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# SINGULAR INFINITE HORIZON LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEM FOR SYSTEMS WITH KNOWN DISTURBANCES: A REGULARIZATION APPROACH

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We consider an infinite horizon linear-quadratic optimal control problem for a system with known additive disturbance. A weight matrix of the control cost in the cost functional of this problem is singular, meaning that the problem itself is singular. Using a regularization method, we obtain the infimum of the cost functional and a minimizing sequence of state-feedback controls in this problem.

## 1. Introduction

In this paper, an optimal control of a linear differential equation with constant coefficients and with a known additive disturbance is considered. The control process is evaluated by an infinite horizon quadratic cost functional to be minimized by a proper choice of the control. A weight matrix of the control cost in this functional is singular. Due to this singularity, the considered problem can be solved neither by application of the Pontriagin's Maximum Principle [13], nor using the Hamilton-Jacobi-Bellman equation approach (Dynamic Programming approach) [2], i.e. this control problem is singular. To the best of our knowledge, five main methods of solution of singular optimal control problems can be

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distinguished. The first method uses higher order optimality conditions (see e.g. [3, 12] and references therein). The second method propose to derive a singular optimal control as a minimizing sequence of open-loop controls, i.e., a sequence of regular control functions of time, along which the cost functional tends to its infimum (see e.g. [10] and references therein). The third method is based on a decomposition of the state space into "regular" and "singular" subspaces, and a design of an optimal open loop control as a sum of impulsive and regular functions (see e.g. [4, 16] and references therein). The fourth method proposes to seek a solution of a singular control problem in a properly defined class of generalized functions (see e.g. [17]). The fifth method is based on a regularization of the original singular problem by a "small" correction of its "singular" cost functional (see e.g. [5, 6] and references therein). Such a regularization is a kind of the Tikhonov's regularization of ill-posed problems [15].

In this paper, the singular control problem is treated by its regularization, yielding an auxiliary partial cheap control problem. An asymptotic analyzes of this auxiliary problem is carried out. Based on this analysis, the expression for the infimum of the cost functional in the singular control problem is derived, and the minimizing sequence of state-feedback controls is designed.

## 2. Problem Statement

### 2.1. Initial control problem

Consider the following controlled differential equation:

$$dZ(t)/dt = \mathcal{A}Z(t) + \mathcal{B}U(t) + \mathcal{F}(t), \quad t \geq 0, \quad Z(0) = Z_0, \quad Z(t) \in E^n, \tag{1}$$

where $U(t) \in E^r$, $(r \leq n)$ is the control; $\mathcal{A}$ and $\mathcal{B}$ are given constant matrices of corresponding dimensions; $\mathcal{B}$ has full column rank $r$; $\mathcal{F}(t)$, $t \geq 0$ is a given vector-valued function, satisfying the inequality $\|\mathcal{F}(t)\| \leq a_1 \exp(-\gamma t)$, $t \geq 0$, $a_1 > 0$ and $\gamma > 0$ are some constants; $Z_0 \in E^n$ is a given vector; for any integer $m > 0$, $E^m$ denotes the real Euclidean space of the dimension $m$.

The cost functional, to be minimized by the control $U(t)$, is

$$\mathcal{J}(U) \triangleq \int_0^{+\infty} \left[Z^T(t)\mathcal{D}Z(t) + U^T(t)GU(t)\right] dt, \tag{2}$$

where $\mathcal{D}$ is a given constant symmetric positive definite matrix of corresponding dimension, $\mathcal{D} > 0$; the given constant $r \times r$-matrix $G$ has the form

$$G = \mathrm{diag}\big(g_1, \dots, g_q, \underbrace{0, \dots, 0}_{r-q}\big), \quad 0 \leq q < r, \quad g_k > 0, \quad k = 1, \dots, q, \quad G \geq 0,$$

the superscript "$T$" denotes the transposition. Since the matrix $G$ is singular, the optimal control problem (1)–(2) is singular. We solve this problem in the class $\mathcal{M}_U$ of admissible controls $U(Z,t)$, $(Z,t) \in E^n \times [0,+\infty)$, satisfying the conditions: (a) $U(Z,t)$ is measurable w.r.t. $t \geq 0$ for any fixed $Z \in E^n$ and satisfies the local Lipschitz condition w.r.t. $Z \in E^n$ uniformly in $t \geq 0$; (b) the initial-value problem (1) for $U(t) = U(Z,t)$ has the unique locally absolutely continuous solution $Z(t)$ on the entire interval $[0,+\infty)$; (c) $Z(t) \in L^2\,[0,+\infty; E^n]$; (d) $U\big(Z(t),t\big) \in L^2\,[0,+\infty; E^r]$.

Let $\mathcal{J}^* \stackrel{\triangle}{=} \inf_{U(Z,t)\in\mathcal{M}_U} \mathcal{J}\big(U(Z,t)\big)$. Since $\mathcal{D} > 0$ and $G \geq 0$, then the infimum $\mathcal{J}^*$ is nonnegative. Moreover, if $\mathcal{M}_U \neq \emptyset$, this infimum is finite.

**Definition 1.** *The control sequence* $\big\{U_k(Z,t)\big\}$, $U_k(Z,t) \in \mathcal{M}_U$, $(k = 1, 2, \dots)$, *is called minimizing in the problem* (1)–(2) *if* $\lim_{k\to+\infty} \mathcal{J}(U_k(Z,t)) = \mathcal{J}^*$. *If there exists* $U^*(Z,t) \in \mathcal{M}_U$, *for which* $\mathcal{J}\big(U^*(Z,t)\big) = \mathcal{J}^*$, *this control is called optimal in the problem* (1)–(2). *In this case there exists a minimizing control sequence, point-wise convergent to* $U^*(Z,t)$ *for a. a.* $(Z,t) \in E^n \times [0,+\infty)$.

### 2.2. Transformation of the problem (1)–(2)

Let us partition the matrix $\mathcal{B}$ into blocks as $\mathcal{B} = \Big(\mathcal{B}_1, \mathcal{B}_2\Big)$, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are of dimensions $n \times q$ and $n \times (r-q)$. We assume that: **(A1)** $\det\big(\mathcal{B}_2^T \mathcal{D} \mathcal{B}_2\big) \neq 0$.

By $\mathcal{B}_c$, we denote a complement matrix to the matrix $\mathcal{B}$, i.e., the matrix of dimension $n \times (n-r)$, and such that the block matrix $(\mathcal{B}_c, \mathcal{B})$ is nonsingular. Hence, the block matrix $\widetilde{\mathcal{B}}_c = (\mathcal{B}_c, \mathcal{B}_1)$ is a complement matrix to $\mathcal{B}_2$. Consider the matrices $\mathcal{H} = (\mathcal{B}_2^T \mathcal{D} \mathcal{B}_2)^{-1} \mathcal{B}_2^T \mathcal{D} \widetilde{\mathcal{B}}_c$, $\mathcal{L} = \widetilde{\mathcal{B}}_c - \mathcal{B}_2 \mathcal{H}$, and partition the matrix $\mathcal{H}$ into blocks $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are of dimensions $(r-q) \times (n-r)$ and $(r-q) \times q$. Now, we transform the state in the control problem (1)–(2) as:

$$Z(t) = (\mathcal{L}, \mathcal{B}_2)\, z(t), \tag{3}$$

where $z(t) \in E^n$ is a new state. The transformation (3) is invertible, [7].

**Remark.** In what follows, we use the notation $O_{n_1 \times n_2}$ for the zero matrix of dimension $n_1 \times n_2$, excepting the cases where the dimension of zero matrix is obvious. In such cases, we use the notation 0 for the zero matrix. By $I_m$, we denote the identity matrix of dimension $m$.

Based on the results of [8] (Lemma 1), we have the following assertion.

**Proposition 1.** *Let the assumption* **(A1)** *hold. Then, transforming the state variable of the problem* (1)–(2) *in accordance with* (3), *and redenoting the control* *as $u(t)$, we obtain the control problem*

$$dz(t)/dt = Az(t) + Bu(t) + f(t), \quad z(0) = z_0, \quad t \geq 0, \tag{4}$$

$$J(u) \overset{\triangle}{=} \int_0^{+\infty} \left[z^T(t)Dz(t) + u^T(t)Gu(t)\right] dt \to \min_u, \tag{5}$$

*where* $A = (\mathcal{L}, \mathcal{B}_2)^{-1} \mathcal{A} (\mathcal{L}, \mathcal{B}_2)$, $f(t) = (\mathcal{L}, \mathcal{B}_2)^{-1} \mathcal{F}(t)$, $z_0 = (\mathcal{L}, \mathcal{B}_2)^{-1} Z_0$,

$$B = (\mathcal{L}, \mathcal{B}_2)^{-1} \mathcal{B} = \begin{pmatrix} O_{(n-r)\times q} & O_{(n-r)\times(r-q)} \\ I_q & O_{q\times(r-q)} \\ \mathcal{H}_2 & I_{r-q} \end{pmatrix},$$

$D = (\mathcal{L}, \mathcal{B}_2)^T \mathcal{D} (\mathcal{L}, \mathcal{B}_2) = \mathrm{diag}\big(D_1, D_2\big)$, $D_1 = \mathcal{L}^T \mathcal{D} \mathcal{L} > 0$, $D_2 = \mathcal{B}_2^T \mathcal{D} \mathcal{B}_2 > 0$. *The function $f(t)$ satisfies the inequality $\|f(t)\| \leq a_2 \exp(-\gamma t)$, $t \geq 0$, $a_2 > 0$ is some constant.*

**Remark.** Since the weight matrix of the control cost in $J(u)$ is singular, the problem (4)–(5) is singular. Moreover, this problem does not have, in general, an optimal control among regular functions. The class $M_u$ of admissible state-feedback controls $u(z, t)$ in (4)–(5) is defined similarly to such a class $\mathcal{M}_U$ in (1)–(2). The infimum $J^* \overset{\triangle}{=} \inf_{u(z,t)\in M_u} J\big(u(z, t)\big)$ is nonnegative. If $M_u \neq \emptyset$, this infimum is finite. The minimizing control sequence $\{u_k(z, t)\}$ and the optimal state-feedback control $u^*(z, t)$ in this problem are defined similarly to those in the problem (1)-(2), (see Definition 1). In the sequel of this paper, we deal with the optimal control problem (4)–(5). We call this problem the Original Optimal Control Problem (OOCP). Since the transformation (3) is invertible, the OOCP (4)-(5) is equivalent to the initial optimal control problem (1)–(2). The latter means that the classes $M_u$ and $\mathcal{M}_U$ are either both nonempty or both empty. If $M_u$ and $\mathcal{M}_U$ are nonempty, the infimum values $J^*$ and $\mathcal{J}^*$ of the cost functionals in these problems are finite and equal to each other.

## 3. Regularization of the OOCP

### 3.1. Partial cheap control problem

Consider the optimal control problem with the dynamics (4) and the performance index

$$J_\varepsilon(u) \overset{\triangle}{=} \int_0^{+\infty} \left[z^T(t)Dz(t) + u^T(t)(G + \mathcal{E})u(t)\right] dt \to \min_u, \tag{6}$$

where $\mathcal{E} = \mathrm{diag}\Big(\underbrace{0, \ldots, 0}_{q}, \underbrace{\varepsilon^2, \ldots, \varepsilon^2}_{r-q}\Big)$, and $\varepsilon > 0$ is a small parameter.

**Remark.** Since the parameter $\varepsilon > 0$ is small, the problem (4), (6) is a partial cheap control problem, i.e., an optimal control problem where a cost of some control coordinates in the cost functional is much smaller than costs of the state and the other control coordinates. In what follows, we call this problem the Partial Cheap Control Problem (PCCP).

### 3.2. Optimal state-feedback control of the PCCP

We look for such a control in the class $M_u$ of the admissible controls, which was introduced earlier for the OOCP. Consider the algebraic matrix Riccati equation

$$PA + A^T P - PS(\varepsilon)P + D = 0, \qquad S(\varepsilon) \stackrel{\triangle}{=} B(G + \mathcal{E})^{-1}B^T. \tag{7}$$

By virtue of the results of [14], if for a given $\varepsilon > 0$ the equation (7) has a symmetric solution $P = P(\varepsilon) \geq 0$ such that the matrix

$$\mathcal{A}(\varepsilon) \stackrel{\triangle}{=} A - S(\varepsilon)P(\varepsilon) \tag{8}$$

is a Hurwitz one, then the optimal control of the PCCP exists in $M_u$, is unique and has the form

$$u^*_\varepsilon(z,t) = -(G + \mathcal{E})^{-1}B^T\Big(P(\varepsilon)z + h(t)\Big), \qquad (z,t) \in E^n \times [0,+\infty). \tag{9}$$

In (9), the vector-valued function $h(t)$ is the unique solution of the problem

$$dh(t)/dt = -\mathcal{A}^T(\varepsilon)h(t) - P(\varepsilon)f(t), \qquad h(+\infty) = 0. \tag{10}$$

The optimal value of the cost functional in the PCCP has the form

$$J^*_\varepsilon = z_0^T P(\varepsilon)z_0 + 2h^T(0)z_0 + s(0), \tag{11}$$

where the scalar function $s(t)$, $t \in [0,+\infty)$ is the unique solution of the problem

$$ds(t)/dt = -2h^T(t)f(t) + h^T(t)S(\varepsilon)h(t), \qquad s(+\infty) = 0. \tag{12}$$

## 4. Asymptotic Analysis of the PCCP

### 4.1. Asymptotic solution of the equation (7)

Similarly to the results of [8], the matrix $S(\varepsilon)$, appearing in this equation, can be represented in the following block form:

$$S(\varepsilon) = \begin{pmatrix} S_1 & S_2 \\ S_2^T & (1/\varepsilon^2)S_3(\varepsilon) \end{pmatrix}, \quad S_1 \stackrel{\triangle}{=} \begin{pmatrix} O_{(n-r)\times(n-r)} & O_{(n-r)\times q} \\ O_{q\times(n-r)} & \widetilde{G}^{-1} \end{pmatrix},$$

$$\widetilde{G} = \mathrm{diag}\big(g_1,\dots,g_q\big), \quad S_2 = \begin{pmatrix} O_{(n-r)\times(r-q)} \\ \widetilde{G}^{-1}\mathcal{H}_2^T \end{pmatrix}, \quad S_3(\varepsilon) = I_{r-q} + \varepsilon^2\mathcal{H}_2\widetilde{G}^{-1}\mathcal{H}_2^T.$$

Due to this block representation, the left-hand side of the equation (7) has a singularity at $\varepsilon = 0$. To remove this singularity, we seek the symmetric solution $P(\varepsilon)$ of the equation (7) in the block form

$$P(\varepsilon) = \begin{pmatrix} P_1(\varepsilon) & \varepsilon P_2(\varepsilon) \\ \varepsilon P_2^T(\varepsilon) & \varepsilon P_3(\varepsilon) \end{pmatrix}, \tag{13}$$

where the blocks $P_1(\varepsilon)$, $P_2(\varepsilon)$ and $P_3(\varepsilon)$ have the dimensions $(n-r+q)\times(n-r+q)$, $(n-r+q)\times(r-q)$ and $(r-q)\times(r-q)$; and $P_1^T(\varepsilon) = P_1(\varepsilon)$, $P_3^T(\varepsilon) = P_3(\varepsilon)$.

We also partition the matrix $A$ into blocks with the same dimensions as in (13) $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$. Substitution of the block representations for the matrices $D$, $S(\varepsilon)$, $P(\varepsilon)$, and $A$ into the equation (7) yields after a routine rearrangement an equivalent set of Riccati-type algebraic matrix equations with respect to $P_1(\varepsilon)$, $P_2(\varepsilon)$ and $P_3(\varepsilon)$. This set does not have a singularity at $\varepsilon = 0$. Setting formally $\varepsilon = 0$ in this set of equations, we obtain the system for its zero-order asymptotic solution $\{P_{10},\ P_{20},\ P_{30}\}$:

$$\begin{gathered} P_{10}A_1 + A_1^T P_{10} - P_{10}S_1P_{10} - P_{20}P_{20}^T + D_1 = 0, \\ P_{10}A_2 - P_{20}P_{30} = 0, \\ \big(P_{30}\big)^2 - D_2 = 0. \end{gathered} \tag{14}$$

Solving the third and second equations of (14) with respect to $P_{30}$ and $P_{20}$, we obtain $P_{30} = \big(D_2\big)^{1/2}$, $P_{20} = P_{10}A_2\big(D_2\big)^{-1/2}$, where the superscript "1/2" denotes the unique symmetric positive definite square root of the corresponding symmetric positive definite matrix, while the superscript "$-1/2$" denotes the inverse matrix for such a square root. Substitution of the above obtained expression for $P_{20}$ into the first equation of (14) yields the algebraic matrix Riccati equation with respect to $P_{10}$

$$P_{10}A_1 + A_1^T P_{10} - P_{10}S_0P_{10} + D_1 = 0, \qquad S_0 \stackrel{\triangle}{=} A_2D_2^{-1}A_2^T + S_1. \tag{15}$$

Based on the results of [8], we represent the matrix $S_0$ as $S_0 = \bar{B}\Theta^{-1}\bar{B}^T$,

$$\bar{B} = \begin{pmatrix} \widetilde{B}\ , & A_2 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} O_{(n-r)\times q} \\ I_q \end{pmatrix}, \quad \Theta = \begin{pmatrix} \widetilde{G} & O_{q\times(r-q)} \\ O_{(r-q)\times q} & D_2 \end{pmatrix}.$$

In what follows, we assume: **(A2)** the pair $(A_1, \bar{B})$ is stabilizable.

Due to this assumption and the results of [1], the equation (15) has the unique symmetric solution $P_{10} > 0$. The matrix $\mathcal{A}_0 \stackrel{\triangle}{=} A_1 - S_0P_{10}$ is a Hurwitz one. Now, using Proposition 1 ($D_2 > 0$), the above mentioned features of (15), and the results of [11] (Sections 3.4 and 3.6.1), we can state the following.

**Lemma 1.** *Let the assumptions* **(A1)**–**(A2)** *hold. Then, there exists a number* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$ *the equation* (7) *has the unique symmetric solution* $P(\varepsilon) > 0$ *of the block form* (13) *satisfying the inequalities* $\|P_i(\varepsilon) - P_{i0}\| \leq a\varepsilon$, $(i = 1, 2, 3)$; $a > 0$ *is some constant independent of* $\varepsilon$. *The matrix* $\mathcal{A}(\varepsilon)$, *given by* (8), *is a Hurwitz one.*

### 4.2. Asymptotic solution of the problem (10)

To derive this asymptotic solution, we represent the matrix $\mathcal{A}(\varepsilon)$, the vector-valued function $f(t)$ and the solution $h(t, \varepsilon)$ to (10) in the block forms as

$$\mathcal{A}(\varepsilon) = \begin{pmatrix} \mathcal{A}_1(\varepsilon) & \mathcal{A}_2(\varepsilon) \\ (1/\varepsilon)\mathcal{A}_3(\varepsilon) & (1/\varepsilon)\mathcal{A}_4(\varepsilon) \end{pmatrix}, \ f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \ h(t, \varepsilon) = \begin{pmatrix} h_1(t, \varepsilon) \\ \varepsilon h_2(t, \varepsilon) \end{pmatrix},$$

where $\mathcal{A}_1(\varepsilon) = A_1 - S_1 P_1(\varepsilon) - \varepsilon S_2 \big(P_2(\varepsilon)\big)^T$, $\mathcal{A}_2(\varepsilon) = A_2 - \varepsilon S_1 P_2(\varepsilon) - \varepsilon S_2 P_3(\varepsilon)$, $\mathcal{A}_3(\varepsilon) = \varepsilon A_3 - \varepsilon S_2 P_1(\varepsilon) - S_3(\varepsilon)\big(P_2(\varepsilon)\big)^T$, $\mathcal{A}_4(\varepsilon) = \varepsilon A_4 - \varepsilon^2 S_2^T P_2(\varepsilon) - S_3(\varepsilon) P_3(\varepsilon)$; $f_1(t)$ and $h_1(t, \varepsilon)$ are of dimension $n - r + q$; $f_2(t)$ and $h_2(t, \varepsilon)$ are of dimension $r - q$.

Substitution of the block representations for $\mathcal{A}(\varepsilon)$, $P(\varepsilon)$, $f(t)$ and $h(t, \varepsilon)$ into the problem (10) yields an equivalent terminal-value problem with respect to $h_1(t, \varepsilon)$ and $h_2(t, \varepsilon)$. Setting formally $\varepsilon = 0$ in this problem, we obtain the problem for its zero-order asymptotic solution $\{h_{10}(t), h_{20}(t)\}$

$$\begin{aligned} & dh_{10}(t)/dt = -\mathcal{A}_1^T(0) h_{10}(t) - \mathcal{A}_3^T(0) h_{20}(t) - P_{10} f_1(t), \quad h_{10}(+\infty) = 0, \\ & 0 = -\mathcal{A}_2^T(0) h_{10}(t) - \mathcal{A}_4^T(0) h_{20}(t). \end{aligned} \tag{16}$$

Solving the second equation of (16) with respect to $h_{20}(t)$, and taking into account that $\mathcal{A}_4(0) = -P_{30} = -\big(D_2\big)^{1/2}$ and $\mathcal{A}_2(0) = A_2$, we obtain $h_{20}(t) = \big(D_2\big)^{-1/2} A_2^T h_{10}(t)$. Substitution of the latter into the first equation of (16) and using the expressions $\mathcal{A}_1(0) = A_1 - S_1 P_{10}$, $\mathcal{A}_3(0) = -\big(P_{20}\big)^T$ yield the problem for $h_{10}(t)$: $dh_{10}(t)/dt = -\mathcal{A}_0^T h_{10}(t) - P_{10} f_1(t)$, $h_{10}(+\infty) = 0$, which has the unique solution $h_{10}(t) = \int_0^{+\infty} \exp\big(\mathcal{A}_0^T \zeta\big) P_{10} f_1(\zeta + t) d\zeta$ satisfying the inequality $\big\|h_{10}(t)\big\| \leq a \exp(-\gamma t)$, $t \geq 0$, $a > 0$ is some constant.

**Lemma 2.** *Let the assumptions* **(A1)**–**(A2)** *hold. Then, there exists a number* $0 < \varepsilon_1 \leq \varepsilon_0$, *such that for all* $\varepsilon \in (0, \varepsilon_1]$ *the solution* $h(t, \varepsilon) = \begin{pmatrix} h_1(t, \varepsilon) \\ \varepsilon h_2(t, \varepsilon) \end{pmatrix}$ *of the problem* (10) *satisfies the inequalities* $\big\|h_i(t, \varepsilon) - h_{i0}(t)\big\| \leq c\varepsilon \exp\left(-\mu t\right)$, $(i = 1, 2)$, $t \geq 0$; $c > 0$ *and* $\mu > 0$ *are some constants independent of* $\varepsilon$.

The detailed proof of the lemma can be found in [9].

### 4.3. Asymptotic solution of the problem (12)

Substitution of the block representations of the matrix $S(\varepsilon)$ and the vectors $f(t)$ and $h(t,\varepsilon)$ into this problem, and setting formally $\varepsilon = 0$ in the resulting problem yield after some rearrangement the problem for the zero-order asymptotic solution $s_0(t)$ to the problem (12):

$$ds_0(t)/dt = -2h_{10}^T(t)f_1(t) + h_{10}^T(t)S_0h_{10}(t), \quad s_0(+\infty) = 0.$$

The latter has the solution

$$s_0(t) = \int_t^{+\infty} \Big(2h_{10}^T(\sigma)f_1(\sigma) - h_{10}^T(\sigma)S_0h_{10}(\sigma)\Big)d\sigma$$

satisfying the inequality $\|s_0(t)\| \le a\exp(-2\gamma t)$, $t \ge 0$; $a > 0$ is some constant. Similarly to Lemma 2, we obtain the following lemma.

**Lemma 3.** *Let the assumptions* **(A1)**–**(A2)** *hold. Then, there exists a number* $0 < \varepsilon_2 \le \varepsilon_1$, *such that for all* $\varepsilon \in (0, \varepsilon_2]$ *the solution* $s(t,\varepsilon)$ *of the problem* (12) *satisfies the inequality* $\|s(t,\varepsilon) - s_0(t)\| \le c\varepsilon\exp(-2\mu t)$, $t \ge 0$, *where* $c > 0$ *is some constant independent of* $\varepsilon$.

### 4.4. Asymptotic expansion of the value (11)

Let us partition the vector $z_0$ into blocks $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $x_0 \in E^{n-r+q}$, $y_0 \in E^{r-q}$, and introduce the value $\bar{J}^* \triangleq x_0^T P_{10}^* x_0 + 2h_{10}(0)x_0 + s_0(0)$. As a direct consequence of the equation (11) and Lemmas 1,2,3, we obtain the following assertion.

**Lemma 4.** *Let the assumptions* **(A1)**–**(A2)** *hold. Then,* $|J_\varepsilon^* - \bar{J}^*| \le c\varepsilon$, $\varepsilon \in (0, \varepsilon_2]$, *where* $c > 0$ *is some constant independent of* $\varepsilon$.

Note, that $\bar{J}^*$ is the optimal value of the cost functional in the following regular optimal control problem, called the Reduced Optimal Control Problem (ROCP):

$$d\bar{x}(t)/dt = A_1\bar{x}(t) + \bar{B}\bar{u}(t) + f_1(t), \quad t \ge 0, \quad \bar{x}(0) = x_0, \quad \bar{x}(t) \in E^{n-r+q},$$

$$\bar{J}(\bar{u}) \triangleq \textstyle\int_0^{+\infty} \big(\bar{x}^T(t)D_1\bar{x}(t) + \bar{u}^T(t)\Theta\bar{u}(t)\big)dt \to \min_{\bar{u}}, \qquad \bar{u}(t) \in E^r.$$

## 5. Main Results: Solution of the OOCP

**Theorem 1.** *Let the assumptions* **(A1)**–**(A2)** *hold. Then,* $J^* = \bar{J}^*$, *where* $J^*$ *is the infimum of the cost functional in the OOCP.*

Consider a numerical sequence $\{\varepsilon_k\}$: $\varepsilon_k > 0$, $(k = 1, 2, \dots)$, $\lim_{k\to+\infty} \varepsilon_k = 0$. For any $k = 1, 2, \dots,$ and $x \in E^{n-r+q}$, $y \in E^{r-q}$, consider the state-feedback control

$$u_k(z,t) = -\begin{pmatrix} \Theta^{-1}\bar{B}^T\Big[P_{10}x + h_{10}(t)\Big] \\ (1/\varepsilon_k)\Big[P_{20}^T x + P_{30}y + h_{20}(t)\Big] \end{pmatrix}, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

**Theorem 2.** *Let the assumptions* **(A1)**–**(A2)** *hold. Then,* $\lim_{k\to+\infty} J\big(u_k(\cdot)\big) = J^*$*, i.e., the sequence of controls* $\big\{u_k(z,t)\big\}$ *is a minimizing sequence in the OOCP.*

Proofs of Theorems 1 and 2 are based on the asymptotic analysis of the PCCP, presented in Section 4. The details of these proofs can be found in [9].

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