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EFFECTS OF TIME RESCALING FOR THE GAS DYNAMICS EQUATIONS

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We show that a special change of time variable allows to obtain splitting by physical processes in the barotropic gas dynamics equations. Namely, on the first time step the gas dynamics can be reduced to nonlinear heat equation, whereas on the second step the same system can be reduced to the pressureless gas dynamics equations with a friction term. We discuss also how the Cauchy data can be used on every step. Further, we show that in several cases there exists analytical representation of solution.

1. Preliminaries

The system of the barotropic gas dynamics consists of the following equations for density $\varrho(t,x)$, velocity $\mathbf{u}(t,x)$, and pressure $p(t,x)$, $t \geq 0$, $x \in \mathbb{R}^n$:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1}$$

$$\partial_t \varrho \mathbf{u} + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = 0, \tag{2}$$

corresponding to conservation of mass and momentum. Equation (2) is vectorial. In the barotropic model the pressure $p(t,x)$ is a function of density:

$$p = \Psi(\varrho). \tag{3}$$

Often they use

$$\Psi(\varrho) = \frac{C}{\gamma}\varrho^{\gamma}, \quad \gamma \geq 1. \tag{4}$$

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The case $\gamma = 1$ corresponds to so called isothermal case [2]. We consider the Cauchy problem

$$(\mathbf{u}(x,0), \varrho(x,0)) = (\mathbf{u}_0(x), \varrho_0(x) \geq 0) \in C^1(\mathbb{R}^n) \cap C_b(\mathbb{R}^n). \tag{5}$$

The system (1), (2) is symmetric hyperbolic, therefore the Cauchy problem locally in t has a solution as smooth as initial data [1].

Below we consider solutions before the moment of singularity formation. In 1D case the system (1), (2) can be written in the Riemann invariants, and the class of data, such that the Cauchy problem has globally smooth solution, can be found explicitly. Namely, the Cauchy problem (1), (2), (3), (4) has a globally smooth solution if and only if

$$\min_x (\mathbf{u}_0' \mp \varrho_0^{\frac{\gamma-3}{2}} \varrho_0') \geq 0$$

Otherwise, the derivatives of solution go to infinity within a finite time $T > 0$ (see [2]).

2. Time rescaling: a formal derivation

Let us perform the following change of time variable: $\tau = \tau(t)$. As the velocity of a particle is the time derivative of its position, then in the new variables (τ, x) we get $\mathbf{u}(t,x) = \tau'(t)\bar{\mathbf{u}}(\tau, x)$, whereas in the component of density we have to change the time variable as $\varrho(t,x) = \bar{\varrho}(\tau, x)$. Here the bar denotes functions of the new time variable. Thus, system (1), (2) will be changed as

$$(\tau'(t))^2 \left[\partial_\tau(\bar{\varrho}\bar{\mathbf{u}}) + \mathrm{div}_x(\bar{\varrho}\bar{\mathbf{u}} \otimes \bar{\mathbf{u}})\right] + \tau''(t)\bar{\varrho}\bar{\mathbf{u}} + \nabla_x \bar{p} = 0, \tag{6}$$

$$\tau'(t)[\partial_\tau \bar{\varrho} + \mathrm{div}_x(\bar{\varrho}\bar{\mathbf{u}})] = 0. \tag{7}$$

Below we discuss different choices of the function τ. First of all let us notice that the time scaling leaves the continuity equation (1) unchanged. Nevertheless, the conservation of momentum is strongly influenced by the time transform. To avoid cumbersome notation we omit the bar in what follows.

2.1. Case I. Small t: the nonlinear diffusion equation

Following [3], for small t we choose $\tau_1(t) = k\dfrac{t^2}{2}$, $\tau(0) = 0$, where k is a constant of appropriate dimension. Thus, from (6) we get

$$q_1(t)\left[\partial_{\tau_1}(\varrho\mathbf{u}) + \mathrm{div}(\varrho\mathbf{u} \otimes \mathbf{u})\right] + [k\varrho\mathbf{u} + \nabla p] = 0, \tag{8}$$

with $q_1 = k^2t^2$.

If $t \ll 1$, and the derivatives of solution are bounded, then (8) implies

$$k\varrho\mathbf{u} + \nabla_x p = 0. \tag{9}$$

Further, taking into account (9), (3) and (7) we obtain

$$\partial_{\tau_1}\varrho = \frac{1}{k}\mathrm{div}_x(\nabla_x\Psi(\varrho)) = \frac{1}{k}\mathrm{div}_x(\Psi'(\varrho)\nabla_x\varrho). \tag{10}$$

This is the nonlinear diffusion equation (known also as the nonlinear heat equation or the nonlinear porous media equation). It arises in many applications (see, e.g.[4]).

2.2. Case II. Fixed time approximation: the advection with dry friction

Let us choose $t = \lambda > 0$, and assume that the solution is smooth for $t \in [0, \lambda)$. The new change of variables can be found as a solution of equation $\tau_2'' = \lambda(\tau_2')^2$. The solution, satisfying conditions $\tau_2(0) = 0$ and $\tau_2 \to +\infty$ as $t \to \lambda - 0$, is the following:

$$\tau_2 = -\frac{1}{\lambda}\ln\left(1 - \frac{t}{\lambda}\right).$$

For this case

$$[\partial_{\tau_2}(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \lambda\varrho\mathbf{u}] + q_2(t)\,\nabla_x p = 0, \tag{11}$$

with $q_2 = \lambda^2(\lambda - t)^2$.

Therefore (11) implies

$$\partial_{\tau_2}(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) = -\lambda\varrho\mathbf{u}, \tag{12}$$

$$\partial_{\tau_2}\varrho + \mathrm{div}(\varrho\mathbf{u}) = 0, \tag{13}$$

as $t \to \lambda - 0$.

2.3. Case III. Large t: the pressureless gas dynamics

Let us choose $\tau_3(t) = \tau_1(t)$ and re-write equation (2) as

$$[\partial_{\tau_3}(\varrho\mathbf{u}) + \mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u})] + q_3(t)\,[k\varrho\mathbf{u} + \nabla p] = 0, \tag{14}$$

with $q_3 = \dfrac{1}{kt^2}$. One can see that (14), (7) implies as $t \to \infty$ the pressureless gas dynamics equations

$$\partial_{\tau_3}(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) = 0,$$
$$\partial_{\tau_3}\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0.$$

Remark. All resulting equations are well-studied. In particular, the discussion on existence, uniqueness and boundedness of classical and generalized solution to the Cauchy problem for the nonlinear diffusion equation, classes of exact self-similar and traveling wave solutions can be found in [4] and references therein. Let us notice that if the solution has a compact support, then the front propagates with a finite speed if and only if the integral

$$\int_0^\delta \frac{\Psi'(\varrho)}{\varrho}\, d\varrho, \quad \delta > 0,$$

converges. For $p = C\varrho^\gamma$ this implies $\gamma > 1$.

For $\gamma = 1$ (isothermal process) the resulting equation is the linear heat equation, the solution can be found by the classical integral formula.

We will show below that for resulting system in cases II and III an integral representation exist as well.

Let us also notice that the gas dynamics equations (1), (2) belong to hyperbolic type (strictly hyperbolic for $\varrho > \delta > 0$), this always implies a finite speed of propagation of perturbations. However, the type of resulting equations is different: parabolic for the case I and non-strictly hyperbolic for cases II and III.

3. The Cauchy problem

Below we will concentrate on the Cauchy problem (1), (2), (3), (5) for $t \in [0, \lambda)$. Thus, we consider the time re-scaling only for the cases I and II. If we want to use the initial data (5), we encounter the following problems.

1. For small time approximation (Case I) we need only $\varrho_0(x)$. The velocity can be found as

$$\mathbf{U} := -\frac{\nabla\Psi(\varrho)}{k\varrho}. \tag{15}$$

It is necessarily potential and

$$\mathbf{U}_0 = -\frac{\nabla\Psi(\varrho_0)}{k\varrho_0} \tag{16}$$

has no relations with initial data $\mathbf{u}_0$.

2. For approximation as $t \to \lambda$ (Case II) we do not have initial data at all.

Therefore we propose the following splitting by physical processes on every intermediate time step, which in principal can be used for numerical procedure.

Let us choose sufficiently small $\lambda > 0$. The interval $(0, \lambda)$ is divided into zones of influence of Cases I and II. They are separated by the point $t_* = \dfrac{\lambda^2}{\lambda + 1} < \lambda$, see Fig. 1.

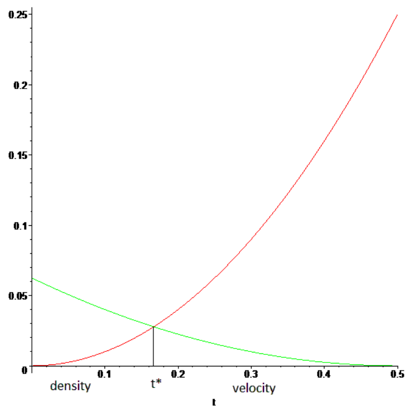

Figure 1: $\lambda = 0.5$: graphs of $q_1 = t^2$ and $q_2 = \lambda^2(\lambda - t)^2$, $t_* = \dfrac{\lambda^2}{\lambda+1} < \lambda$

- Intermediate step I. For $t \in (0, t^*)$ we use equation (10) and solve the Cauchy problem

  $$\partial_{\tau_1} \varrho = \frac{1}{k} \mathrm{div}_x (\Psi'(\varrho) \nabla_x \varrho), \quad \varrho(0, x) = \varrho_0(x), \tag{17}$$

  ignoring $\mathbf{u}_0$. Let us notice that since

  $$\mathbf{u}_0(x) = \tau_1'(t) \bar{\mathbf{u}}(\tau_1(t), x)\big|_{t=0},$$

  and $\tau_1'(0) = 0$, we cannot define the Cauchy data for velocity inherited from (5).

  We denote $\rho(t^*, x) = \varrho(\tau_1(t), x)|_{t=t^*}$, where $\varrho(\tau_1(t), x)$ is a solution to (17).

- Intermediate step II. For $t \in (t^*, \lambda)$ we solve the Cauchy problem for (12), (13) with initial data

  $$\varrho(\tau_2(t^*), x) = \rho(t^*, x), \tag{18}$$

  $$\mathbf{u}(t^*, x) = [\mathbf{u}_0(x) - \mathbf{U}_0(x) + \mathbf{U}(\tau_1(t^*), x)]/\tau_2'(t^*), \tag{19}$$

  where $\tau_2'(t^*) = \dfrac{\lambda+1}{\lambda^2} \neq 0$, $\mathbf{U}_0$ and $\mathbf{U}$ are defined as in (16) and (15). Condition (18) is obtained on the first step, whereas (19) takes into account the difference between genuine initial velocity and initial velocity, inherited from initial density by formula (9).

4. Analytical representation

If $\gamma = 1$ (the isothermal case), then the solution to problem (17) can be found as a convolution of the heat kernel with initial data (see, e.g. [5]). However, for general state equation (3) we cannot find the solution on the 1st intermediate step analytically. Nevertheless, for the solution on the 2nd step one can always obtain analytical representation as an asymptotical limit. We use the method of [6], the details can be also found in [7].

First of all let us note that on the smooth solutions the system (12), (13) of Case II is equivalent to

$$\partial_\tau \mathbf{u} + (\mathbf{u}, \nabla_x)\mathbf{u} = -\lambda \mathbf{u},$$
$$\partial_\tau \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0.$$

In what follows we skip the index and denote $\tau_2 = \tau$, $\mathbf{u} = (U_1, \dots, U_n)$.

Let us introduce the Lagrangian coordinate $x(\tau)$ to label a point which moves together with the medium, that is $\dfrac{dx(\tau)}{d\tau} = \mathbf{u}(\tau, x(\tau))$. If we consider a medium with random particles paths, described by a $2n$ dimensional Itô stochastic differential system of equations, we get:

$$dX_k(\tau) = U_k(\tau)dt + \sigma d(W_k)_\tau, \tag{20}$$
$$dU_k(\tau) = -\lambda U_k(\tau) d\tau. \tag{21}$$

Here $k = 1, \dots, n$, $X(0) = x$, $\quad U(0) = u$, where $(X(\tau), U(\tau))$ runs in the phase space $\mathbb{R}^{2n}$, $\sigma > 0$ is constant, $(W)_\tau = (W)_{k,\tau}$, $k = 1, \dots, n$, is the n - dimensional Brownian motion.

The Fokker-Planck equation for the probability density in position and velocity space $P = P(\tau, x, u)$, corresponding to (20), (21), has the form

$$\frac{\partial P(\tau,x,u)}{\partial \tau} = \left[-\sum_{k=1}^{n} u_k \frac{\partial}{\partial x_k} + \lambda \sum_{k=1}^{n} \left(u_k \frac{\partial}{\partial u_k} + \frac{\partial}{\partial u_k} \right) + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x_k^2} \right] P(\tau,x,u),$$

subject to the initial data

$$P(0, x, u) = P_0(x, u).$$

Let us denote by $\hat{u}(\tau, x)$ the conditional expectation of the velocity $U(\tau)$ at time τ given the position $X(\tau)$ at time τ. Thus, if

$$\hat{\rho}(\tau, x) = \int\limits_{\mathbb{R}^n} P(\tau, x, u)\, du,$$

then

$$\hat{\mathbf{u}}(\tau,x) = \frac{1}{\hat\rho(\tau,x)}\int\limits_{\mathbb{R}^n} \mathbf{u}\, P(\tau,x,u)\,du,$$

$x \in \mathbb{R}^n, \quad \tau \ge \tau_2(t^*)$. If we choose

$$P_0(x,u) = \delta(u - v_0(\tau_2(t^*),x))\,f(x) = \prod_{k=1}^{n} \delta(u_k - (v_0(\tau_2(t^*),x))_k)\,f(x),$$

with an arbitrary sufficiently regular $f(x)$, then

$$\hat\rho(\tau_2(t^*),x) = f(x), \quad \hat{\mathbf{u}}(\tau_2(t^*),x) := v_0(\tau_2(t^*),x) = \mathbf{u}(t^*,x)/\tau_2'(t^*).$$

The scalar function $\hat\rho(t,x)$ and the vector-function $\hat{\mathbf{u}}(\tau,x) = (\hat u_1,\dots,\hat u_n)$ solve the following system [7]:

$$\frac{\partial\hat\rho}{\partial\tau} + \mathrm{div}_x(\hat\rho\hat{\mathbf{u}}) = \frac{1}{2}\sigma^2\sum_{k=1}^{n}\frac{\partial^2\hat\rho}{\partial x_k^2},$$

$$\frac{\partial(\hat\rho\hat u_i)}{\partial\tau} + \nabla_x(\hat\rho\,\hat u_i\,\hat u) =$$

$$-\lambda\hat\rho\hat u_i + \frac{1}{2}\sigma^2\sum_{k=1}^{n}\frac{\partial^2(\hat\rho\hat u_i)}{\partial x_k^2} - \int\limits_{\mathbb{R}^n}(u_i - \hat u_i)\big((u - \hat u), \nabla_x P(t,x,u)\big)\,du,$$

$i = 1,\dots,n,\ t \ge \tau_2(t^*)$.

We apply the Fourier transform with respect to the variable x to obtain the following equation for $\tilde P(\tau,\xi,u)$

$$\frac{\partial\tilde P}{\partial\tau} = \lambda\sum_{k=1}^{n} u_k\,\frac{\partial\tilde P}{\partial u_k} + (\lambda - \frac{\sigma^2}{2}\,\xi^2 - i\,(\xi,u))\,\tilde P, \tag{22}$$

subject to initial data

$$\tilde P(0,\xi,u) = \int\limits_{\mathbb{R}^n} e^{-i(\xi,s)}\delta(u - v_0(s))\,f(s)\,ds, \qquad \xi \in \mathbb{R}^n. \tag{23}$$

The solution to (22), (23) is

$$\tilde P(\tau,\xi,u) = e^{\lambda\tau - \frac{\sigma^2\xi^2\tau}{2} + \frac{i(u,\xi)}{\lambda}}\,F(\xi, ue^{\lambda\tau}),$$

with an arbitrary differentiable function $F : \mathbb{R}^2 \mapsto \mathbb{R}$.

$$F(\xi,u) = e^{-\frac{i(u,\xi)}{\lambda}}\int\limits_{\mathbb{R}^n} e^{-i(\xi,s)}\delta(u - v_0(s))\,f(s)\,ds, \qquad \xi \in \mathbb{R}^n,$$

$$\tilde{P}(\tau,\xi,u)=e^{\lambda\tau-\frac{\sigma^2|\xi|^2\tau}{2}+\frac{i(u,\xi)(1-e^{\lambda\tau})}{\lambda}}\int\limits_{\mathbb{R}^n}e^{-i(\xi,s)}\delta(ue^{\lambda\tau}-v_0(s))\,f(s)\,ds,$$

$$P(\tau,x,u)=$$

$$\frac{e^{\lambda\tau}}{(\sigma\sqrt{2\pi\tau})^n}\int\limits_{\mathbb{R}^n}f(s)\,\delta(e^{\lambda\tau}u-v_0(s))\int\limits_{\mathbb{R}^n}e^{-\frac{\sigma^2|\xi|^2\tau}{2}+i(\xi,(x-s-\phi(\tau)e^{\lambda\tau}u))}\,d\xi\,ds=$$

$$\frac{e^{\lambda\tau}}{(\sigma\sqrt{2\pi\tau})^n}\int\limits_{\mathbb{R}^n}f(s)\,\delta(e^{\lambda\tau}\,u-v_0(s))\,e^{-\frac{|s-x+\phi(\tau)e^{\lambda\tau}u|^2}{2\sigma^2\tau}}\,ds,$$

where $\phi(\tau)=\dfrac{1-e^{-\lambda\tau}}{\lambda}$.

Thus,

$$\hat{\rho}(t,x)=\frac{1}{(\sigma\sqrt{2\pi\tau_2(t)})^n}\int\limits_{\mathbb{R}^n}f(s)\,e^{-\frac{\left(s-x+\phi(\tau_2(t))\mathbf{u}(t^*,s)/\tau_2'(t^*)\right)^2}{2\sigma^2\tau_2(t)}}\,ds,$$

$$\hat{\mathbf{u}}(t,x)=\frac{e^{-\lambda\tau_2(t)}\int\limits_{\mathbb{R}^n}f(s)\mathbf{u}(t^*,s)\,e^{-\frac{\left(s-x+\phi(\tau_2(t))\mathbf{u}(t^*,s)/\tau_2'(t^*)\right)^2}{2\sigma^2\tau_2(t)}}\,ds}{\int\limits_{\mathbb{R}^n}f(s)\,e^{-\frac{\left(s-x+\phi(\tau_2(t))\mathbf{u}(t^*,s)/\tau_2'(t^*)\right)^2}{2\sigma^2\tau_2(t)}}\,ds}.$$

If $(\rho(\tau,x),\mathbf{U}(\tau,x))$, the limits of $(\hat{\rho},\hat{\mathbf{u}})$ as $\sigma\to0$, are C^1 – smooth bounded functions for $(\tau,x)\in[\tau_2(t^*),T)\times\mathbb{R}^n$, $T\le\infty$, then they solve the damped pressureless gas dynamics system

$$\partial_\tau\rho+\operatorname{div}_x(\rho\mathbf{U})=0,\quad \partial_\tau(\rho\mathbf{U})+\nabla_x(\rho\mathbf{U}\otimes\mathbf{U})=-\lambda\rho\mathbf{U},$$

which coincides with (12), (13).

5. Discussion

Our results imply that the structure of initial boundary layer for the gas dynamics equations is the following: first the density reacts on the initial data and then the velocity does. Thus, it one of the form of splitting by physical processes, see [8]. In fact, the same idea was used in the famous "the particle-in-cell" computing method by Harlow [9], see also [10]. Indeed, on the 1st intermediate step one have to take into account the influence of pressure by solving the system

$$\partial_t\varrho=0,\quad \varrho\partial_t\mathbf{u}+\nabla_x p=0.$$

Then, on the 2nd intermediate step one have to take into account the processes of advection and solve

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) = 0,$$
$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0.$$

The second system describes the influence of velocity. The respective numerical scheme has the 1st order of approximation, so has the scheme based on our method.

It should be also noticed that the nonlinear porous medium equation arises as the long time asymptotics for compressible gas dynamics with damping. Namely, there exists the following hypothesis, based on experiment: as $t\to\infty$, solution to the system

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0,$$
$$\partial_t\varrho\mathbf{u} + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p = -\varrho\mathbf{u},$$

tends to the solution of

$$\partial_t\varrho = \Delta_x p(\varrho), \quad \varrho\mathbf{u} + \nabla_x p = 0.$$

Important advances to support this hypothesis were made by P.Marcati, A.Milani, L.Hsiao, T.Liu, T.Yang, K.Nishihara, T.Luo, H.Zhao starting from 1990. In some sense definitive result for $n = 1$, $1 < \gamma < 3$, was obtained in [11]. Namely, it was proved that L^∞-entropy weak finite mass solution of compressible gas dynamics with damping tends in the sense of distributions to the Barenblatt solution of the porous media equation. Let us recall that the Barenblatt solution is the following positive solution with a finite mass M:

$$U_\gamma(t,x;C) := \frac{1}{t^a}\big(C - \frac{b(\gamma-1)}{2\gamma}\frac{|x|^2}{t^{2b}}\big)_+^{\frac{1}{\gamma-1}},$$

$$a = \frac{n}{n(\gamma-1)+2}, \quad b = \frac{a}{n}, \quad C = C(\gamma,n,M) > 0.$$

$$\lim_{t\to 0} U_\gamma(t,x,C) = M\delta(0).$$

REFERENCES

[1] A. Majda. Compressible fluid flow and systems of conservation laws in several space variables. *Appl.Math.Sci.* **53** (1984), 1–159.

[2] B. L. Rozhdestvenskij, N. N. Yanenko. Systems of quasilinear equations and their applications to gas dynamics. Providence, R.I., AMS, 1983.

[3] Y. Brenier, X. L. Duan. From conservative to dissipative systemsthrough quadratic change of time, with application to the curve-shortening flow, arXiv:1703.03404

[4] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailov. Blow-up in quasilinear parabolic equations (De Gruyter Expositions in Mathematics). Berlin-New York, Walter de Gruyter, 1995.

[5] L. C. Evans.Partial Differential Equations. 2nd edition. Providence, American Mathematical Soc., 2010.

[6] S. Albeverio, O. Rozanova. The non-viscous Burgers equation associated with random position in coordinate space: a threshold for blow up behaviour. *Mathematical Models and Methods in Applied Sciences* **19** (2009), 749–767.

[7] S. Albeverio, A. Korshunova, O. Rozanova. A probabilistic model associated with the pressureless gas dynamics. *Bulletin des Sciences Mathematiques* **137** (2013), 902–922.

[8] G. I. Marchuk. Splitting and alternating direction methods. Handbook Numer. Analys., Vol. **1**, Amsterdam etc., North-Holland, 1990.

[9] F. H. Harlow. Hydrodynamic problems involving large fluid distortion. *Journal of the Association for Computing Machinery* **4** (1957), 137.

[10] M. B. Liu, G. R. Liu. Particle Methods for Multi-Scale and Multi-Physics. Singapore, World Scientific, 2017.

[11] F. Huang, P. Marcati, R. Pan. Convergence to the Barenblatt solution for the compressible Euler equations with damping and vacuum. *Archive for Rational Mechanics and Analysis* **176** (2004), 1–24.

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