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ON AN EXAMPLE OF DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION WITH $\mathbb{D}_2$ REDUCTION

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We briefly analyze the integrable derivative nonlinear Schrödinger (DNLS) equations paying attention to mainly to the one known as GI equation [5, 6, 4]. Using Mikhailov's reduction group we impose on the Lax pair additional $\mathbb{Z}_2$ invariance and derive new integrable DNLS eq. having an additional cubic nonlinearity. We analyze the spectral properties of the new $\mathbb{D}_2$-invariant Lax operator. We construct the fundamental analytic solutions of the reduced Lax operator and formulate the Riemann-Hilbert problem that they satisfy.

1. Introduction

The well known non-linear Schrödinger equation

$$i\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + |u|^2 u = 0, \quad u = u(x,t) \tag{1}$$

solved by Zakharov and Shabat [13] finds numerous applications [3] – for example in optics, laser physics, bose-einstein condensates. It allows for derivative generalizations with the most famous being, the family of three derivative NLS equations (DNLS) admitting a Lax representation

• DNLS-I, or Kaup-Newell eq. [8]

$$i\partial_t q + \partial_x^2 q - \partial_x (q|q|^2) = 0, \tag{2}$$

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- DNLS-II, or Chen-Lee-Liu eq. [2]

$$i\partial_t q + \partial_x^2 q + i|q|^2 \partial_x q = 0, \tag{3}$$

- DNLS-III or GI eq. [5, 6, 4]

$$i\partial_t q + \partial_x^2 q + iq^2 \partial_x q^* + \frac{1}{2} q|q|^4 = 0, \tag{4}$$

It can be shown that those equations are related by a chain of gauge transformation, but it is often more convenient to treat them separately [12].

Each of the DNLS can be written down as the compatibility condition $[L(\cdot,\lambda), M(\cdot,\lambda)]$ of two linear operators:

$$\begin{aligned} L\psi &\equiv i\frac{\partial \psi}{\partial x} + U(x,t,\lambda)\psi(x,t,\lambda) = 0, \\ M\psi &\equiv i\frac{\partial \psi}{\partial t} + V(x,t,\lambda)\psi(x,t,\lambda) = 0. \end{aligned} \tag{5}$$

where the potentials $U(x,t,\lambda)$ and $V(x,t,\lambda)$ depend polynomially on λ.

The aim of the present paper is to demonstrate that by imposing additional $\mathbb{Z}_2$ symmetry on L and M, which takes λ into λ^{-1}, it is possible to derive new Lax pairs resulting in new integrable versions of DNLS. Such approach generically allows one, starting from Lax pair with $\mathbb{Z}_h$ Mikhailov's reduction [11] to construct new Lax pair which will be invariant under the group $\mathbb{D}_h$.

The paper is a natural extension of [7] and is structured as follows: In Section 2 we formulate some preliminary facts relevant to our purposes. In Section 3, following [7] we derive the corresponding GI eq. with $\mathbb{D}_2$ reduction group. Compared to the original GI eq. the new one contains additional cubic nonlinearity and additional linear terms. In Section 4 we derive the spectral properties of the new Lax operator. We end with some concluding remarks.

2. Preliminaries

The Lax pair (5) is of generic form. In what follows we will specify the potentials $U(x,t,\lambda)$ and $V(x,t,\lambda)$ to be polynomial in the spectral parameter λ with coefficients taking values in a simple Lie algebra. We will fix them to be:

$$\begin{aligned} U(x,t,\lambda) &= \left(Q_0 + \lambda Q_1 - \lambda^2 J\right) \\ V(x,t,\lambda) &= \left(V_0 + \lambda V_1 + \lambda^2 V_2 + \lambda^3 V_3 - \lambda^4 K\right), \end{aligned} \tag{6}$$

where $V_i(x,t)$, $Q_i(x,t)$, J and K take values in the Lie algebra $\mathfrak{sl}(2)$. We will fix up the gauge by choosing

$$J = K = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q_1(x,t) = \begin{pmatrix} 0 & 2q \\ -2p & 0 \end{pmatrix}. \tag{7}$$

2.1. Reductions of Lax pairs

The reduction groups introduced by [11] are a powerful tool for deriving new integrable equations, admitting a Lax representation.

A reduction group G_R is a finite group acting on the solution set of (5) which preserves the Lax representation [10], i.e. it ensures that the reduction constraints are automatically compatible with the evolution. G_R must have two realizations: i) $G_R \subset \mathrm{Aut}\mathfrak{g}$ and ii) $G_R \subset \mathrm{Conf}\,\mathbb{C}$, i.e. as conformal mappings of the complex λ-plane. To each $g_k \in G_R$ we relate a reduction condition for the Lax pair as follows [11]:

$$C_k(L(\Gamma_k(\lambda))) = \eta_k L(\lambda), \quad C_k(M(\Gamma_k(\lambda))) = \eta_k M(\lambda), \tag{8}$$

where $C_k \in \mathrm{Aut}\ \mathfrak{g}$ and $\Gamma_k(\lambda) \in \mathrm{Conf}\,\mathbb{C}$ are the images of g_k and $\eta_k = 1$ or -1 depending on the choice of C_k. Since G_R is a finite group then for each g_k there exist an integer N_k such that $g_k^{N_k} = \mathbb{1}$.

The finite subgroups of $\mathrm{Conf}\,(\mathbb{C})$ were classified by Klein [9]. They consist of two infinite series: i) $\mathbb{Z}_h$ - cyclic group of order h; ii) $\mathbb{D}_h$ - dihedral group of order $2h$; and the groups related to the Platonic solids: tetrahedron, cube, octahedron, dodecahedron and icosahedron.

It is important to note that the form of the equations depends not only on the chosen reduction group, but also on its realization.

The effect of some typical reductions on the matrix-valued functions of the Lax representation [11]:

$$\begin{array}{lll} 1) & C_1(U^\dagger(\kappa_1(\lambda))) = U(\lambda), & C_1(V^\dagger(\kappa_1(\lambda))) = V(\lambda), \\ 2) & C_2(U^T(\kappa_2(\lambda))) = -U(\lambda), & C_2(V^T(\kappa_2(\lambda))) = -V(\lambda), \\ 3) & C_3(U^*(\kappa_1(\lambda))) = -U(\lambda), & C_3(V^*(\kappa_1(\lambda))) = -V(\lambda), \\ 4) & C_4(U(\kappa_2(\lambda))) = U(\lambda), & C_4(V(\kappa_2(\lambda))) = V(\lambda), \end{array} \tag{9}$$

Let us consider a $\mathbb{D}_h$ type reduction. The dihedral group $\mathbb{D}_h$ has two generating elements satisfying the generating relations:

$$r^2 = s^h = \mathbb{1}, \qquad srs^{-1} = s^{-1}. \tag{10}$$

The group has $2h$ elements: $\{s^k, rs^k, k=1,\dots,h\}$ and allows several inequivalent realization on the complex λ-plane. Some of them are:

$$\begin{aligned} &\text{(i)} \quad s(\lambda)=\lambda\omega, \quad r(\lambda)=\epsilon\lambda^*, &&\text{(ii)} \quad s(\lambda)=\lambda\omega, \quad r(\lambda)=\frac{\epsilon}{\lambda^*},\\ &\text{(iii)} \quad s(\lambda)=\lambda\omega, \quad r(\lambda)=\epsilon\lambda, &&\text{(iv)} \quad s(\lambda)=\lambda\omega, \quad r(\lambda)=\frac{\epsilon}{\lambda}, \end{aligned}$$

where $\omega=\exp(2\pi i/h)$ and $\epsilon=\pm1$. An important realization in the case of a $\mathbb{D}_2$ reduction group is given by

$$\text{(v)} \quad s(\lambda)=\lambda^*, \quad r(\lambda)=\frac{\epsilon}{\lambda}. \tag{11}$$

3. Generalization of the DNLS-III equation

We will impose two reductions (types one and four from (9)). Their effect on the potential of the Lax operator is given by

$$1)\quad U^\dagger(x,t,\lambda^*)=U(x,t,\lambda), \qquad 2)\quad \tilde{U}\left(x,t,\frac{1}{\lambda}\right)=U(x,t,\lambda). \tag{12}$$

where by "tilde" we mean

$$\tilde{X}=-BX^TB^{-1}, \quad B=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}. \tag{13}$$

The same holds for $V(x,t,\lambda)$. It is not difficult to show that, in order to satisfy the reductions, the new potentials of the Lax operators must be invariant with respect to G_R, i.e. they take the form

$$\begin{aligned} \boldsymbol{U}(x,t,\lambda)&=U(x,t,\lambda)-BU^T(x,t,\lambda^{-1})B^{-1}\\ &=Q_0+\lambda Q_1-\lambda^2J+\frac{1}{\lambda}\tilde{Q}_1-\frac{1}{\lambda^2}\tilde{J}, \end{aligned} \tag{14}$$

$$\begin{aligned} \boldsymbol{V}(x,t,\lambda)&=V(x,t,\lambda)-VU^T(x,t,\lambda^{-1})B^{-1}\\ &=V_0+\lambda V_1+\lambda^2V_2+\lambda^3V_3-\lambda^4K+\frac{1}{\lambda}\tilde{V}_1+\frac{1}{\lambda^2}\tilde{V}_2+\frac{1}{\lambda^3}\tilde{V}_3-\frac{1}{\lambda^4}\tilde{K}. \end{aligned} \tag{15}$$

The compatibility condition $[L,M]=0$ leads to the following set of recursion relations:

$$\begin{aligned} &\lambda^6: \quad [J,K]=0, \qquad \lambda^5: \quad [J,V_3]=[K,Q_2],\\ &\lambda^4: \quad [J,V_2]=[K,Q_0]+[Q_1,V_3],\\ &\lambda^3: \quad [J,V_1]=i\frac{\partial}{\partial x}V_3+[Q_1,V_2]+[Q_0,V_3]+[\tilde{Q}_1,K],\\ &\lambda^2: \quad [J,V_0]=i\frac{\partial}{\partial x}V_2+[Q_0,V_2]+[Q_1,V_1]+[\tilde{Q}_1,V_3]+[\tilde{J},K]. \end{aligned} \tag{16}$$

There are also analogous conditions for the negative powers of λ which are automatically satisfied, provided that the above are true.

Solving these recurrent relations we determine V_3, V_2, V_1 and V_0 in terms of Q_1 and Q_1 and their derivatives.

$$
\begin{aligned}
&Q_0 = \begin{pmatrix} -2qp & 0 \\ 0 & 2qp \end{pmatrix}, \qquad V_3 = \begin{pmatrix} 0 & 2q \\ -2p & 0 \end{pmatrix} \\
&V_2 = \begin{pmatrix} -2qp & 0 \\ 0 & 2qp \end{pmatrix}, \qquad V_1 = \begin{pmatrix} 0 & i\partial_x q \\ i\partial_x p & 0 \end{pmatrix}, \\
&V_0 = -\frac{1}{2}Q_0^2 + \frac{1}{4}[Q_1,[V_1,J]] + 2Q_0,
\end{aligned} \tag{17}
$$

where $p = q^*$.

Finally, inserting them in the last two relations:

$$
\begin{aligned}
\lambda^1 : \quad & i\frac{\partial}{\partial t}Q_1 = i\frac{\partial}{\partial x}V_1 + [Q_0, V_1] + [Q_1, V_0] + [\tilde{Q}_1, V_2] - [J, \tilde{V}_1] + [\tilde{J}, V_3], \\
\lambda^0 : \quad & i\frac{\partial}{\partial t}Q_0 = i\frac{\partial}{\partial x}V_0 + [Q_0, V_0] + [Q_1, \tilde{V}_1] + [\tilde{Q}_1, V_1] - [J, \tilde{V}_2] + [\tilde{J}, V_2].
\end{aligned} \tag{18}
$$

we obtain the relevant NLEE. The λ^1 terms in (16) give the following equation

$$
i\frac{\partial q}{\partial t} + \frac{1}{2}\frac{\partial^2 q}{\partial x^2} + 2iq^2\frac{\partial q^*}{\partial x} + 4q|q|^4 - 8q\,|q|^2 + 2i\frac{\partial q}{\partial x} - 4q = 0. \tag{19}
$$

This is DNLS-III equation with an additional cubic nonlinearity (and a linear term). The λ-independent term vanishes, provided that q is a solution of (19).

4. Spectral properties of the Lax operator

Here we will briefly outline the spectral properties of the new Lax operators:

$$
\boldsymbol{L}\psi \equiv i\frac{\partial \psi}{\partial x} + \boldsymbol{U}(x,t,\lambda)\psi(x,t,\lambda) = 0, \quad \boldsymbol{M}\psi \equiv i\frac{\partial \psi}{\partial t} + \boldsymbol{V}(x,t,\lambda)\psi(x,t,\lambda) = 0, \tag{20}
$$

assuming that the the potential $Q_1(x)$ is a smooth function of x tending to 0 fast enough for $x \to \pm\infty$. To this end we introduce the Jost solutions of $\boldsymbol{L}$. These are fundamental solutions of (20) defined by their asymptotics for $x \to \infty$ and for $x \to -\infty$:

$$
\lim_{x \to -\infty} \boldsymbol{\phi} \exp(i\mathcal{J}(\lambda)x) = \mathbb{1}, \qquad \lim_{x \to \infty} \boldsymbol{\psi} \exp(i\mathcal{J}(\lambda)x) = \mathbb{1}, \tag{21}
$$

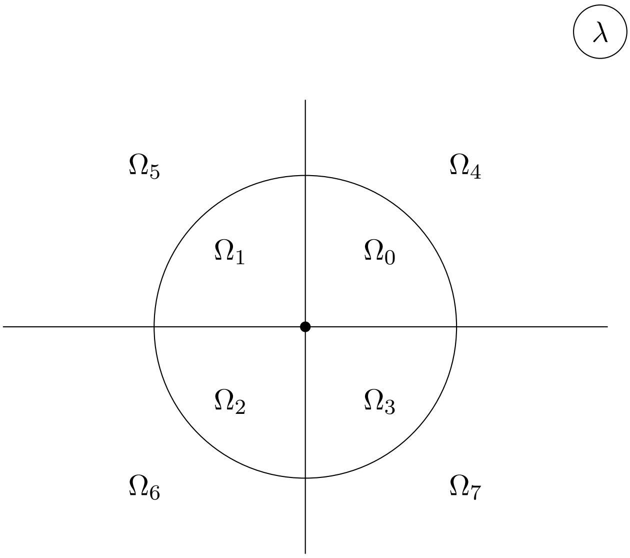

Figure 1: The continuous spectrum of the Lax operator (20) and the contour of the related Riemann-Hilbert problem.

where $\mathcal{J}(\lambda) = (\lambda^2 + \lambda^{-2})\sigma_3$. Then one can derive the integral equations for the Jost solutions. The result is the following set of Volterra-type integral equations:

(22) $$\boldsymbol{X}^{\pm}(x,\lambda) = \mathbb{1} + i\int_{\pm\infty}^{x} dy\, \mathcal{E}^{-1}(x,y,\lambda)\left(Q_0(y) + \lambda Q_1(x) + \lambda^{-1}\tilde{Q}_1(y)\right)\boldsymbol{X}^{\pm}(y,\lambda)\mathcal{E}(x,y,\lambda),$$

where $\mathcal{E}(x,y,\lambda) = \exp(i\mathcal{J}(\lambda)\sigma_3(x-y))$ and

$$\boldsymbol{X}^{+}(x,\lambda) = \boldsymbol{\psi}(x,t,\lambda)\exp(i\mathcal{J}(\lambda)\sigma_3 x), \qquad \boldsymbol{X}^{-}(x,\lambda) = \boldsymbol{\phi}(x,t,\lambda)\exp(i\mathcal{J}(\lambda)\sigma_3 x).$$

The next step is to determine the values of λ on the complex λ-plane for which eq. (22) allows solution, i.e. we need to find those values of λ for which the exponentials $\mathcal{E}(x,y,\lambda)$ oscillate:

(23) $$\operatorname{Im}\mathcal{J}(\lambda) = \operatorname{Im}(\lambda^2 + \lambda^{-2}) = 0.$$

Let us parametrize $\lambda = \rho e^{i\alpha}$ where $\rho = |\lambda|$ and $\alpha = \arg(\lambda)$. Then eq. (23) goes into:

(24) $$(\rho^2 - \rho^{-2})\sin(2\alpha) = 0.$$

Ω_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7
$-$	$+$	$-$	$+$	$+$	$-$	$+$	$-$

Table 1: The signs of $\operatorname{Im}\mathcal{J}(\lambda)$ in each of the sectors Ω_a

Obviously eq. (24) has three solutions: i) $\alpha = 0$; ii) $\alpha = \pi/2$ and iii) $\rho = 1$. Thus the continuous spectrum of $\boldsymbol{L}$ will consist of the real axis, the imaginary axis and the unit circle, see Figure 1. Inside each of the sectors Ω_a, $a = 0, \dots, 8$ the function $\operatorname{Im}\mathcal{J}(\lambda)$ keeps its sign, see the Table 1.

The scattering matrix of the operator $\boldsymbol{L}$ is determined by:

$$T(\lambda, t) = \boldsymbol{\psi}^{-1}\boldsymbol{\phi}(x, t, \lambda) = \begin{pmatrix} a^+ & -b^- \\ b^+ & a^- \end{pmatrix}. \tag{25}$$

Its t-dependence is fixed by the second operator $\boldsymbol{M}$; in our case it is:

$$i\frac{\partial T}{\partial t} = (\lambda^4 + \lambda^{-4})[\sigma_3, T(\lambda, t)]. \tag{26}$$

which means that the functions $a^{\pm}(\lambda)$ are t-independent, while

$$i\frac{\partial b^{\pm}}{\partial t} = \mp 2(\lambda^4 + \lambda^{-4})b^{\pm}(\lambda, t). \tag{27}$$

The inverse scattering problem for $\boldsymbol{L}$ can be reduced to a Riemann-Hilbert problem (RHP). Indeed, coming back to the equations (22) it is not difficult to check that only one of the columns of $\boldsymbol{X}^+(x, \lambda)$ and $\boldsymbol{X}^-(x, \lambda)$ will allow analytic extension in each of the sectors. Combining them we will be able to introduce a fundamental analytic solution (FAS) in each of the sectors. For example, the FAS in the sectors Ω_0 and Ω_4 are constructed as follows:

$$\xi_0(x, t, \lambda) = \big(\boldsymbol{X}^+_{(1)}, \boldsymbol{X}^-_{(2)}\big)(x, t, \lambda), \qquad \xi_4(x, t.\lambda) = \big(\boldsymbol{X}^-_{(1)}, \boldsymbol{X}^+_{(2)}\big)(x, t, \lambda). \tag{28}$$

Similarly in each of the sectors Ω_k one can construct the corresponding FAS $\xi_k(x, t, \lambda)$. These FAS $\xi_k(x, t, \lambda)$ and $\xi_p(x, t, \lambda)$ will be linearly related on the intersection of the sectors:

$$\xi_k(x, t, \lambda) = \xi_p(x, t, \lambda) G_{kp}(x, t, \lambda), \qquad \lambda \in \Omega_k \cap \Omega_p, \tag{29}$$

and the x, t dependence of the sewing functions G_{kp} is determined by:

$$i\frac{\partial G_{kp}}{\partial x} = (\lambda^2 + \lambda^{-2})[\sigma_3, G_{kp}(x, t, \lambda)], \quad i\frac{\partial G_{kp}}{\partial t} = (\lambda^4 + \lambda^{-4})[\sigma_3, G_{kp}(x, t, \lambda)]. \tag{30}$$

k	0	1	2	3	4	5	6	7
$\bar{k}$	2	3	0	1	6	7	4	5

Table 2: The correspondence between k and $\bar{k}$ in eq. (32)

Elaborating further one can express G_{kp} in terms of the scattering matrix elements. For example,

$$G_{04}(x,t,\lambda) = \frac{1}{a^{-}(\lambda)} \begin{pmatrix} 1 & b^{-}(\lambda,t) \\ b^{+}(\lambda,t) & 1 \end{pmatrix}, \qquad (31)$$

where $\lambda \in \Omega_0 \cap \Omega_4$, i.e. $\lambda = e^{i\alpha}$ and $0 \leq \alpha \leq \pi/2$.

The set of relations (29) can be viewed as a RHP. One can check that it is canonically normalized, i.e. $\lim_{\lambda\to\infty} \xi_k(x,t,\lambda) = \mathbb{1}$ for $k = 4,\dots,7$. In addition the FAS satisfy

$$\xi_k(x,t,-\lambda) = \xi_{\bar{k}}^{-1}(x,t,\lambda). \qquad (32)$$

where the correspondence between k and $\bar{k}$ is given in Table 2.

5. Discussion and conclusions

We demonstrated how one can obtain new integrable equations applying additional reductions to the initial Lax pair. We have analyzed the spectral properties of the new $\mathbb{D}_2$-invariant Lax operator $\boldsymbol{L}$ and constructed its fundamental analytic solutions. This allows one to apply the dressing Zakharov-Shabat method [14, 15] and construct the soliton solutions of the new system.

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