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# NECESSARY AND SUFFICIENT CONDITION FOR FINITE TIME BLOW UP OF THE SOLUTIONS TO SIXTH ORDER DOUBLE DISPERSIVE EQUATIONS*

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The nonlinear double dispersive equation of sixth order with linear restoring force is investigated. Necessary and sufficient condition for finite time blow up of the solution with arbitrary positive energy is obtained. New very general sufficient conditions for blow up of the solution are proved. Explicit choice of initial data with arbitrary positive initial energy, satisfying all conditions of the theorems, are given.

## 1. Introduction

The aim of this paper is to prove necessary and sufficient condition for finite time blow up of the solutions to Cauchy problem for sixth order double dispersive equation with linear restoring force

$$u_{tt} - u_{xx} - u_{ttxx} + u_{xxxx} + u_{ttxxxx} + u + f(u)_{xx} = 0, \tag{1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}. \tag{2}$$

The initial data $u_0$, $u_1$ satisfy the regularity conditions

$$u_0 \in \mathrm{H}^1(\mathbb{R}), \quad u_1 \in \mathrm{H}^1(\mathbb{R}), \quad (-\Delta)^{-1/2} u_0 \in \mathrm{L}^2(\mathbb{R}), \quad (-\Delta)^{-1/2} u_1 \in \mathrm{L}^2(\mathbb{R}) \tag{3}$$

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where $(-\Delta)^{-s}u = \mathcal{F}^{-1}\left(|\xi|^{-2s}\mathcal{F}(u)\right)$ for $s>0$ and $\mathcal{F}(u)$, $\mathcal{F}^{-1}(u)$ are the Fourier transform and the inverse Fourier transform, respectively.

The nonlinear term in (1) has one of the following forms

$$
\begin{aligned}
&f(u) = \sum_{k=1}^{l} a_k|u|^{p_k-1}u - \sum_{j=1}^{s} b_j|u|^{q_j-1}u,\\
(4)\qquad &\\
&f(u) = a_1|u|^{p_1} + \sum_{k=2}^{l} a_k|u|^{p_k-1}u - \sum_{j=1}^{s} b_j|u|^{q_j-1}u,
\end{aligned}
$$

where the constants $a_k$, $p_k$ $(k=1,2,\dots,l)$ and $b_j$, $q_j$ $(j=1,2,\dots,s)$ fulfil the conditions

$$
\begin{aligned}
&a_1>0,\ a_k\ge 0,\ b_j\ge 0 \ \text{ for }\ k=2,\dots,l,\quad j=1,\dots,s,\\
(5)\qquad &\\
&1<q_s<q_{s-1}<\dots<q_1<p_1<p_2<\dots<p_l<\infty.
\end{aligned}
$$

The nonlinear term (4), (5) includes the quadratic-cubic nonlinearity ($f(u) = u^2+u^3$) and the cubic-quintic nonlinearity ($f(u)=u^3+u^5$) which appears in a number of mathematical models of physical processes, e.g. propagation of longitudinal strain waves in an isotropic cylindrical compressible elastic rod [9], water wave problems with nonzero tension [10] and others.

It is wellknown that every weak solution to (1)–(4) with nonpositive initial energy, except the trivial one, blows up for a finite time. The global behaviour of the solutions to (1)–(4) with positive initial energy is basically investigated by the potential well method introduced by Sattinger and Payne [8] for nonlinear wave equation.

For special nonlinearities

$$
(6)\qquad f(u)=a|u|^p \quad\text{or}\quad f(u)=a|u|^{p-1}u, \qquad p>1 \quad a>0
$$

and for combined power nonlinearities (4) global existence or finite time blow up of the solutions to (1)–(2) is proved in [11, 12, 13] under the sign condition of the Nehari functional $I(u_0)>0$ or $I(u_0)<0$ respectively. The main assumption in the potential well method is that the initial energy is subcritical, i.e. $0<E(0)\le d$, where $d$ is the critical energy constant.

For supercritical initial energy $E(0)>d$ there are only sufficient conditions for finite time blow up of the solutions to (1)–(2), see [3]. The proof of the finite time blow up is based on the concavity method of Levine [7].

In the present paper, in Theorem 7, we give a necessary and sufficient condition for finite time blow up of the solutions to (1)–(4). The result sheds light on the genesis of the blow up of the solutions to (1)–(4) and gives better understanding of the different sufficient conditions and their analysis.

The paper is organized in the following way. In Section 2 some preliminary results are given, while in Section 3 the main result is formulated and proved. In Section 4 explicit choice of initial data satisfying the sufficient conditions in theorems in Section 3 is proposed.

## 2. Preliminaries

For functions $u(t,x)$, depending on $t$ and $x$, we use the following notations

$$||u|| := ||u(t,\cdot)||_{\mathrm{L}^2(\mathbb{R})},\ ||u||_1 := ||u(t,\cdot)||_{\mathrm{H}^1(\mathbb{R})},\ (u,v) = \int_{\mathbb{R}} u(t,x)v(t,x)\,dx,$$

$$\langle u,v\rangle = \langle u(t,x),v(t,x)\rangle = (u,v) + (u_x,v_x) + ((-\Delta)^{-1/2}u,(-\Delta)^{-1/2}v). \tag{7}$$

We recall the definition for blow up of the solutions to (1)–(4).

**Definition 1.** *Suppose $u(t,x)$ is a weak solution to (1)–(4) in the maximal existence time interval $[0,T_m)$, $0 < T_m \le \infty$. Then $u(t,x)$ blows up at $T_m$ if*

$$\limsup_{t\to T_m, t<T_m} ||u||_1 = \infty \tag{8}$$

Let us formulate the local existence result for problem (1)–(4).

**Theorem 1.** *If (3), (4) hold, then problem (1)–(2) has a unique local solution $u(t,x) \in \mathrm{C}^1([0,T_m);\mathrm{H}^1(\mathbb{R}))$, $(-\Delta)^{-1/2}u \in \mathrm{C}^1([0,T_m);\mathrm{L}^2(\mathbb{R}))$, $(-\Delta)^{-1/2}u_t \in \mathrm{C}^1([0,T_m);\mathrm{L}^2(\mathbb{R}))$ on a maximal existence time interval $[0,T_m)$, $T_m \le \infty$. Moreover:*

(*i*) *The solution $u(t,x)$ satisfies the conservation law*

$$E(t) = E(0) \quad \textit{for every} \quad t \in [0,T_m), \tag{9}$$

*where*

$$E(t) := E(u(t,\cdot),u_t(t,\cdot)) = \frac{1}{2}\left(\langle u_t,u_t\rangle + \langle u,u\rangle\right) - \int_{\mathbb{R}}\int_0^{u(t,x)} f(y)\,dy\,dx.$$

(*ii*) *If* $\limsup_{t\to T_m, t<T_m} ||u||_1 < \infty$, *then* $T_m = \infty$.

The proof of Theorem 1 is similar to the proofs of local existence results in (Th 2.4, [11]), (Th 23, [13]) and we omit it.

We will use the results from [2] for finite time blow up of the solutions to the following ordinary differential equation

$$\begin{aligned} &\Psi''(t)\Psi(t) - \gamma\Psi'^2(t) = \alpha\Psi^2(t) - \beta\Psi(t) + H(t),\ t \in [0, T_m),\ 0 < T_m \le \infty, \\ &\gamma > 1,\ \alpha > 0,\ \beta > 0,\ H(t) \in C([0, T_m)),\ H(t) \ge 0 \text{ for } t \in [0, T_m). \end{aligned} \tag{10}$$

**Theorem 2.** ([2], Th 2.2) *Suppose* $\Psi(t) \in C^2([0, T_m))$ *is a nonnegative solution of the equation* (10), *where* $[0, T_m)$, $0 < T_m \le \infty$ *is the maximal existence time interval for* $\Psi(t)$. *If* $\Psi(t)$ *blows up at* $T_m$ *then* $T_m < \infty$.

**Theorem 3.** ([2], Th 2.3) *Suppose* $\Psi(t) \in C^2([0, T_m))$ *is a nonnegative solution of the equation* (10), *where* $[0, T_m)$, $0 < T_m \le \infty$ *is the maximal existence time interval for* $\Psi(t)$, $H(t) \in C([0, \infty))$, *and* $H(t) \ge 0$ *for* $t \in [0, \infty)$. *Then* $\Psi(t)$ *blows up at* $T_m$ ***if and only if***

$$\text{there exists } b \in [0, T_m) \text{ such that } \beta \le \alpha\Psi(b) \text{ and } \Psi'(b) > 0. \tag{11}$$

*Moreover, if* (11) *holds, then the estimate*

$$T_m \le b + \frac{\Psi(b)}{(\gamma - 1)\Psi'(b)} < \infty \tag{12}$$

*is satisfied.*

**Theorem 4.** ([2], Th 3.1) *Suppose* $\Psi(t) \in C^2([0, T_m))$ *is a nonnegative solution of* (10) *in the maximal existence time interval* $[0, T_m)$, $0 < T_m \le \infty$, $H(t) \in C([0, \infty))$ *and* $H(t) \ge 0$ *for* $t \in [0, \infty)$. *If*

$$\beta < \frac{2\gamma - 1}{2}\frac{\Psi'^2(0)}{\Psi(0)} + \frac{\alpha(2\gamma - 1)}{2(\gamma - 1)}\Psi(0) - \frac{\alpha^{2\gamma-1}\Psi^{2\gamma-1}(0)}{2(\gamma - 1)\beta^{2\gamma-2}}, \tag{13}$$

$$\Psi'(0) > 0, \tag{14}$$

*then* $\Psi(t)$ *blows up at* $T_m < \infty$.

**Theorem 5.** ([2], Th 3.2) *Suppose $\Psi(t) \in C^2([0,T_m))$ is a nonnegative solution of* (10) *in the maximal existence time interval $[0,T_m)$, $0 < T_m \le \infty$, $H(t) \in C([0,\infty))$ and $H(t) \ge 0$ for $t \in [0,\infty)$. If $\Psi'(0) > 0$ and one of the following conditions*

(*i*)

$$\beta < \alpha\Psi(0); \tag{15}$$

(*ii*) [4]

$$\beta < \frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)} + \alpha\Psi(0); \tag{16}$$

(*iii*) [1]

$$\beta < \frac{2\gamma-1}{2}\frac{\Psi'^2(0)}{\Psi(0)} + \alpha\Psi(0) + \frac{\alpha\Psi(0)}{2(\gamma-1)}(1-A^{2-2\gamma}),\ A = \frac{\gamma-1}{\alpha}\frac{\Psi'^2(0)}{\Psi^2(0)} + 1 \tag{17}$$

*is satisfied, then $\Psi(t)$ blows up at $T_m < \infty$.*

## 3. Main results

In this section we formulate and prove the main results in this paper.

**Theorem 6.** *Suppose $u(t,x)$ is the weak solution to* (1)–(4) *with $E(0) > 0$ defined in the maximal existence time interval $[0,T_m)$, $0 < T_m \le \infty$. If $u(t,x)$ blows up at $T_m$ then $T_m < \infty$.*

**Theorem 7.** *Suppose $u(t,x)$ is the weak solution to* (1)–(4) *with $E(0) > 0$ defined in the maximal existence time interval $[0,T_m)$, $0 < T_m \le \infty$. Then $u(t,x)$ blows up at $T_m$* ***if and only if*** *there exists $b \in [0,T_m)$ such that*

$$E(0) \le \frac{p_1-1}{2(p_1+1)}\langle u(b,\cdot),u(b,\cdot)\rangle \text{ and } \langle u(b,\cdot),u_t(b,\cdot)\rangle > 0. \tag{18}$$

*Moreover, if* (18) *holds, then the estimate*

$$T_m \le b + \frac{2}{(p_1-1)}\frac{\langle u(b,\cdot),u(b,\cdot)\rangle}{\langle u(b,\cdot),u_t(b,\cdot)\rangle} \tag{19}$$

*is satisfied.*

In order to prove the main result we need the following auxiliary statements.

**Lemma 8.** *Suppose $u(t,x)$ is the weak solution to (1)–(4) with $E(0) > 0$ defined in the maximal existence time interval $[0,T_m)$, $0 < T_m \le \infty$. Then the blow up of $H^1$ norm of $u(t,x)$ at $T_m$ is equivalent to the blow up of $\langle u(t,\cdot), u(t,\cdot)\rangle$ at $T_m$, i.e. $\limsup_{t\to T_m, t<T_m} ||u||_1 = \infty$ if and only if*

$$\limsup_{t\to T_m, t<T_m} \langle u(t,\cdot), u(t,\cdot)\rangle = \infty. \tag{20}$$

P r o o f. If $\limsup_{t\to T_m, t<T_m} ||u||_1 = \infty$, then from (7) it follows that $||u||_1^2 \le \langle u,u\rangle$ and $\langle u,u\rangle$ blows up at $T_m$.

Conversely, suppose that (20) holds but

$$\limsup_{t\to T_m, t<T_m} ||u||_1 < \infty. \tag{21}$$

From Definition 1 we get $\limsup_{t\to T_m}((-\Delta)^{-1/2}u, (-\Delta)^{-1/2}u) = \infty$. By means of the conservation law (9) it follows that at least one of the norms $||u||_{L^{p_k+1}}$ tends to infinity for $t \to T_m$. Hence from the embedding of $H^1(\mathbb{R})$ into $L^{p_k+1}(\mathbb{R})$, $p_k > 1$ we get that $\limsup_{t\to T_m, t<T_m} ||u||_1 = \infty$, which contradicts (21). Lemma 8 is proved. □

**Lemma 9.** *Suppose $u(t,x)$ is the weak solution to (1)–(4) in the maximal existence time interval $[0,T_m)$, $0 < T_m \le \infty$. Then function $\Psi(t) = \langle u,u\rangle$ satisfies the equation*

$$\Psi''(t)\Psi(t) - \frac{p_1+3}{4}\Psi'^2(t) = (p_1-1)\Psi^2(t) - 2(p_1+1)E(0)\Psi(t) + H(t), \tag{22}$$

*where*

$$H(t) = (p_1+3)\left(\langle u_t,u_t\rangle\langle u,u\rangle - \langle u,u_t\rangle^2\right) + 2(p_1+1)B(t)\langle u,u\rangle \ge 0 \tag{23}$$

*and*

$$B(t) = \sum_{k=2}^{l} \frac{a_k(p_k-p_1)}{(p_k+1)(p_1+1)} \int_{\mathbb{R}} |u|^{p_k+1}\,dx + \sum_{j=1}^{s} \frac{b_j(p_1-q_j)}{(q_j+1)(p_1+1)} \int_{\mathbb{R}} |u|^{q_j+1}\,dx. \tag{24}$$

P r o o f. By means of (1) and (9), we get the following identities for $\Psi(t)$:

$$\Psi'(t) = 2\langle u, u_t\rangle,$$

$$\Psi''(t) = 2\langle u_t, u_t\rangle + 2\langle u, u_{tt}\rangle = 2\langle u_t, u_t\rangle - 2\|u\|_1^2 - 2\|(-\Delta)^{-1/2}u\|^2 + 2\int_{\mathbb{R}} uf(u)\,dx$$

$$= 2\langle u_t, u_t\rangle - 2\langle u, u\rangle + 2\int_{\mathbb{R}} uf(u)\,dx$$

$$= (p_1+3)\langle u_t, u_t\rangle - 2(p_1+1)E(0) + (p_1-1)\langle u, u\rangle + 2(p_1+1)B(t).$$

Here $B(t)$ is given by (24) and from (5) we have

$$B(t) \geq 0 \qquad \text{for} \qquad t \in [0, T_m). \tag{25}$$

Substituting $\Psi'(t)$ and $\Psi''(t)$ in the lhs of (22), we get that $\Psi(t)$ is a solution to (22). Here $H(t)$ is given in (23) and $H(t) \geq 0$ from (25) and the Cauchy–Schwarz inequality. □

P r o o f o f T h e o r e m 6. From Lemma 9 it follows that the function $\Psi(t)$ satisfies in $[0, T_m)$ equation (10). Hence, $\Psi(t)$ is a solution to (10) for

$$\alpha = p_1 - 1, \qquad \beta = 2(p_1+1)E(0) > 0, \qquad \gamma = \frac{p_1+3}{4} > 1 \tag{26}$$

and $H(t)$ defined in (23). If $u(t,x)$ blows up at $T_m$, i.e. (8) holds, then from Lemma 8 we get that $\Psi(t) = \langle u, u\rangle$ blows up at $T_m$. Applying Theorem 2, we obtain that $T_m < \infty$. Theorem 6 is proved. □

P r o o f o f T h e o r e m 7. *(Necessity).* Suppose $u(t,x)$ blows up at $T_m$ and hence from Lemma 8, $\Psi(t) = \langle u, u\rangle$ blows up at $T_m$. Then from Lemma 2.1 in [2] for $M = 2(p_1+1)E(0)/(p_1-1)$ and $b = t_0$ condition (18) is satisfied.

*(Sufficiency).* Suppose (18) holds. We assume by contradiction that $u(t,x)$ does not blow up at $T_m$, i.e

$$\limsup_{t\to T_m, t<T_m} ||u||_1 < \infty. \tag{27}$$

From Theorem 1(ii) it follows that $T_m = \infty$. According to Lemma 9 $\Psi(t) = \langle u, u\rangle$ satisfies (22) in $[0,\infty)$ for $\alpha$, $\beta$, $\gamma$ defined in (26). Note, that $H(t)$, given in (23), is a nonnegative function for every $t \geq 0$. Moreover, condition (11) in Theorem 3 is fulfilled from (18). Applying Theorem 3 we get that $\Psi(t) = \langle u, u\rangle$ blows up at $T_m$. Hence from Lemma 8 $u(t,x)$ also blows up at $T_m$, which contradicts (27). Theorem 7 is proved. □

## 4. Sufficient conditions for finite time blow up

In this section we give explicit sufficient conditions on $u_0$, $u_1$ for finite time blow up of the solutions to (1)–(4).

**Theorem 10.** *Suppose $u(t,x)$ is the weak solution to (1)–(4) with $E(0)>0$ defined in the maximal existence time interval $[0,T_m)$, $0<T_m\le\infty$ . If $\langle u_0,u_1\rangle > 0$ and one of the following conditions*

*(i)*

$$E(0) < \frac{p_1-1}{2(p_1+1)}\langle u_0,u_0\rangle$$

*(ii)* [5]

$$E(0) < \frac{1}{2}\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle} + \frac{p_1-1}{2(p_1+1)}\langle u_0,u_0\rangle \tag{28}$$

*(iii)* [6]

$$E(0) < \frac{1}{2}\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle} + \frac{p_1-1}{2(p_1+1)}\langle u_0,u_0\rangle + \frac{\langle u_0,u_0\rangle}{p_1+1}\left[1-\left(1+\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle^2}\right)^{\frac{1-p_1}{2}}\right]$$

*is satisfied, then $u(t,x)$ blows up at $T_m<\infty$.*

**Theorem 11.** *Suppose $u(t,x)$ is the weak solution to (1)–(4) with $E(0)>0$ defined in the maximal existence time interval $[0,T_m)$, $0<T_m\le\infty$ . If $\langle u_0,u_1\rangle > 0$ and*

$$E(0) < \frac{1}{2}\frac{\langle u_0,u_1\rangle^2}{\langle u_0,u_0\rangle} + \frac{1}{2}\langle u_0,u_0\rangle - \left(\frac{p_1-1}{2}\right)^{\frac{p_1-1}{2}}\left(\frac{\langle u_0,u_0\rangle}{p_1+1}\right)^{\frac{p_1+1}{2}} E^{\frac{1-p_1}{2}}(0), \tag{29}$$

*then $u(t,x)$ blows up at $T_m<\infty$.*

The proof of Theorem 10 and Theorem 11 follows from Theorem 4 and Theorem 5, respectively, for $\alpha$, $\beta$, $\gamma$ defined in (26) and $\Psi(t)=\langle u,u\rangle$, $\Psi(0)=\langle u_0,u_0\rangle$.

Example: For $f(u)=a_1u^3+a_2u^5$, $a_1>0$, $a_2>0$ conditions (iii) of Theorem 10 and (29) become

$$\langle u_0,u_1\rangle > 0 \tag{30}$$

$$E(0) < E_0 = \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} + \frac{1}{4}\langle u_0, u_0\rangle + \frac{1}{4}\frac{\langle u_0, u_0\rangle\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle^2 + \langle u_0, u_1\rangle^2}$$

$$E(0) < E_0 + \frac{1}{4}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle}\left\{\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle^2 + \langle u_0, u_1\rangle^2} + \left(1 + \frac{2\langle u_0, u_0\rangle^2}{\langle u_0, u_1\rangle^2}\right)^{\frac{1}{2}}\right\} \tag{31}$$

respectively.

## 5. Choice of initial data

We will chose explicitly initial data $u_0$, $u_1$ with arbitrary large positive energy, satisfying (30), (31). For this purpose we rewrite (31) in the following equivalent way

$$\kappa = \frac{1}{2}\langle u_1, u_1\rangle - \frac{1}{4}\langle u_0, u_0\rangle - \frac{1}{2}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle} - \frac{1}{4}\frac{\langle u_0, u_0\rangle\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle^2 + \langle u_0, u_1\rangle^2} \tag{32}$$

$$- \frac{1}{4}\frac{\langle u_0, u_1\rangle^2}{\langle u_0, u_0\rangle}\left\{\frac{\langle u_0, u_1\rangle}{\langle u_0, u_0\rangle^2 + \langle u_0, u_1\rangle^2} + \left(1 + \frac{2\langle u_0, u_0\rangle^2}{\langle u_0, u_1\rangle^2}\right)^{\frac{1}{2}}\right\} \tag{33}$$

$$- \frac{a_1}{4}\int_R u^4 dx - \frac{a_2}{6}\int_R u^6 dx < 0 \tag{34}$$

Let $v \in \mathrm{H}^1(\mathbb{R})$, $w \in \mathrm{H}^2(\mathbb{R})$ be arbitrary functions satisfying the conditions

$$(v, w) = 0, \quad (v', w') = 0, \quad (v'', w'') = 0, \quad ||v||_2 = ||w||_2 = 1 \tag{35}$$

where $||f||_2 = ||f||_{\mathrm{H}^2(\mathbb{R})}$. For example, if $v$ is odd and $w$ is an even function, then orthogonality conditions (35) will be satisfied.

We fix an arbitrary constant $M > 0$ and $\epsilon \leq 0$ and chose the initial data $u_0 = w'$, $u_1 = \sigma w' + \mu v'$. For suitable chosen constant $\sigma > 0$ and $\mu > 0$ we will show that

$$\kappa = \epsilon, |\epsilon| << 1 \tag{36}$$

$$E(0) \geq M \tag{37}$$

Since $\langle u_0, u_1\rangle = \sigma||w||_2^2 > 0$ and $\sigma > 0$, condition (30) is satisfied. For $u_0$, $u_1$ we get the following identities

$$\langle u_0, u_0\rangle = ||w||_2^2 = 1, \quad \langle u_1, u_1\rangle = \sigma^2||w||_2^2 + \mu^2||v||_2^2 = \sigma^2 + \mu^2, \qquad (38)$$
$$\langle u_0, u_1\rangle = \sigma||w||_2^2 = \sigma,$$

$$\begin{aligned}\kappa =& \frac{1}{2}(\sigma^2+\mu^2) - \frac{1}{4} - \frac{1}{2}\sigma^2 - \frac{1}{4}\frac{\sigma^2}{1+\sigma^2} - \frac{1}{4}\sigma^2\left\{\frac{\sigma}{1+\sigma^2} + \left(1+\frac{2}{\sigma^2}\right)^{\frac{1}{2}}\right\}\\ &- \frac{a_1}{4}\int_R w'^4dx - \frac{a_2}{6}\int_R w'^6dx\\ =& \frac{1}{4}\left\{2\mu^2 - 1 - \frac{\sigma^2}{1+\sigma^2} - \frac{\sigma^3}{1+\sigma^2} - \sigma^2\left(1+\frac{2}{\sigma^2}\right)^{\frac{1}{2}}\right.\\ &\left. -a_1\int_R w'^4dx - \frac{2a_2}{3}\int_R w'^6dx\right\} = \epsilon,\end{aligned}$$

$$E(0) = \frac{1}{4}\left\{2\sigma^2 + 2\mu^2 + 1 - a_1\int_R w'^4dx - \frac{2a_2}{3}\int_R w'^6dx\right\}.$$

We fix $\sigma = \sigma_0$,

$$\sigma_0^2 = 2M + \frac{1}{2}a_1\int_R w'^4dx + \frac{1}{3}a_2\int_R w'^6dx \qquad (39)$$

so that $E(0) \ge M + \dfrac{2\mu^2+1}{4} \ge M$ and (37) holds. Finally, the constant $\mu$ is chosen as

$$\mu^2 = \frac{1}{2}\left[4\epsilon + 1 + \frac{\sigma_0^2+\sigma_0^3}{1+\sigma_0^2} + \sigma_0^2\left(1+\frac{2}{\sigma_0^2}\right)^{\frac{1}{2}} + a_1\int_R w'^4dx + \frac{2a_2}{3}\int_R w'^6dx\right]$$

and (36) is fulfilled. Under the above choice of initial data all conditions in (30), (31) are satisfied.

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