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NOTE FOR GLOBAL EXISTENCE OF SEMILINEAR HEAT EQUATION IN WEIGHTED L^∞ SPACE

K. Fujiwara, V. Georgiev, T. Ozawa

The local and global existence of the Cauchy problem for semilinear heat equations with small data is studied in the weighted $L^\infty(\mathbb{R}^n)$ framework by a simple contraction argument. The contraction argument is based on a weighted uniform control of solutions related with the free solutions and the first iterations for the initial data of negative power.

1. Introduction

We consider the following Cauchy problem for semilinear heat equation:

$$\begin{cases} \partial_t u - \Delta u = F(u), & t \in [0,T), \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $F(u) = |u|^p$ or $|u|^{p-1}u$ with $p > p_F = 1 + 2/n$, $T > 0$, and u_0 has a singularity localized near the origin so that $u_0 \in \dot{L}^\infty_k(\mathbb{R}^n)$, where

$$\dot{L}^\infty_k(\mathbb{R}^n) = |x|^{-k} L^\infty(\mathbb{R}^n) = \{f \in L^\infty_{loc}(\mathbb{R}^n \setminus \{0\}); |x|^k f \in L^\infty(\mathbb{R}^n)\}.$$

Here p_F is known as the Fujita exponent and L^q with $1 \le q \le \infty$ denotes the usual Lebesgue space on $\mathbb{R}^n$. We shall assume that the order k of the singularity of u_0 satisfies $0 \le k \le 2/(p-1)$. The aim of the present paper is to study local and global existence of solutions of (1) with initial data u_0 having small $\dot{L}^\infty_k$ norm.

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Let us recall some known results on (1). In the pioneer work of Fujita [6], he showed that if $p < p_F$, then any positive solution blows up at a finite time and if $p > p_F$, then for some initial data, solutions exist globally. Hayakawa [9] and Kobayashi, Sirao, and Tanaka [11] showed that when $p = p_F$, any positive solution also blows up at a finite time.

It is also known that the integrability and the size of the initial data affect local and global existence of solutions. For instance, Weissler [15, 16] showed that

Proposition 1. ([15]) *Let $p > 1$ and $1 \leq q < \infty$.*

1. *[Theorem 1] If $q > n(p-1)/2$ or $q = n(p-1)/2 > 1$, then for any $u_0 \in L^q$(which may not be non-negative), there is a local solution to (1) in L^q sense. Here, the time derivative of (1) is regarded as a derivative of L^q-valued function and Laplacian is taken in the Sobolev sense.*

2. *[Corollary 5.2] If $q < n(p-1)/2$, then for some $u_0 \in L^q$, there is no solution to (1) in $C([0,T); L^q)$ for any $T > 0$.*

Proposition 2. ([16, Theorem 3])

1. *Let $u_0 \geq 0$ be in L^q with $1 \leq q < \infty$. Suppose that*

$$(p-1)\int_0^\infty \|e^{t'\Delta}u_0\|_{L^\infty}^{p-1} dt' \leq 1.$$

Then there exists a global L^q-valued solution to the integral form of (1);

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-t')\Delta}F(u(t'))dt'. \tag{2}$$

Moreover the solutions above satisfy

$$0 \leq u(t,x) \leq \frac{e^{t\Delta}(u_0)(x)}{1-(p-1)\int_0^t \|e^{t'\Delta}u_0\|_{L^\infty}^{p-1} dt'}$$

for all $t \geq 0$ and $x \in \mathbb{R}^n$.

2. *For $p > p_F$ and small $L^{n(p-1)/2}$ initial data, there exists a non-negative global $L^{n(p-1)/2}$-valued solution.*

In short, $L^{n(p-1)/2}$ gives the criteria for local existence and non-existence for (1). Furthermore, there is also another criteria for global existence and non-existence in $L^{n(p-1)/2}$. We also refer [2] for related subjects.

The Cauchy problem (1) has also been studied from the view point of the pointwise condition of initial data. For instance, according to the result of Baras and Pierre [1, Proposition 3.2], it is shown that there exists a positive constant M such that if initial data u_0 satisfies that

$$u_0(x) \geq \begin{cases} M|x|^{-n}\Big(\log\Big(e+\dfrac{1}{x}\Big)\Big)^{-n/2-1}, & \text{if} \quad p = p_F, \\ M|x|^{-2/(p-1)}, & \text{if} \quad p > p_F \end{cases}$$

on a neighborhood of the origin, then there is no local solution to (4). We remark that

$$|x|^{-2/(p-1)} \in L^{n(p-1)/2,\infty} \backslash L^{n(p-1)/2},$$

where $L^{q,\infty}$ is the usual weak L^q space for $1 < q < \infty$. Later, Lee and Ni [12] showed the following:

Proposition 3. ([12])

1. *[Theorem 3.8] Let $p > p_F$. There exists a positive constant δ such that if*

$$0 \leq u_0(x) \leq \frac{\delta}{1+|x|^{2/(p-1)}},$$

then we have a global solution u to (2) satisfying

$$0 \leq u(t,x) \leq \frac{C_1}{1+(t+|x|^2)^{1/(p-1)}}. \tag{3}$$

2. *[Theorem 3.2] For $p > p_F$, there exists a positive constant M such that if*

$$\liminf_{|x|\to\infty} |x|^{2/(p-1)} u_0(x) \geq M,$$

then there is no non-negative global solutions to (2).

For related subjects, we refer the reader [7, 13, 14] and reference therein.

Since these conditions are described by $L^{n(p-1)/2,\infty}$ functions, it is also natural to consider the local and global existence criteria in $L^{n(p-1)/2,\infty}$. Ferreira and Villamizar-Roa [5, Thorem 3.4] showed that, if $n \geq 3$ and $p > n/(n-2) > p_F$, then for small $L^{n(p-1)/2,\infty}$ data(which may be sign changing), there exists a global solution to (2) in $BC([0,\infty); L^{n(p-1)/2,\infty})$, where BC means bounded continuous. We remark that their approach is a contraction argument based on $BC([0,\infty); L^{n(p-1)/2,\infty})$. Later, Ishige, Kawakami, and Sierżęga [10] studied supersolutions of some parabolic system and their argument implies that for any $n \geq 1$, $p > p_F$, and positive small $L^{n(p-1)/2,\infty}$ data, there exists global solution. For details, see [10, Corollary 3.1].

Here we recall that $L^{n(p-1)/2}$ and $L^{n(p-1)/2,\infty}$ arise naturally from the scaling invariance of (1). It is seen that (1) in invariant under

$$A(u,\lambda)(t,x) = \lambda^{1/(p-1)} u(\lambda t, \lambda^{1/2} x) \tag{4}$$

for any positive λ. Then we see that for any $\lambda > 0$,

$$\begin{aligned} \|A(u,\lambda)(0,\cdot)\|_{L^{n(p-1)/2}} &= \|u_0\|_{L^{n(p-1)/2}}, \\ \|A(u,\lambda)(0,\cdot)\|_{L^{n(p-1)/2,\infty}} &= \|u_0\|_{L^{n(p-1)/2,\infty}}. \end{aligned}$$

Since, $u_0(x) = |x|^{-2/(p-1)}$ is also invariant under (4), so is u. Therefore, if the solution u exists, then by choosing $\lambda = 1/t$, u is rewritten by

$$u(t,x) = t^{-1/(p-1)} g(x/t^{1/2}) \tag{5}$$

with some radial function $g(x) = G(|x|)$ satisfying

$$\begin{cases} G''(r) + \left(\dfrac{n-1}{r} + \dfrac{r}{2}\right) G'(r) + \dfrac{1}{p-1} G(r) + |G(r)|^{p-1} G(r) = 0, \\ G(0) > 0, \quad G'(0) = 0. \end{cases} \tag{6}$$

The Cauchy problem (6) is known to be well-posed. For details, see [8, Theorem 5']. This family of self-similar solutions gives supersoluition to (1), which is the main idea to study (1) with $L^{n(p-1)/2,\infty}$ data in previous works.

On the other hand, recently, Cazenave, Dickstein, Naumkin, and Weissler [3, Theorem 1.1.] showed that for $n \geq 2$, $1 < p < 1 + 4/(n-2)$ $(> p_F)$, $F(u) = |u|^{p-1}u$, and $u_0(x) = \mu |x|^{-2/(p-1)}$, one can find at least two different sign changing self-similar solutions to (1) so that for any t and x,

$$|u(t,x)| \leq C(t + |x|^2)^{-1/(p-1)}. \tag{7}$$

Therefore, one can pose the natural question to find appropriate conditions that guarantee the existence and uniqueness of sign changing solutions having general initial data in L^∞_k, i.e. we have singular initial data with singularity of order $|x|^{-k}$ near the origin.

However, in order to show the small data global existence for sign changing solutions, the construction of supersolutions may be insufficient.

In this paper, we prove the following existence result:

Proposition 4. *Let $p > p_F$, $0 \le k \le 2/(p-1)$, and $u_0 \in \dot{L}^\infty_k$ be sufficiently small. If $k = 2/(p-1)$, then we have a unique global solution to* (2) *satisfying* (7) *with sufficiently small C. If $k < 2/(p-1)$, we have a local solution to* (2) *satisfying* (7) *for any $(t,x) \in [0,T) \times \mathbb{R}^n$ with some positive T.*

We remark that since solutions of Proposition 4 satisfy (7) at some t, they also satisfy (1) in classical sense for $t \in (0,\infty)$ if $k = 2/(p-1)$ and for $t \in (0,T)$ if $k < 2/(p-1)$. In addition, in the case where $k < 2/(p-1)$, Proposition 3 (2) implies that some solutions cannot be extended globally.

We also remark that it may be possible to consider small data global existence in the $L^{n(p-1)/2,\infty}$ framework, where $\dot{L}^\infty_{-2/(p-1)} \subset L^{n(p-1)/2,\infty}$. However, in the weighted L^∞ framework, we may see that solutions are simply controlled by free solutions or self-similar solutions with $u_0(x) = |x|^{-k}$ and some positive k. We here recall that the shape of initial data is not taken into account in the $L^{n(p-1)/2,\infty}$ framework. On the other hand, in the weighted L^∞ framework, the shape of weight functions and corresponding free solutions may explain the behavior of solutions directly. Namely, Proposition 4 is a simple consequence of the following weighted L^∞ estimates.

Proposition 5. *Let $p > p_F$, $0 \le k \le 2/(p-1) < n$, $u_0 \in \dot{L}^\infty_k$, and $\widetilde{F} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$. Let u satisfy*

$$u(t,x) = e^{t\Delta}u_0(x) + \int_0^t e^{(t-t')\Delta}\widetilde{F}(t')(x)dt'. \tag{8}$$

If $k = 2/(p-1)$ and $\widetilde{F} \in (t+|x|^2)^{-k/2-1}L^\infty(0,\infty;L^\infty)$, then

$$\begin{aligned} &\|(t+|x|^2)^{k/2}u\|_{L^\infty(0,\infty;L^\infty)} \\ &\le C_0\||x|^k u_0\|_{L^\infty(\mathbb{R}^n)} + C_1\|(t+|x|^2)^{k/2+1}\widetilde{F}\|_{L^\infty(0,\infty;L^\infty)}. \end{aligned} \tag{9}$$

If $k = 2\theta/(p-1)$ with $0 < \theta < 1$, $T > 0$, and $\widetilde{F} \in (t+|x|^2)^{-k/2-\theta}L^\infty(0,T;L^\infty)$, then

$$\begin{aligned} &\|(t+|x|^2)^{k/2}u\|_{L^\infty(0,T;L^\infty)} \\ &\le C_0\||x|^k u_0\|_{L^\infty} + C_1T^{1-\theta}\|(t+|x|^2)^{k/2+\theta}\widetilde{F}\|_{L^\infty(0,T;L^\infty)}. \end{aligned} \tag{10}$$

Remark. Following the proof of the estimate (10) combined with rescaling argument one can show that for any $k > 0$

$$e^{t\Delta} : \dot{L}^\infty_k \to L^\infty_k \subset \dot{L}^\infty_k, \forall t > 0, \tag{11}$$

where

$$L_k^\infty(\mathbb{R}^n) = (1+|x|)^{-k}L^\infty(\mathbb{R}^n) = \{f \in L^\infty_{loc}(\mathbb{R}^n); (1+|x|)^k f \in L^\infty(\mathbb{R}^n)\}$$

is the inhomogeneous weighted L^∞ space.

Remark. After finishing the preparation of this work, the authors found in [4] similar results and brief sketch of the proofs. However, the proofs presented here are more detailed and self - contained. Additional remark concerns the fact that the a priori estimates used in the proof of Proposition 5 can be applied to show that $u(t,x) = e^{t\Delta} f$ is not in $C([0,1]; \dot{L}_k^\infty)$ if $f = |x|^{-k} \in \dot{L}_k^\infty$.

We remark that the weight functions, $(t+|x|^2)^{k/2}$ come from $1/(e^{t\Delta}(|\cdot|^{-k})(x))$. For details, see Lemma 1 below. The weight functions for (9) and (10) are also natural from the view point of scaling argument. Let

$$\widetilde{A}(u,\lambda)(t,x) = \lambda^{k/2} u(\lambda t, \lambda^{1/2} x)$$

and consider the differential form of (8);

$$\begin{cases} \partial_t u - \Delta u = \widetilde{F}, & t \in [0,\infty), x \in \mathbb{R}^n, \\ u(0,x) = |x|^{-k}, & t \in [0,\infty), x \in \mathbb{R}^n, \end{cases} \tag{12}$$

for $0 < k < n$. We remark that $\widetilde{A}$ coincides with A when $\widetilde{F} = F$ and $k = 2/(p-1)$. For any $\lambda > 0$,

$$\begin{aligned} \widetilde{A}(e^{t\Delta}u_0, \lambda) &= e^{t\Delta} u_0, \\ (\partial_t - \Delta)\widetilde{A}(u,\lambda) &= \lambda \widetilde{A}((\partial_t - \Delta)u, \lambda) = \lambda \widetilde{A}(\widetilde{F}, \lambda). \end{aligned} \tag{13}$$

Therefore, if solution u to (12) satisfies (8), then

$$\widetilde{A}(u,\lambda)(t,x) = e^{t\Delta}u_0(x) + \int_0^t e^{(t-t')\Delta} \lambda \widetilde{A}(\widetilde{F},\lambda)(t')(x) dt'.$$

This implies that Proposition 3 must hold with $\widetilde{A}(u,\lambda)$ and $\lambda\widetilde{A}(\widetilde{F},\lambda)$ for any $\lambda > 0$. Actually, we have

$$\begin{aligned} &\|(t+|x|^2)^{k/2} u\|_{L^\infty(0,\infty;L^\infty)} \\ &= \|(t+|x|^2)^{k/2} \widetilde{A}(u,\lambda)\|_{L^\infty(0,\infty;L^\infty)} \\ &\leq \||x|^k e^{t\Delta} u_0\|_{L^\infty} + \|(t+|x|^2)^{k/2+1} \lambda \widetilde{A}(\widetilde{F},\lambda)\|_{L^\infty(0,\infty;L^\infty)} \\ &= \||x|^k e^{t\Delta} u_0\|_{L^\infty} + \|(t+|x|^2)^{k/2+1} \widetilde{F}\|_{L^\infty(0,\infty;L^\infty)}. \end{aligned}$$

We also remark that Proposition 4 does not violate the non-uniqueness result of [3] because Proposition 4 is based on a contraction argument on a ball in $(t+|x|^2)^{1/(p-1)}L^\infty(0,\infty;L^\infty(\mathbb{R}^n))$ with sufficiently small radius. Therefore we have other solutions outside of our ball. This phenomena can be restated as the uniqueness of small solutions to (6), which is the following Corollary of Proposition 4:

Corollary 1. *There exist $\delta_0, \delta_1 > 0$ such that for any $0 < G(0) < \delta_0$, there exists a unique solution G to* (6) *satisfying*

$$|G(x)| \le \delta_1(1+|x|^2)^{-1/(p-1)}.$$

Proposition 5 is a conclusion of pointwise estimates for 0-th and first iteration of the self-similar solution (5). We introduce them and prove Proposition 3 in the next section. In Section 3, we show Proposition 2. Section 4 is devoted for the proof of Corollary 1.

2. Proof of Proposition 3

Lemma 1. (0-th Iteration) *For $0 \le k < n$, any $t \ge 0$, and $x \in \mathbb{R}^n$,*

$$e^{t\Delta}(|\cdot|^{-k})(x) \sim t^{-k/2}(1+|x|^2/t)^{-k/2}.$$

Especially, there exists a positive constant C_0 such that

$$e^{t\Delta}(|\cdot|^{-k})(x) \le C_0(t+|x|^2)^{-k/2}.$$

Proof. By (13), it is enough to show

$$e^{\Delta}(|\cdot|^{-k})(\xi) \sim (1+|\xi|^2)^{-k/2}.$$

For $|\xi| \le 1$,

$$\int_{|y|\le 2} e^{-|\xi-y|^2}|y|^{-k}dy \le \int_{|y|\le 2} |y|^{-k}dy \le \omega_{n-1}(n-k)^{-1}2^{n-k},$$

$$\int_{|y|\le 2} e^{-|\xi-y|^2}|y|^{-k}dy \ge \omega_{n-1}(n-k)^{-1}e^{-9}2^{n-k},$$

$$\int_{|y|\ge 2} e^{-|\xi-y|^2}|y|^{-k}dy \le \int_{|y|\ge 2} e^{-|y|^2/4}|y|^{-k}dy,$$

$$\int_{|y|\ge 2} e^{-|\xi-y|^2}|y|^{-k}dy \ge \int_{|y|\ge 2} e^{-4|y|^2}|y|^{-k}dy,$$

where ω_{n-1} is the volume of the unit sphere S_{n-1}. For $|\xi| \ge 1$,

$$\int_{|\xi|\ge 2|y|} e^{-|y|^2}|\xi-y|^{-k}dy \le 2^k|\xi|^{-k}\int_{\mathbb{R}^n} e^{-|y|^2}dy,$$

$$\begin{aligned}
\int_{|\xi|\geq 2|y|} e^{-|y|^2}|\xi-y|^{-k}dy &\geq 3^{-k}2^k|\xi|^{-k}\int_{|y|<1/2} e^{-|y|^2}dy,\\
\int_{|y|\geq 2|\xi|} e^{-|y|^2}|\xi-y|^{-k}dy &\leq 2^k\int_{|y|\geq 2|\xi|} e^{-|y|^2}|y|^{-k}dy\\
&\leq |\xi|^{-k}\int_{\mathbb{R}^n} e^{-|y|^2}dy,\\
\int_{|y|\geq 2|\xi|} e^{-|y|^2}|\xi-y|^{-k}dy &\geq 5^{-k}|\xi|^{-k}\int_{2|\xi|\leq|y|\leq 4|\xi|} e^{-|y|^2}dy,\\
\int_{|\xi|/2\leq|y|\leq 2|\xi|} e^{-|y|^2}|\xi-y|^{-k}dy &\leq e^{-|\xi|^2/4}\int_{|z|\leq 3|\xi|}|z|^{-k}dz\\
&\leq \omega_{n-1}(n-k)^{-1}3^{n-k}e^{-|\xi|^2/4}|\xi|^{n-k},\\
\int_{|\xi|/2\leq|y|\leq 2|\xi|} e^{-|y|^2}|\xi-y|^{-k}dy &\geq \omega_{n-1}(n-k)^{-1}2^{k-n}e^{-4|\xi|^2}|\xi|^{n-k}.
\end{aligned}$$

This proves Lemma 1. □

Lemma 2. (First Iteration) *For $0 < q < 1+n/2$, any $t \geq 0$, and $x \in \mathbb{R}^n$,*

$$\int_0^t e^{(t-t')\Delta}(t'+|\cdot|^2)^{-q}dt'(x) \leq C_1 t(t+|x|^2)^{-q}$$

Proof. We divide the integral domain into 5 parts.

At first,

$$\begin{aligned}
&\int_{t/2}^t\int_{|y|\leq|x|/2}(t-t')^{-n/2}e^{-|y|^2/(t-t')}(t'+|x-y|^2)^{-q}dydt'\\
&\lesssim (t+|x|^2)^{-q}\int_{t/2}^t\int_{\mathbb{R}^n} e^{-|y|^2}dydt'\\
&\lesssim t(t+|x|^2)^{-q}.
\end{aligned}$$

At second,

$$\begin{aligned}
&\int_{t/2}^t\int_{|y|\geq|x|/2}(t-t')^{-n/2}e^{-|y|^2/(t-t')}(t'+|x-y|^2)^{-q}dydt'\\
&\lesssim t^{-q}\int_{t/2}^t\int_{|y|\geq|x|/2(t-t')^{1/2}} e^{-|y|^2}dy\,dt'\\
&\lesssim t^{1-q}\int_{|y|\geq|x|/(2t)^{1/2}} e^{-2|y|^2}dy
\end{aligned}$$

$$
\begin{aligned}
&\lesssim t^{1-q} e^{-|x|^2/t} \\
&\lesssim t(t+|x|^2)^{-q}.
\end{aligned}
$$

At third, if $|x| \geq t^{1/2}$,

$$
\begin{aligned}
&\int_0^{t/2} \int_{|y| \leq |x|/2} (t-t')^{-n/2} e^{-|y|^2/(t-t')} (t' + |x-y|^2)^{-q} dy dt' \\
&\lesssim \int_0^{t/2} \int_{|y| \leq |x|/4} (t-t')^{-n/2} e^{-|y|^2/(t-t')} |x|^{-2q} dy dt' \\
&\lesssim t^{1-q} (|x|^2/t)^{-q} \lesssim t(t+|x|^2)^{-q}.
\end{aligned}
$$

Otherwise, by changing $t^{1/2}$ as τ,

$$
\begin{aligned}
&\int_0^{t/2} \int_{|y| \leq |x|/2} (t-t')^{-n/2} e^{-|y|^2/(t-t')} (t' + |x-y|^2)^{-q} dy dt' \\
&\lesssim t^{-n/2} \int_0^{t/2} \int_{|y| \leq |x|/2} (t' + |y|^2)^{-q} dy dt' \\
&\lesssim t^{-n/2} \int_0^{t^{1/2}} \int_{|y| \leq |x|/2} (\tau^2 + |y|^2)^{-q} \tau dy d\tau \\
&\lesssim t^{-n/2} \int_0^{(t+|x|^2)^{1/2}} \rho^{-2q+n+1} d\rho \\
&\lesssim t^{-n/2} (t+|x|^2)^{-q+n/2+1} \lesssim t(t+|x|^2)^{-q}.
\end{aligned}
$$

At fourth,

$$
\begin{aligned}
&\int_0^{t/2} \int_{|x|/2 \leq |y| \leq 2|x|} (t-t')^{-n/2} e^{-|y|^2/(t-t')} (t' + |x-y|^2)^{-q} dy dt' \\
&\lesssim t^{-n/2} e^{-|x|^2/4t} \int_0^{t/2} \int_{|z| \leq 3|x|} (t' + |z|^2)^{-q} dz dt' \\
&\lesssim t^{-n/2} e^{-|x|^2/4t} (t+|x|^2)^{-q+n/2+1} \lesssim t(t+|x|^2)^{-q},
\end{aligned}
$$

where we have used the fact that

$$
e^{-r} \lesssim (1+r)^{-n/2-1}.
$$

At last,

$$
\int_0^{t/2} \int_{|y| \geq 2|x|} (t-t')^{-n/2} e^{-|y|^2/(t-t')} (t' + |x-y|^2)^{-q} dy dt'
$$

$$
\begin{aligned}
&\lesssim t^{-n/2}\int_0^{t/2}\int_{|y|\geq 2|x|} e^{-(t'+|y|^2)/t}(t'+|y|^2)^{-q}dydt' \\
&\lesssim t^{-n/2}\int_{\rho\geq|x|} e^{-\rho^2/t}\rho^{-q+n/2+1}d\rho \\
&\lesssim t^{-q+1}\int_{\rho\geq|x|/t} e^{-\rho^2}\rho^{-q+n/2+1}d\rho \\
&\lesssim t^{-q+1}(1+|x|^2/t)^{-q}.
\end{aligned}
$$

This proves Lemma 2. $\square$

From now on, let

$$
\begin{aligned}
X &= (t+|x|^2)^{-k}L^\infty(0,\infty;L^\infty), & Y &= L^\infty(0,\infty;L^\infty),\\
X'_T &= (t+|x|^2)^{-k}L^\infty(0,T;L^\infty), & Y'_T &= L^\infty(0,T;L^\infty).
\end{aligned}
$$

P r o o f o f P r o p o s i t i o n 3. By Lemma 1, for $0\leq k<n$,

$$
\begin{aligned}
|(t+|x|^2)^{k/2}e^{t\Delta}u_0(x)| &\leq \||x|^k u_0\|_{L^\infty}(t+|x|^2)^{k/2}e^{t\Delta}(|\cdot|^{-k})(x)\\
&\leq C_0\||x|^k u_0\|_{L^\infty}.
\end{aligned}
$$

By Lemma 2,

$$
\begin{aligned}
&(t+|x|^2)^{k/2}\left|\int_0^t e^{(t-t')\Delta}F(t')dt'(x)\right|\\
&\leq \|(t+|x|^2)^{k/2+1}F\|_Y(t+|x|^2)^{k/2}\int_0^t e^{(t-t')\Delta}(t'+|\cdot|^2)^{-k/2-1}(x)dt',\\
&\leq C_1\|(t+|x|^2)^{k/2+1}F\|_Y t(t+|x|^2)^{-1}.
\end{aligned}
$$

Moreover, if $0\leq t\leq T$, with $0\leq\theta<1$,

$$
\begin{aligned}
&(t+|x|^2)^{k/2}\left|\int_0^t e^{(t-t')\Delta}F(t')dt'(x)\right|\\
&\leq \|(t+|x|^2)^{k/2+\theta}F\|_{Y'_T}(t+|x|^2)^{k/2}\int_0^t e^{(t-t')\Delta}(t'+|\cdot|^2)^{-k/2-\theta}(x)dt',\\
&\leq C_1\|(t+|x|^2)^{k/2+\theta}F\|_{Y'_T}t(t+|x|^2)^{-\theta}\\
&\leq C_1T^{1-\theta}\|(t+|x|^2)^{k/2+\theta}F\|_{Y'_T}.
\end{aligned}
$$

This proves Proposition 3. $\square$

3. Proof of Proposition 2

At first, assume $k = 2/(p-1)$. Put $\||x|^k u_0\|_{L^\infty} = \varepsilon$. If

$$\varepsilon < \sup_{1<L<p'} \left(\frac{L-1}{C_0^{p-1} C_1 L^p} \right)^{1/(p-1)}, \tag{14}$$

we have $1 < L < p'$ such that

$$C_0 L \varepsilon = C_0 \varepsilon + C_1 C_0^p L^p \varepsilon^p.$$

Let

$$\Phi(u_0, u) = e^{t\Delta} u_0 + \int_0^t e^{(t-t')\Delta} F(u(t')) dt',$$

where we recall $F(u) = |u|^{p-1}u$ or $|u|^p$. By (3),

$$\begin{aligned} \|\Phi(u_0, u)\|_X &\leq C_0 \||x|^k u_0\|_{L^\infty(\mathbb{R}^n)} + C_1 \|(t + |x|^2)^{k/2+1} |u|^p\|_Y \\ &= C_0 \||x|^k u_0\|_{L^\infty(\mathbb{R}^n)} + C_1 \|(t + |x|^2)^{pk/2} |u|^p\|_Y \\ &\leq C_0 \||x|^k u_0\|_{L^\infty(\mathbb{R}^n)} + C_1 \|u\|_X^p. \end{aligned}$$

Then $\Phi(u_0, \cdot)$ is a contraction map on

$$B(LC_0\varepsilon) = \{v;\ \|v\|_X \leq LC_0\varepsilon\}.$$

Moreover,

$$\begin{aligned} ||u|^p - |v|^p| &\leq p \int_0^1 (\sigma|u| + (1-\sigma)|v|)^{p-1} d\sigma ||u| - |v|| \\ &\leq p \max(|u|, |v|)^{p-1} |u - v| \end{aligned}$$

and

$$\begin{aligned} &||u|^{p-1}u - |v|^{p-1}v| \\ &\leq \max(|u|, |v|)^{p-1} |u - v| + (|u|^{p-1} - |v|^{p-1}) \min(|u|, |v|) \\ &\leq \max(|u|, |v|)^{p-1} |u - v| + (p-1) \int_0^1 (\sigma|u| + (1-\sigma)|v|)^{p-2} d\sigma \min(|u|, |v|) ||u| - |v|| \\ &\leq p \max(|u|, |v|)^{p-1} |u - v|. \end{aligned}$$

Therefore, there exists $\mu < 1$ such that

$$\|\Phi(u_0, u) - \Phi(u_0, v)\|_X \leq pC_1 (LC_0\epsilon)^{p-1} \|u - v\|_X$$

$$= \frac{(L-1)p}{L}\|u-v\|_X \leq \mu\|u-v\|_X,$$

where we have used the fact that L can be given by $p/(p-\mu) < p'$ by (14).

Assume $k = 2\theta/(p-1)$ for $0 \leq \theta < 1$. By (10),

$$\begin{aligned}\|\Phi(u_0,u)\|_{X_1'} &\leq C_0\||x|^k u_0\|_{L^\infty} + C_1 T^{1-\theta}\|(t+|x|^2)^{k/2+\theta}|u|^p\|_{Y_1'} \\ &= C_0\||x|^k u_0\|_{L^\infty} + C_1 T^{1-\theta}\|(t+|x|^2)^{pk/2}|u|^p\|_{Y_1'} \\ &\leq C_0\||x|^k u_0\|_{L^\infty} + C_1 T^{1-\theta}\|u\|_{X_1'}^p.\end{aligned}$$

Then $\Phi(u_0,\cdot)$ has a fixed point u as in the previous case.

4. Proof of Corollary 1

We show only the uniqueness. Let $Z = (1+|x|^2)^{1/(p-1)}L^\infty(\mathbb{R}^n)$. (5) implies that if G is a solution to (6), then $u_G(t,x) = t^{-1/(p-1}G(|x|/t^{1/2})$ satisfies (1) globally in time. Let G_1 and G_2 satisfy (6) with initial data $G(0)$ and that

$$\max(\|G_1\|_Z, \|G_2\|_Z) \leq \mu(pC_1)^{-1/(p-1)}.$$

with $0 < \mu < 1$. Then by Proposition 4,

$$\begin{aligned}\|G_1 - G_2\|_Z &= \|u_{G_1} - u_{G_2}\|_X \\ &= C_1\|u_{G_1}^p - u_{G_2}^p\|_X \\ &\leq pC_1 \max(\|u_{G_1}\|_X, \|u_{G_2}\|_X)^{p-1}\|u_{G_1} - u_{G_2}\|_X \\ &\leq \mu\|G_1 - G_2\|_Z.\end{aligned}$$

Therefore, $G_1 = G_2$.

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K. Fujiwara
Centro di Ricerca Matematica Ennio De Giorgi
Scuola Normale Superiore
Piazza dei Cavalieri, 3, 56126 Pisa, Italy
e-mail: `kazumasa.fujiwara@sns.it`

V. Georgiev
Dipartimento di Matematica, Università di Pisa
Largo B. Pontecorvo 5, 56100 Pisa, Italy
and
Faculty of Science and Engineering, Waseda University
3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
and
Institute of Mathematics and Informatics–BAS
Acad. G. Bonchev Str., Block 8, 1113 Sofia, Bulgaria
e-mail: `georgiev@dm.unipi.it`

T. Ozawa
Department of Applied Physics, Waseda University
3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
e-mail: `txozawa@waseda.jp`