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RECENT DEVELOPMENTS OF THE METHODOLOGY OF THE MODIFIED METHOD OF SIMPLEST EQUATION WITH APPLICATION

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We discuss an extension of the modified method of simplest equation for obtaining exact analytical solutions of nonlinear partial differential equations. The extension includes the possibility for use of: (i) more than one simplest equation; (ii) relationship that contains as particular cases the relationship used by Hirota [1] and the relationship used in the previous version of the methodology; (iii) transformation of the solution that contains as particular case the possibility of use of the Painlevé expansion; (iv) more than one balance equation. The discussed version of the methodology allows: obtaining multi-soliton solutions of nonlinear partial differential equations if such solutions do exist and obtaining particular solutions of nonintegrable nonlinear partial differential equations. Examples for the application of the methodology are discussed.

1. Introduction

Differential equations occur in the process of mathematical study of many problems from natural and social sciences as these equations relate quantities to their changes and such relationships are frequently encountered. Nonlinear differential equations are used for modeling of processes in many branches of science such as fluid mechanics, atmospheric and ocean sciences, mathematical biology, social dynamics, etc. [2]–[5]. In many cases the model equations are nonlinear partial

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differential equations and their exact solutions help us to understand complex nonlinear phenomena such as existence and change of different regimes of functioning of complex systems, spatial localization, etc. Because of the above the exact solutions of nonlinear partial differential equations are studied intensively [6]–[14]. In the yearly years of the research on the methodology for obtaining exact solutions of nonlinear partial differential equations one has searched for transformations that can transform the solved nonlinear partial differential equation to a linear differential equation. Numerous attempts for obtaining such transformations have been made and in 1967 Gardner, Green, Kruskal and Miura [9] managed to connect the Korteweg – de Vries equation to the inverse scattering problem for the linear Schödinger equation. This methodology is known today as *Method of Inverse Scattering Transform*, [6]. Below we are interested in another line of research that was followed by Hirota who developed a direct method for obtaining such exact solutions – *Hirota method* [1], [15]. Hirota method is based on bilinearization of the solved nonlinear partial differential equation by means of appropriate transformations. Truncated Painleve expansions may lead to many of these appropriate transformations [14], [16], [17] and the study of the applications of these truncated expansions [18] led to the formulation of the *Method of Simplest Equation (MSE)* [19], [20]. We refer to the articles of Kudryashov and co-authors for further results connected to *MSE* [21]–[29].

I have started my work on the method of simplest equation by proposing the use of the ordinary differential equation of Bernoulli as simplest equation [30] and by application of the method to ecology and population dynamics [31] where the concept of the balance equation has been used. Today the method of simplest equation has two versions. The original version of Kudryashov is called *Method of Simplest Equation – MSE* and there the determination of the truncation of the corresponding series of solutions of the simplest equation is based on the first step in the algorithm for detection of the Painleve property. An equivalent version is called *Modified Method of Simplest Equation – MMSE* or *Modified Simple Equation Method – MSEM* [20], [32], [33]. It is based on determination of the kind of the simplest equation and truncation of the series of solutions of the simplest equation by means of application of a balance equation. Up to now our contributions to the methodology and its application are connected to this version of the method [34]–[39] and in [40] where we have extended the methodology of the *MMSE* to simplest equations of the class

$$(1) \quad \left(\frac{d^k g}{d\xi^k} \right)^l = \sum_{j=0}^m d_j g^j$$

where $k = 1, \dots, l = 1, \dots$, and m and d_j are parameters. The solution of Eq. (1) defines a special function that contains as particular cases, e.g.: trigonometric functions; hyperbolic functions; elliptic functions of Jacobi; elliptic function of Weierstrass. Our goal is to extend the methodology of *MSE* and *MMSE* in order to make it applicable to larger classes of nonlinear partial differential equations.

The text below is organized as follows. In Sect. 2 we discuss a version the modified method of simplest equation that makes the methodology capable to obtain multi-soliton solutions of nonlinear partial differential equations. Sect.3 contains examples of applications of the method. Several concluding remarks are given in Sect. 4.

2. Extended version of the modified method of simplest equation

In the previous version of the method we have used a representation of the searched solution of a nonlinear partial differential equation as power series of a solution of a simplest equation. This approach does not work for the case of search for bisoliton, trisoliton, and multisoliton solutions because the previous version of the modified method of simplest equation was connected to the use of a single simplest equation. If we allow for use of more than one simplest equation then the modified method of simplest equation can be formulated in a way that makes obtaining of multisoliton solutions possible. Below we formulate such a version of the modified method of simplest equation. The schema of the old and the new version of the methodology is shown in Fig. 1.

Let us consider a nonlinear partial differential equation

$$(2) \quad \mathcal{R}(u, \dots) = 0$$

where $\mathcal{R}(u, \dots)$ depends on the function $u(x, \dots, t)$ and some of its derivatives participate in (u can be a function of more than 1 spatial coordinate). The 7 steps of the extended modified method of simplest equation are as follows.

Step 1.) We apply a transformation

$$(3) \quad u(x, \dots, t) = H(F(x, \dots, t))$$

where $H(F)$ is some function of another function F . In general $F(x, \dots, t)$ is a function of the spatial variables as well as of the time. The transformation $H(F)$ may be the Painleve expansion [1], [11], [18], [41], [42] or another transformation, e.g., $u(x, t) = 4 \tan^{-1}[F(x, t)]$ for the case of the sine – Gordon equation, etc. [43]—[47]. In many particular cases one may skip this step (then we have just $u(x, \dots, t) = H(x, \dots, t)$) but in some cases the step is necessary for obtaining

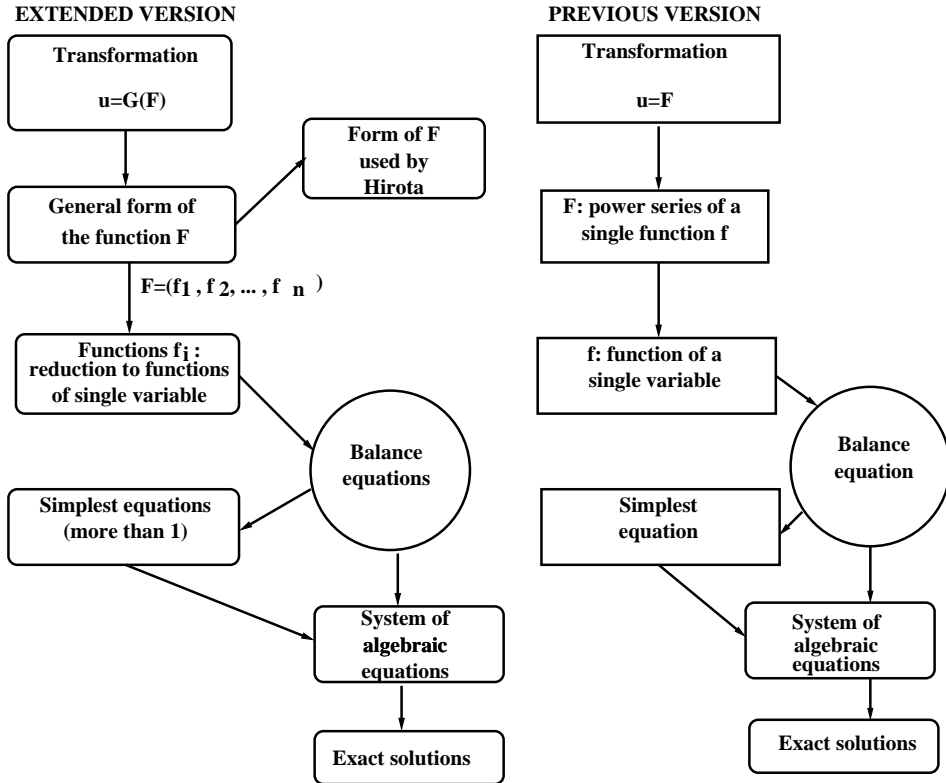


Figure 1: Schema of the previous version (right-hand side) and of the extended version (left-hand side) of the Modified method of simplest equation. The transformation used in the extended version of the method contains as particular case the transformation used in the previous version of the method. The form of the function F used in the extended version of the method contains as particular case the form of the function used in the previous version of the method. The new form of the function F contains as particular case also the form of the function used by Hirota. In the extended form of the method more than one simplest equation can be used and more than one balance equation can arise. In the previous version of the method one uses one simplest equation and one balance equation.

a solution of the studied nonlinear PDE. The application of Eq. (3) to Eq. (2) leads to a nonlinear PDE for the function $F(x, \dots, t)$.

Step 2.) The function $F(x, \dots, t)$ is represented as a function of other func-

tions f_1, \dots, f_N ($N = 1, 2, \dots$) that are connected to solutions of some differential equations (these equations can be partial or ordinary differential equations) that are more simple than Eq. (2). We stress that the forms of the function $F(f_1, \dots, f_N)$ can be different. One example is

$$(4) \quad F = \alpha + \sum_{i_1=1}^N \beta_{i_1} f_{i_1} + \sum_{i_1=1}^N \sum_{i_2=1}^N \gamma_{i_1, i_2} f_{i_1} f_{i_2} + \dots + \sum_{i_1=1}^N \dots \sum_{i_N=1}^N \sigma_{i_1, \dots, i_N} f_{i_1} \dots f_{i_N}$$

where $\alpha, \beta_{i_1}, \gamma_{i_1, i_2}, \sigma_{i_1, \dots, i_N} \dots$ are parameters. The relationship (4) contains as particular case the relationship used by Hirota [1]. The power series $\sum_{i=0}^N \mu_n f^n$ (where μ is a parameter) used in the previous versions of the methodology of the modified method of simplest equation are a particular case of the relationship (4) too.

Step 3.) In general the functions f_1, \dots, f_N are solutions of partial differential equations. By means of appropriate ansätze (e.g., traveling-wave ansätze such as $\xi = \hat{\alpha}x + \hat{\beta}t$; $\zeta = \hat{\gamma}x + \hat{\delta}t$, $\eta = \hat{\mu}y + \hat{\nu}t \dots$) the solved differential equations for f_1, \dots, f_N may be reduced to differential equations D_l , containing derivatives of one or several functions

$$(5) \quad D_l [a(\xi), a_\xi, a_{\xi\xi}, \dots, b(\zeta), b_\zeta, b_{\zeta\zeta}, \dots] = 0; \quad l = 1, \dots, N$$

If the equations for the functions f_1, \dots are ordinary differential equations one may skip this step but the step may be necessary if the equations for f_1, \dots are partial differential equations.

Step 4.) We assume that the functions $a(\xi)$, $b(\zeta)$, etc., are functions of other functions, e.g., $v(\xi)$, $w(\zeta)$, etc., i.e.

$$(6) \quad a(\xi) = A[v(\xi)]; \quad b(\zeta) = B[w(\zeta)]; \dots$$

We note that the functions A , B , \dots are not prescribed. One may use a finite-series relationship, e.g.,

$$(7) \quad a(\xi) = \sum_{\mu_1=-\nu_1}^{\nu_2} q_{\mu_1} [v(\xi)]^{\mu_1}; \quad b(\zeta) = \sum_{\mu_2=-\nu_3}^{\nu_4} r_{\mu_2} [w(\zeta)]^{\mu_2}, \dots$$

(where $q_{\mu_1}, r_{\mu_2}, \dots$ are coefficients) but other kinds of relationships may be used too.

Step 5.) The functions $v(\xi)$, $w(\zeta)$, ... are solutions of simpler ordinary differential equations called *simplest equations*, i.e., the extended version of the methodology allows for the use of more than one simplest equation.

Step 6.) The application of the steps 1.)–5.) to Eq. (2) transforms the left-hand side of this equation. Let the result of this transformation be a function that is a sum of terms where each term contains some function multiplied by a coefficient. This coefficient contains some of the parameters of the solved equation and some of the parameters of the solution. In the most cases a balance procedure must be applied in order to ensure that the above-mentioned relationships for the coefficients contain more than one term. This balance procedure may lead to one or more additional relationships *balance equations* among the parameters of the solved equation and parameters of the solution.

Step 7.) We may obtain a nontrivial solution of Eq. (2) if all coefficients mentioned in Step 6.) are set to 0. This condition usually leads to a system of nonlinear algebraic equations for the coefficients of the solved nonlinear PDE and for the coefficients of the solution. Any nontrivial solution of the above algebraic system leads to a solution the studied nonlinear partial differential equation.

3. Examples

3.1. The most simple example: bisoliton solution of the Korteweg-de Vries equation

We shall describe this example very briefly in order to show the capacity of the extended version of the methodology to lead to multi-soliton solutions of integrable partial differential equations. We consider a version of the Korteweg – de Vries equation

$$(8) \quad u_t + \sigma uu_x + u_{xxx} = 0$$

where σ is a parameter. The 7 steps of the application of the version of the modified method of simplest equation from Sect. 2 are as follows.

Step 1.) *The transformation*

We set $u = p_x$ in Eq. (8). The result is integrated and we apply the transformation $p = \frac{12}{\sigma}(\ln F)_x$. The result is

$$(9) \quad FF_{tx} + FF_{xxxx} - F_tF_x + 3F_{xx}^2 - 4F_xF_{xxx} = 0$$

Step 2.) *Relationship among $F(x, t)$ and two functions $f_{1,2}$ that will be connected below to two simplest equations*

We shall use two functions $f_1(x, t)$ and $f_2(x, t)$ and the relationship for F is assumed to be a particular case of Eq. (4):

$$(10) \quad F(x, t) = 1 + f_1(x, t) + f_2(x, t) + cf_1(x, t)f_2(x, t)$$

where c is a parameter. The substitution of Eq. (10) in Eq. (9) leads to a nonlinear partial differential equation containing 64 terms.

Step 3.) *Equations for the functions $f_1(x, t)$ and $f_2(x, t)$*

The structure of the obtained allow us to assume a the simple form of the equations for the functions $f_{1,2}$:

$$(11) \quad \frac{\partial f_1}{\partial x} = \alpha_1 f_1; \quad \frac{\partial f_1}{\partial t} = \beta_1 f_1; \quad \frac{\partial f_2}{\partial x} = \alpha_2 f_2; \quad \frac{\partial f_2}{\partial t} = \beta_2 f_2;$$

Eq. (11) transforms the solved nonlinear partial differential equation to a polynomial of f_1 and f_2 . Further we assume that $\xi = \alpha_1 x + \beta_1 t + \gamma_1$ and $\zeta = \alpha_2 x + \beta_2 t + \gamma_2$ and $f_1(x, t) = a(\xi)$; $f_2(x, t) = b(\zeta)$ where $\alpha_{1,2}$, $\beta_{1,2}$ and $\gamma_{1,2}$ are parameters.

Step 4.) *Relationships connecting $a(\xi)$ and $b(\zeta)$ to the functions $v(\xi)$ and $w(\zeta)$ that are solutions of the simplest equations*

In the discussed here case the relationships are quite simple. We can use Eq. (16) for the cases $\mu_1 = \nu_2 = 1$ and $\mu_2 = \nu_4 = 1$. The result is: $a(\xi) = q_1 v(\xi)$; $b(\zeta) = r_1 w(\zeta)$.

Step 5.) *Simplest equations for $v(\xi)$ and $w(\zeta)$*

The simplest equations are

$$(12) \quad \frac{dv}{d\xi} = v; \quad \frac{dw}{d\zeta} = w$$

and the corresponding solutions are $v(\xi) = \omega_1 \exp(\xi)$; $w(\zeta) = \omega_2 \exp(\zeta)$. Below we shall omit the parameters $\omega_{1,2}$ as they can be included in the parameters q_1 and r_1 respectively. We shall omit also q_1 and r_1 as they can be included in ξ and ζ .

Step 6.) *Transformation of the nonlinear PDE that contains 64 terms*

The substitution of all above in the nonlinear partial differential equation that contains 64 terms leads to a sum of exponential functions and each exponential function is multiplied by a coefficient. Each of these coefficients is a relationship containing the parameters of the solution and all of the relationships contain more than one term. Thus we don't need to perform a balance procedure.

Step 7.) *Obtaining and solving the system of algebraic equations*

The system of algebraic equations is obtained by setting of above-mentioned relationships to 0. Thus we obtain the following system:

$$\alpha_1^3 + \beta_1 = 0, \quad \alpha_2^3 + \beta_2 = 0,$$

$$(13) \quad (c+1)\alpha_1^4 + 4\alpha_2(c-1)\alpha_1^3 + 6\alpha_2^2(c+1)\alpha_1^2 + [(4c-4)\alpha_2^3 + (\beta_1 + \beta_2)c + \beta_1 - \beta_2]\alpha_1 + [(c+1)\alpha_2^3 + (\beta_1 + \beta_2)c - \beta_1 + \beta_2]\alpha_2 = 0.$$

The non-trivial solution of this system is: $\beta_1 = -\alpha_1^3$; $\beta_2 = -\alpha_2^3$; $c = \frac{(\alpha_1 - \alpha_2)^2}{(\alpha_1 + \alpha_2)^2}$ and the corresponding solution of Eq. (8) is

$$(14) \quad u(x, t) = \frac{12}{\sigma} \frac{\partial^2}{\partial x^2} \left[1 + \exp(\alpha_1 x - \alpha_1^3 t + \gamma_1) + \exp(\alpha_2 x - \alpha_2^3 t + \gamma_2) + \frac{(\alpha_1 - \alpha_2)^2}{(\alpha_1 + \alpha_2)^2} \exp((\alpha_1 + \alpha_2)x - (\alpha_1^3 + \alpha_2^3)t + \gamma_1 + \gamma_2) \right]$$

Eq. (14) describes the bisoliton solution of the Korteweg – de Vries equation.

3.2. Second example: the generalized Maxwell-Cataneo equation

As a second example we shall consider a nonlinear partial differential equation that is a generalization of the Maxwell-Cataneo kind of equation

$$(15) \quad u_t + ru^a u_{tt} = pu^{b-1} u_x^2 + qu^b u_{xx}$$

The original Maxwell-Cataneo kind of equation is obtained when $a = 0$, $b = 1$ [48]. We follow the methodology for the case $u(x, t) = H[F(x, t)] = F(x, t)$. and search for traveling waves by using the traveling-wave ansatz $F(x, t) = F(\xi) = F(\alpha x + \beta t)$ that reduces the solved nonlinear PDE to a nonlinear ODE. Then the solution $F(\xi)$ is searched as some function of another function $f(\xi)$, i.e.,

$$(16) \quad F(\xi) = \sum_{\mu=-\nu}^{\nu_1} p_\mu [f(\xi)]^\mu,$$

p_μ are coefficients and $f(\xi)$ is a solution of simpler ordinary differential equation (the simplest equation):

$$(17) \quad f_\xi = n \left[f^{(n-1)/n} - f^{(n+1)/n} \right],$$

where n is an appropriate positive real number. The solution of this equation is $f(\xi) = \tanh^n(\xi)$. n must be such real number that $\tanh^n(\xi)$ exists for $\xi \in (-\infty, +\infty)$ ($n = 1/5$ is an appropriate value for n and $n = 1/4$ is not an appropriate value for n). Following the steps of the methodology we obtain two

balance equations: $a = b = \frac{1}{n}$, and the nonlinear partial differential equation (15) is reduced to the system of nonlinear algebraic equations

$$(18) \quad \begin{cases} r(n+1)\beta^2 - [(p+q)n+q]\alpha^2 = 0 \\ \left\{ r(n-1)\beta^2 - \left[\left((p+q)n - q \right) \right] \alpha^2 \right\} \delta^{1/n} + \beta = 0 \\ \left[(p+q)\alpha^2 - r\beta^2 \right] n\delta^{1/n} - \frac{1}{2}\beta = 0. \end{cases}$$

One non-trivial solution of this system is

$$(19) \quad \alpha = \frac{1}{2npr^{1/2}\delta^{1/n}} \left[(n+1)(np+nq+q) \right]^{1/2}; \quad \beta = \frac{np+nq+q}{2npr\delta^{1/n}},$$

and the corresponding solution of Eq. (15) is

$$(20) \quad u(x, t) = \delta \tanh^n \left\{ \frac{1}{2npr^{1/2}\delta^{1/n}} \left[\left((n+1)(np+nq+q) \right)^{1/2} x + \frac{np+nq+q}{r^{1/2}} t \right] \right\}.$$

Several particular cases are as follows. For $n = 1$: the equation

$$(21) \quad u_t + ruu_{tt} = pu_x^2 + quu_{xx},$$

has the solution

$$(22) \quad u(x, t) = \delta \tanh \left\{ \frac{1}{2pr^{1/2}\delta} \left[\left(2(p+2q) \right)^{1/2} x + \frac{p+2q}{r^{1/2}} t \right] \right\}.$$

For $n = 2$: the equation

$$(23) \quad u_t + ru^{1/2}u_{tt} = pu^{-1/2}u_x^2 + qu^{1/2}u_{xx},$$

has the solution

$$(24) \quad u(x, t) = \delta \tanh^2 \left\{ \frac{1}{4pr^{1/2}\delta^{1/2}} \left[\left(3(2p+3q) \right)^{1/2} x + \frac{2p+3q}{r^{1/2}} t \right] \right\}.$$

Finally let $n = 1/3$. Then the equation

$$(25) \quad u_t + ru^3u_{tt} = pu^2u_x^2 + qu^3u_{xx},$$

has the solution

$$(26) \quad u(x, t) = \delta \tanh^{1/3} \left\{ \frac{3}{2pr^{1/2}\delta^3} \left[\left(\frac{4}{9}(p+4q) \right)^{1/2} x + \frac{p+4q}{3r^{1/2}} t \right] \right\}.$$

4. Concluding remarks

Above we have discussed an extended version of the methodology of the modified method of simplest equation. The extension is based on the possibility of use of more than one simplest equation, on a transformation connected to the searched solution and on a possibility of use of more general relationship among the solution of the solved nonlinear partial differential equation and the solutions of the simplest equations. These possibilities add the capability for obtaining multisolitons to the discussed extended methodology by keeping its ability to lead to particular exact solutions of nonintegrable nonlinear partial differential equations. Two examples of application of the methodology are presented and it is demonstrated that the balance procedure can lead to more than one balance equation.

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