,t_0) \in Q$, i.e. (x_0,t_0) is internal point for the domain, then $w_t(x_0,t_0) = 0$. If $(x_0,t_0) \in \Omega \times |T|$, i.e. (x_0,t_0) belongs to the upper lid of the parabolic cylinder, then $w_t(x_0,t_0) \leq 0$. Therefore $w_t(x_0,t_0) \leq 0$.

On the other hand, following the proof of Theorem (3) in [3], one can prove that $0 > L_M w(x_0,t_0)$. Actually, in the proof of Theorem (3) in [3], which concerns elliptic systems, we need one additional condition

$$\lambda(t) + c_k^+(x,t) \geq 0 \tag{8}$$

for every $x \in \Omega$ and k=1…,N. But in the case of parabolic systems, one can substitute in system (1)

$$u^k = v^k . e^{(\lambda_0+b)t},$$

where $\lambda_0 = \sup|\lambda(t)|$ and b is defined in (4) and we obtain that (8) is fulfilled.

Subtitution of $w_t(x_0,t_0) \leq 0$ and $0 > L_M w(x_0,t_0)$ in (7) yields the contradiction

$$0 > L_M w(x_0,t_0) \geq -w_t(x_0,t_0) \geq 0$$

and therefore comparison principle holds for system (1). □

Example 2. Let the coefficients of system (1) depend only on x, i.e. consider systems of the type

$$u_t^k - \sum_{i,j=1}^n D_j\left(a_k^{ij}(x) D_i u^k\right) + \sum_{i=1}^n b_k^i(x) D_i u^k + c_k(x) u^k + \sum_{l=1}^N m_{lk}(x) u^k = f^l(x)$$

$l = 1, \dots, N$, with boundary conditions on Γ

$$u^k(x,t) = g^k(x).$$

Let

$$\left(\lambda + \sum_{k=1}^N m_{kj}^+(x_0)\right) > 0 \texttt{ for } j = 1, \dots, N$$

and

$$\lambda + m_{jj}^+(x) \geq 0 \text{ for every } x \in \Omega \text{ and } j = 1, \dots, N$$

where λ is the principal eigenvalue of the operator L_{M^-} in Ω. Then comparison principle holds for system (1), (2).

If $f^l(x) = g^k(x) \equiv 0$ we can write explicitly the solution in the form

$$u(x,t) = \exp^{-\lambda t} v(x)$$

where v is the principal eigenfunction of

$$-\sum_{i,j=1}^{n} D_j \left(a_k^{ij}(x) D_i v^k \right) + \sum_{i=1}^{n} b_k^i(x) D_i v^k + c_k(x) v^k + \sum_{l=1}^{N} m_{lk}(x) v^k = \lambda v(x).$$

If the system is cooperative, then $v(x) > 0$.

The following theorem gives some conditions when comparison principle fails. It is based on Theorem 7 in [3]. The idea is that if we fix t_0 and there is no comparison principle for the elliptic system $Pu(t_0, x) = f$ then there is no comparison principle for system (1), (2).

Theorem 3. *Let* (1) *be a weakly coupled, uniformly parabolic system with fully coupled cooperative part of* $L^*_{M^-}$. *Suppose there is* t_0 *and index* $j \in \{1, \dots, N\}$ *such that* $\left(\lambda + m^+_{jj}(x, t_0)\right) < 0$ *for some point* $x \in \Omega$, *where* λ *is the principal eigenvalue of* L_{M^-}, *and* $m^+_{jl}(x, t_0) = 0$ *for* $l \neq j$, $l = 1, \dots, N$. *Then the comparison principle does not hold for system (1).*

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