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EXISTENCE OF CLASSICAL SOLUTIONS OF LINEAR NON-COOPERATIVE ELLIPTIC SYSTEMS

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In this paper is considered the solvability in classical $C^2(\Omega)\bigcap C(\overline{\Omega})$ sense of linear non-cooperative weakly coupled systems of elliptic second-order PDE. It is based on the validity of comparison principle and method of sub- and super-solutions. The existence theorem is proved firstly for competitive systems and then for general non-cooperative ones.

1. Introduction

In this paper is studied the solvability of linear non-cooperative weakly coupled systems of elliptic second-order PDE. The cooperative case is proved in general case in [2], and many authors build solutions for particular systems, for instance in [7]. In non-cooperative case existence theorem is proved firstly for competitive systems and then for general non-cooperative ones. The main purpose of this article is to give a new, correct prove of existence theorem in [3].

Let $\Omega \in R^n$ be a bounded domain with smooth boundary $\partial\Omega$. In this paper are considered weakly coupled linear elliptic systems of the form

$$L_M u = f(x) \text{ in } \Omega \tag{1}$$

and boundary data

$$u(x) = g(x) \text{ on } \partial\Omega, \tag{2}$$

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where $L_M = L + M$, L is a matrix operator with null off-diagonal elements $L = \operatorname{diag}(L_1, L_2, \dots, L_N)$, and matrix $M = \{m_{ki}(x)\}_{k,i=1}^N$. Scalar operators

$$L_k u^k = -\sum_{i,j=1}^n D_j\left(a_{ij}^k(x) D_i u^k\right) + \sum_{i=1}^n b_i^k(x) D_i u^k + c^k u^k \ \texttt{in} \ \Omega$$

are supposed uniformly elliptic ones for $k = 1, 2, \dots, N$, i.e. there are constants $\lambda, \Lambda > 0$ such that

$$\lambda |\xi|^2 \le \sum_{i,j=1}^n a_{ij}^k(x) \xi_i \xi_j \le \Lambda |\xi|^2$$

for every k and any $\xi = (\xi_1, \dots, \xi_n) \in R^n$.

Right-hand side $f(x)$ is supposed a bounded vector-function, that is

$$|f^l(x)| \le C \ \text{ in } \ \Omega \tag{3}$$

for every $l = 1, \dots, N$, where C is a positive constant.

Coefficients c^k and m_{ik} in (1) are supposed continuous ones in $\overline{\Omega}$, $a_{ij}^k(x) \in C^1(\Omega) \cap C(\overline{\Omega})$ and $\dfrac{\partial a_{ij}^k}{\partial x_j}, b_i^k(x)$ are Holder continuous with Holder constant $0 < \alpha < 1$.

Hereafter by $f^-(x) = \min(f(x), 0)$ and $f^+(x) = \max(f(x), 0)$ are denoted the non-negative and, respectively, the non-positive part of the function f. The same convention is valid for matrices as well. For instance, we denote by M^+ the non-negative part of M, i.e. $M^+ = \{m_{ij}^+(x)\}_{i,j=1}^N$.

We recall that system (1) is called cooperative one if $m_{jk} \le 0$, and competitive one if $m_{jk} \ge 0$. Analogously we call $L_M^+ u$ the cooperative part of the operator, and $L_M^- u$ – the competitive one.

Solvability of system (1), (2) could be studied using the theorem of Lere–Shauder and the classical method of continuation of solutions along a parameter, since the coefficients of (1), (2) are smooth. In order to apply the theorem of Lere–Schauder one have to find a priori estimates for $\max\limits_{\Omega} |u(x)|$, $\max\limits_{\Omega} |\nabla u(x)|$, $\max\limits_{\partial\Omega} |\nabla u(x)|$ and of Holder norms $|u(x)|_{\alpha,\Omega}$. This approach is well described in [5] for systems with the same principal symbol for all equations of the system, which is significant constrain for many applications. Transfer of the results of O. A. Ladyzhenskaya and N. Ural'tseva ([5]) for elliptic systems with arbitrary principal symbol is not trivial and faces a number of technical obstacles.

Another approach to the solvability problem is the method of sub- and super-solutions. It is widely used for scalar equations and reads that if comparison principle holds for system (1), (2), and there are super- and sub-solutions of the same system, then exists a solution of system (1), (2).

In order to use the method of sub- and super-solutions we need some constrains on the growth of the coefficients. Assume that for every $k = 1, \dots, N$

$$\left\{ \sum_{i=1}^{n} \left(\sum_{j=1}^{n} D_j a_{ij}^k(x) + b_i^k(x) \right)^2, |c^k| \right\} \le b \tag{4}$$

holds for $x \in \overline{\Omega}$, where b is a positive constant,

$$\left[\sum_{i=1}^{n} b_i^k(x).p_i.u^k + c^k u^k + \sum_{i=1}^{n} m_{k,i}(x).u_i(x) \right] u^k \ge c_1|u|^2 - c_2 \tag{5}$$

for every $x \in \Omega$, $l = 1, \dots, N$ and arbitrary vectors u and bounded p, where $c_1 = \text{const} > 0$ and $c_2 = \text{const} \ge 0$,

$$\left| \sum_{i=1}^{n} b_i^k(x).p_i.u^k + c^k u^k + \sum_{i=1}^{n} m_{k,i}(x).u_i(x) \right| \le \varepsilon(C_M) + P(p, C_M)(1 + |p|^2), \tag{6}$$

where $P(p, C_M) \to 0$ for $|p| \to \infty$ and $\varepsilon(C_M)$ is sufficiently small and depends only on n, N, C_M, λ and Λ. λ and Λ are the constants from elliptic condition and

$$C_M = \max \left\{ \max_{\partial\Omega} |u|, \frac{2 \max |f(x)|}{c_1 n}, \sqrt{\frac{2c_2}{c_1 n}} \right\}. \tag{7}$$

Furthermore, in the case of competitive systems we need some a priori estimates. Let $\min_{\Omega} c^k > 0$ and

$$\max_{\Omega} \frac{\sum_{i=1}^{n} m_{ki}^+(x)}{c^k} \le K < 1. \tag{8}$$

Then system (1), (2) is solvable in $C^2(\Omega) \bigcap C(\overline{\Omega})$ as it is shown in the next two chapters. In the following one is proved the existence theorem for competitive elliptic systems with, roughly speaking, small with respect to c^k coupling coefficients $m_{ki}(x)$ for all $k = 1, \dots, N$ and in this case we need no CP. Then we use the result about competitive systems and CP in order to prove existence theorem for general non-cooperative system.

2. Existence of classical solution for competitive elliptic systems

Suppose system (1) is competitive one, i.e. $m_{ij}(x) \geq 0$. Then the following theorem holds:

Theorem 1. *Suppose conditions* (3) *to* (8) *hold for competitive system* (1), (2). *Then the vector function* $\overline{m} = (C_M, C_M, \dots, C_M)$ *is a super-solution of* (1), (2), *where* C_M *is the constant from* (5) *and there exists a classical* $C^2(\Omega)\bigcap C(\overline{\Omega})$ *solution* $v(x)$ *of the problem* (1), (2) *with null boundary data.*

Since the system (1) is a linear one, we assume in the following proof without loss of generality that $g(x) = 0$.

Proof of Theorem 1. Consider the sequence of vector-functions $v_0, v_1, \dots, v_l, \dots$, where v_0 is a super-solution and $v_l \in H_0^1(\Omega)$ defines v_{l+1} by induction as a solution of the problem with null boundary conditions

$$Lv_{l+1} + \sigma v_{l+1} = f(x) - M^+ v_l + \sigma v_l \tag{9}$$

or in details

$$-\sum_{i,j=1}^{N} D_i(a_{ij}^k(x) D_j v_{l+1}^k) + \sum_{i=1}^{N} b_i^k(x) D_j v_{l+1}^k + c^k v_{l+1}^k + \sigma v_{l+1}^k =$$

$$= f^k(x) - \sum_{i=1}^{n} m_{ki}^+(x) v_l^i + \sigma v_l^k \texttt{ in } \Omega$$

for every $k = 1, \dots, N$, and σ is positive constant.

1. By Theorem 1 in [6] conditions (3)–(6) are sufficient for solvability of the corresponding PDEs, while by Theorem 4 in [6], p. 120 conditions (10)-(12) below are derived in every subset of the domain where the coefficients of the diffraction problem are smooth. In our case this is the whole domain Ω. Therefore for the solution $v_{l+1}^k(x) \in C^2(\overline{\Omega})$ there is constant $\beta = \beta(l+1) \in (0,1)$ such that

$$\|v_l^k\|_{C^\beta(\overline{\Omega})} < c, \tag{10}$$

$$\left\|\frac{\partial v_l^k}{\partial x_i}\right\|_{C^\beta(\overline{\Omega})} < c_1 \quad \text{for every } i = 1, \dots, n,\ \gamma = 1, \dots, m. \tag{11}$$

$$\text{For every compact set } K \subset \Omega \text{ holds } \left\|\frac{\partial^2 v_k^l}{\partial x_i \partial x_j}\right\|_{C^\beta(K)} < c_7(\rho) \tag{12}$$

for every $i, j = 1, \dots, n$, $\rho = \operatorname{dist}(K, \partial\Omega)$, and constants $c_4 - c_7$ are independent on k.

2. Since system (9) is linear one then $v_{l+1}^k - v_l^k$ is a solution as well. Furthermore, $c^k + \sigma > 0$ and by inequality (1.5), page 145 in [5] we have

$$\max_{\Omega} |v_{l+1}^k - v_l^k| \leq \max_{\Omega} |\frac{\sum_{i=1}^n (m_{ki}^+(x) - \delta_{ik}\sigma)(v_l^i - v_{l-1}^i)}{c^k + \sigma}|$$

since we consider the problem with null boundary data and then $\max_{\partial\Omega} |v_{l+1}^k - v_l^k| = 0$. Therefore, by a priori estimate (8) we have

$$\max_{\Omega} |v_{l+1}^k - v_l^k| \leq \max_{\Omega} |\frac{\sum_{i=1}^n (m_{ki}^+(x) - \delta_{ik}\sigma)}{c^k + \sigma}| \max_{\Omega} |(v_l^i - v_{l-1}^i)| \leq$$

$$\leq \max_{\Omega} \frac{\sum_{i=1}^n (m_{ki}^+(x))}{c^k} \max_{\Omega} |(v_l^i - v_{l-1}^i)| \leq K. \max_{\Omega} |v_l^i - v_{l-1}^k|$$

Hence the operator used for construction of sequence $v_0, v_1, \dots, v_l, \dots$ is contracting one since

$$\max_{\Omega} |v_{l+1}^k - v_l^k| \leq K. \max_{\Omega} |v_l^i - v_{l-1}^k|. \tag{13}$$

3. The sequence of vector-functions $\{v^k\}$ is contracting in Ω by (13). Therefore there is a function v such that $v^k(x) \to v(x)$ point-wise in Ω. Furthermore, (10) yields $\{v^k\}$ is uniformly bounded and equicontinuous in $\overline{\Omega}$ and $\{v^k\} <$ const, since $v_l^k(x)$ is Holder continuous and therefore $|v_l^k(x) - v_l^k(x_0)| \leq c(|x - x_0|^\beta)$ for every $l = 1, \dots, N$. By Arzela–Ascoli compactness criterion there is a subsequence $\{v_{k_j}\}$ that converges uniformly to $v \in C(\overline{\Omega})$. For convenience we denote $\{v_{k_j}\}$ by $\{v^k\}$.

Since $v \in C(\overline{\Omega})$ and all functions $\{v_{k_j}\}$ satisfy the null boundary conditions, then v satisfies the boundary conditions as well.

The functions v^k are Holder continuous with the same Holder constant, therefore v is Holder continuous as well with the same Holder constant, i.e. $v \in C^\beta(\overline{\Omega})$.

Since $v_{l+1}(x)$ is contracting and $v(x)$ is continuous, then $\{(v^k)^2\} \to v^2$ in Ω. Then the Dominated Convergence Theorem (Theorem 5 at p.648 in [4]) yields $v^k \to v(x)$ in $(L^2(\Omega))^N$.

4. Analogously to the previous step, (11) yields $\{D_i v^k\}$ is uniformly bounded and equicontinuous in $\overline{\Omega}$ and $\{D_i v^k\} <$ const. According to Arzela–Ascoli compactness criterion there is sub-sequence $\{D_i v_{k_j}\}$ that converges uniformly to $D_i v \in C(\overline{\Omega})$. For convenience we denote $\{v_{k_j}\}$ by $\{v^k\}$.

5. For every $0 < \eta(x) = (\eta^1(x), \dots, \eta^N(x)) \in (H_0^1(\Omega))^N$ we have

$$\int_\Omega \left(\sum_{i,j=1}^N a_{ij}^k(x) D_j v_{l+1}^k D_i \eta^k(x) + \sum_{i=1}^N b_i^k(x) D v_{l+1}^k \eta^k(x) + (c^k + \sigma) v_{l+1}^k \eta^k(x) \right) dx =$$

$$= \int_\Omega (f^k(x) - \sum_{i=1}^n m_{ki}^+(x) v_l^i + \sigma v_l^k) \eta^k(x) dx$$

holds and for $k \to \infty$ we obtain

$$\int_\Omega \left(\sum_{i,j=1}^N a_{ij}^k(x) D_j v^k D_i \eta^k(x) + \sum_{i=1}^N b_i^k(x) D v^k \eta^k(x) + c^k . v^k \eta^k(x) \right) dx =$$

$$= \int_\Omega (f^k(x) - \sum_{i=1}^n m_{ki}^+(x) v^i) \eta^k(x) dx$$

that is $v(x)$ is a weak solution of (1), (2).

6. Since the coefficients $a_{ij}^k(x)$ of the principal symbol in (1) are $C^{1+\alpha}(\Omega)$ smooth and $D_x^2 v_k(x)$ are locally bounded, then $D_x^2 v(x) \in C(\Omega)$.

In fact by the exhaustion of Ω by compact sets κ_r, $\kappa_r \subset \kappa_{r+1} \subset \Omega$ and $\bigcup \kappa_r = \Omega$, and by (12) we have $D_x^2 u_k \in C^\beta(K_r)$ are uniformly bounded and equicontinuous in κ_r. Applying Arzela–Ascoli theorem and Cantor diagonal process (for sub-sequence and compact) yields C^2 smoothness in Ω of the limit $v(x)$.

Therefore $v(x) \in C^2(\Omega))^N$ is classical solution of (1), (2). $\square$

3. Existence of classical solution for general non-cooperative elliptic systems

For general non-cooperative elliptic system the following theorem holds:

Theorem 2. *Suppose conditions* (3) *to* (15) *hold for system* (1), (2)*. Then the vector function* $\overline{m} = (C_M, C_M, \dots, C_M)$ *is a super-solution of* (1), (2)*, where* C_M *is the constant from* (5) *and there exists a classical* $C^2(\Omega) \bigcap C(\overline{\Omega})$ *solution* $u(x)$ *of the problem* (1), (2) *with null boundary data.*

Theorem 2 is proved by the method of sub- and super-solutions. A key-point of the method is the comparison principle. Unlike the cooperative systems, for non-cooperative ones there is no complete theory for the validity of the comparison principle. In [1] are given some sufficient conditions such that the comparison principle holds. Actually conditions (14) and (15) below enforce the validity of comparison principle for system (1), (2).

In order to apply the method of super- and sub- solutions for general non-cooperative systems we need the validity of CP for the competitive part of the system. Let us recall the following Theorem for CP in this case (Theorem 4 in [1]):

Theorem 3. *Assume $m^-_{ij} \equiv 0$ for $i \neq j$ and (2) is satisfied. Then the comparison principle holds for the classical $C^2(\Omega)\bigcap C(\overline{\Omega})$ solutions of system (1) if there is $x_0 \in \Omega$ such that*

$$\lambda_j + \sum_{k=1}^{N} m^+_{kj}(x_0) > 0 \quad \textit{for every } j = 1, \dots, N, \quad \textit{and} \tag{14}$$

$$\lambda_j + m^+_{jj}(x) \geq 0 \quad \textit{for every } x \in \Omega \quad \textit{and} \quad j = 1, \dots, N, \tag{15}$$

where λ_j is the principal eigenvalue of $\widetilde{L}_j = L_j + m^-_{jj}$ in Ω.

Theorem 3 is formulated for diagonal matrix M^-, but the statement is valid with obvious modification if M^- has Jordan cells on the main diagonal.

Since the system (1) is a linear one, we assume in the following proof without loss of generality that $g(x) = 0$.

P r o o f o f T h e o r e m 2. Let us consider the sequence of vector-functions $u_0, u_1, \dots, u_l, dots$, where $u_0 = \overline{m}$ is the super-solution and $u_l \in H^1_0(\Omega)$ defines u_{l+1} by induction as a solution of the problem

$$L_{M^+} u_{l+1} + \sigma u_{l+1} = f(x) - M^- u_l + \sigma u_l \tag{16}$$

or in details

$$- \sum_{i,j=1}^{N} D_i(a^k_{ij}(x) D_j u^k_{l+1}) + \sum_{i=1}^{N} b^k_i(x) D u^k_{l+1} + c^k u^k + \sum_{i=1}^{n} m^+_{ki}(x) u^i + \sigma u^k_{l+1} =$$

$$= f^k(x) - \sum_{i=1}^{n} m^-_{ki}(x) u^i + c^k u^k + \sigma u^k_l \ \texttt{in}\ \Omega$$

with null boundary conditions

$$u^k_{l+1}(x) = 0 \ \texttt{on}\ \partial\Omega$$

for every $k = 1, \dots, N$, $\sigma < 0$ is a constant.

Let denote the left-hand side of (16) by $A^k(x, u, \sigma)$, and the right-hand side – by $B^k(x, u, \sigma)$, $k = 1, \dots, N$.

1. The problem (16) is competitive system and by Theorem (1) it is solvable. Even more, for the solution $u^k_{l+1}(x) \in C^2(\overline{\Omega})$ there is constant $\beta = \beta(l+1) \in (0,1)$ such that

$$\|u^k_l\|_{C^\beta(\overline{\Omega})} < c, \tag{17}$$

$$\left\| \frac{\partial u^k_l}{\partial x_i} \right\|_{C^\beta(\overline{\Omega})} < c_1 \quad \text{for every} \quad i = 1, \dots, n, \ \gamma = 1, \dots, m. \tag{18}$$

$$\text{For every compact set } K \subset \Omega \text{ holds } \left\| \frac{\partial^2 u^l_k}{\partial x_i \partial x_j} \right\|_{C^\beta(K)} < c_7(\rho) \tag{19}$$

for every $i, j = 1, \dots, n$, $\rho = \operatorname{dist}(K, \partial\Omega)$, and constants $c_4 - c_7$ are independent on k. By Theorem 1 in [6] conditions (3)–(5) are sufficient for solvability of the corresponding PDEs, while by Theorem 4 in [6, p. 120], conditions (17)–(19) are derived in every subset of the domain where the coefficients of the diffraction problem are smooth. In our case this is the whole domain Ω.

2. Furthermore $u^l_0 \geq u^l_1 \geq \cdots \geq u^k_{l+1} \geq \cdots$ by the comparison principle and the fact that

$$f^k(x) - F^+_k(x, u^1_l, \dots, u^N_l) + \sigma u^k_l - f^k(x) - F^+_k(x, u^1_{l-1}, \dots, u^N_{l-1}) + \sigma u^k_{l-1} =$$

$$= -F^+_k(x, u^1_l - u^1_{l-1}, \dots, u^N_l - u^N_{l-1}) + \sigma(u^k_l - u^N_{l-1}) \geq 0$$

since $u^k_l \leq u^N_{l-1}$ and $-m^+_{ki}(x) \leq 0$

The proof of $u^l_0 \geq u^l_1$ is trivial since u^l_0 is a super-solution of (1), (2).

3. Obviously the inequality $u_{l+1}(x) \geq w(x)$ holds for every u_{l+1}, since $w(x)$ is a sub-solution of the same system (1), (2).

4. The sequence of vector-functions $\{u^k\}$ is monotonously decreasing and bounded from below in Ω. Therefore there is a function u such that $u^k(x) \to u(x)$ point-wise in Ω. Furthermore, (17) yields $\{u^k\}$ is uniformly bounded and equicontinuous in $\overline{\Omega}$ and $\{u^k\} < const$, since $u^k_l(x)$ is Holder continuous and therefore $|u^k_l(x) - u^k_l(x_0)| \leq c(|x - x_0|^\beta)$ for every $l = 1, \dots, N$. By Arzela–Ascoli compactness criterion there is a sub-sequence $\{u_{k_j}\}$ that converges uniformly to $u \in C(\overline{\Omega})$. For convenience we denote $\{u_{k_j}\}$ by $\{u^k\}$.

Since $u \in C(\overline{\Omega})$ and all functions $\{u_{k_j}\}$ satisfy the null boundary conditions, then u satisfies the boundary conditions as well.

The functions u^k are Holder continuous with the same Holder constant, therefore u is Holder continuous as well with the same Holder constant, i.e. $u \in C^\beta(\overline{\Omega})$.

Since $u_{l+1}(x)$ is monotone and $u(x)$ is continuous, then $\{(u^k)^2\} \to u^2$ in Ω. Then the Dominated Convergence Theorem (Theorem 5 at p. 648 in [4]) yields $u^k \to u(x)$ in $(L^2(\Omega))^N$.

5. Analogously to the previous step, (18) yields $\{D_i u^k\}$ is uniformly bounded and equicontinuous in $\overline{\Omega}$ and $\{D_i u^k\} <$ const. According to Arzela–Ascoli compactness criterion there is sub-sequence $\{D_i u_{k_j}\}$ that converges uniformly to $D_i u \in C(\overline{\Omega})$. For convenience we denote $\{u_{k_j}\}$ by $\{u^k\}$.

6. For every $0 < \eta(x) = (\eta^1(x), \dots, \eta^N(x)) \in (H_0^1(\Omega))^N$

$$\int_\Omega \left(\sum_{i,j=1}^N a_{ij}^k(x) D_j u_{l+1}^k D_i \eta^k(x) + \sum_{i=1}^N b_i^k(x) D u_{l+1}^k \eta^k(x) + \sigma u_{l+1}^k \eta^k(x) \right) dx =$$

$$= \int_\Omega (f^k(x) - F^k(x, u_k^1, \dots, u_k^N) + \sigma u_l^k) \eta^k(x) dx$$

holds and for $k \to \infty$ we obtain

$$\int_\Omega \left(\sum_{i,j=1}^N a_{ij}^k(x) D_j u^k D_i \eta^k(x) + \sum_{i=1}^N b_i^k(x) D u^k \eta^k(x) \right) dx =$$

$$= \int_\Omega (f^k(x) - F^k(x, u^1, \dots, u^N)) \eta^k(x) dx$$

that is $u(x)$ is solution of (1), (2).

7. Since the coefficients $a_{ij}^k(x)$ of the principal symbol in (1) are $C^{1+\alpha}(\Omega)$ smooth and $D_x^2 u_k(x)$ are locally bounded, then $D_x^2 u(x) \in C(\Omega)$.

In fact by the exhaustion of Ω by compact sets κ_r, $\kappa_r \subset \kappa_{r+1} \subset \Omega$ and $\bigcup \kappa_r = \Omega$, and by (19) we have $D_x^2 u_k \in C^\beta(K_r)$ are uniformly bounded and equicontinuous in κ_r. Applying Arzela–Ascoli theorem and Cantor diagonal process (for sub-sequence and compact) yields C^2 smoothness in Ω of the limit $u(x)$.

Therefore $u(x) \in C^2(\Omega))^N$ is classical solution of (1), (2). $\square$

Note that the vector function $\overline{m} = (C_M, C_M, \dots, C_M)$ is a super-solution of (1), (2), where C_M is the constant from (5). Indeed, $\overline{m}$ is a super-solution of $L_{M^-} u(x) = f(x)$ (see ([2]) and therefore $L_{M^-}\overline{m} + M^+\overline{m} - f(x) \geq M^+\overline{m} \geq 0$, while $\overline{m} \geq g(x)$ on $\partial\Omega$. The solvability of system (1), (2) is complete.

For computational purposes one may use more precise super-solution than $\overline{m}$ in computing the sequence $u_0, u_1, \dots, u_l, \dots$.

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