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ORBITAL STABILITY OF SOLITARY WAVES TO FOURTH ORDER DISPERSIVE EQUATIONS WITH QUADRATIC NONLINEARITY*

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The orbital stability of solitary waves with constant velocity c to fourth order dispersive problem

$$u_{tt} - u_{xx} + h_1 u_{xxxx} - h_2 u_{ttxx} + (au^2)_{xx} = 0, \quad h_1 > 0, \quad h_2 > 0, \quad a > 0,$$

is investigated. Direct proof of the stability of the solitary waves to the equation is proposed. Complete investigation of the regions of stability is given. The precise dependence of the stability on the parameters h_1, h_2 and a is obtained.

1. Introduction

The aim of this paper is to study the orbital stability of solitary waves to fourth order dispersive equation

$$(1) \qquad u_{tt} - u_{xx} + h_1 u_{xxxx} - h_2 u_{ttxx} + (au^2)_{xx} = 0, \quad h_1 > 0, \quad h_2 > 0, \quad a > 0$$

with initial data

$$(2) \qquad u(0,x) = u_0(x), \; u_t(0,x) = u_1(x),$$

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$$u_0(x) \in H^1(\mathbb{R}),\ u_1(x) \in L^2(\mathbb{R}),\ (-\Delta)^{-\frac{1}{2}} u_1(x) \in L^2(\mathbb{R}),$$

where $(-\Delta)^{-s}u = \mathcal{F}^{-1}\left(|\xi|^{-2s}\mathcal{F}(u)\right)$ for $s > 0$ and $\mathcal{F}(u)$, $\mathcal{F}^{-1}(u)$ are the Fourier and the inverse Fourier transform.

Problem (1)–(2) appears in some physical models as a model of propagation of longitudinal strain waves in an isotropic cylindrical compressible elastic rod [6], [7], [8].

Let us mention that the orbital stability and instability of the solitary waves to the equation (1) is partially investigated in [1], [2], [3], [9]. More precisely, in [1] for $c = 0$ and in [2], [9] for $c^2 < c_0^2$ strong instability, i.e. instability by blow up, is proved. However, [1], [2] and [9] do not include the quadratic nonlinearity au^2 and the case $h_1 < h_2$.

In the present paper we give complete results for orbital stability of the solitary waves to (1). The orbital stability is based on a direct proof of stability theory (without Grillakis, Shatach and Strauss's results) and analytical formula for function $d(c)$ and $d''(c)$ connected to some conserved quantities of (1).

The paper is organized in the following way. In Section 2 preliminary results for solitary waves and orbital stability are discussed. In Section 3 the main results are formulated and proved, while in Section 4 some conclusions are given.

2. Preliminaries

The solitary wave $\varphi_c(x - ct)$ to (1) satisfies the problem

$$(h_1 - h_2c^2)\varphi_c''(\zeta) - (1 - c^2)\varphi_c(\zeta) + a\varphi_c^2(\zeta) = 0,\ \varphi_c(\zeta) \to 0 \text{ for } |\zeta| \to \infty.$$

We recall the result from [3] for existence of positive solitary waves to (1).

Theorem 2.1. *[3] There exists a unique (up to translation of the coordinate system) positive solitary wave $\varphi_c(\zeta)$, $\zeta = x - ct$, to (1) with constant velocity c*

$$\varphi_c(\zeta) = 3|1 - c^2| \left(a \operatorname{sgn}(1 - c^2) + |a| \cosh\left(\sqrt{\frac{1 - c^2}{h_1 - h_2c^2}}\zeta \right) \right)^{-1}, \tag{3}$$

when one of the following assumptions is fulfilled:

(A) $a > 0,\ \ h_1 > 0,\ \ h_2 > 0,\quad c^2 \in \left[0, \min\left(1, \frac{h_1}{h_2}\right)\right),$

(B) $a < 0,\ \ h_1 > 0,\ \ h_2 > 0,\ \ c^2 \in \left(\max\left(1, \frac{h_1}{h_2}\right), \infty\right).$

In order to formulate the definition of orbital stability/instability we rewrite problem (1) to the following equivalent Hamiltonian system (see [2])

$$(4) \qquad u_t = w_x, \quad w_t = \left(E - h_2\frac{\partial^2}{\partial x^2}\right)^{-1}\left(\left(E - h_1\frac{\partial^2}{\partial x^2}\right)u_x - (au^2)_x\right),$$
$$u(0,x) = u_0(x), \quad w(0,x) = w_0(x), \quad x \in \mathbb{R}.$$

where E is the identity operator and $w_0(x) = \mathcal{F}^{-1}\left((i\xi)^{-1}\mathcal{F}(u_1)(\xi)\right) \in L^2(\mathbb{R})$. In the space $X = H^1(\mathbb{R}) \times H^1(\mathbb{R})$ with the norm

$$\|\vec{u}\|_X^2 = \|(u,w)\|_X^2 = \|u\|^2_{H^1(\mathbb{R})} + \|w\|^2_{H^1(\mathbb{R})}$$

we have from Theorem 2.1 in [2] that $\vec{u} \in X$ when $\vec{u}_0 = (u_0, w_0) \in X$.

We recall conservation law of the energy H and the momentum M

$$H(\vec{u}(t,\cdot)) = H(u,w) = \frac{1}{2}\int_{\mathbb{R}}\left(u^2 + h_1u_x^2 + w^2 + h_2w_x^2 - 2a\frac{u^3}{3}\right)dx = H(\vec{u}(0,\cdot)),$$

$$M(\vec{u}(t,\cdot)) = M(u,w) = \int_{\mathbb{R}}(uw + h_2u_xw_x)\,dx = M(\vec{u}(0,\cdot)).$$

and some important functionals

$$I_c(u) = \left(h_1 - h_2c^2\right)\|u_x\|^2 + \left(1 - c^2\right)\|u\|^2 - \int_{\mathbb{R}} au^3\,dx,$$

$$J_c(u) = \frac{1}{2}\left(h_1 - h_2c^2\right)\|u_x\|^2 + \frac{1}{2}\left(1 - c^2\right)\|u\|^2 - \int_{\mathbb{R}}\frac{au^3}{3}\,dx,$$

$$N_c = \left\{u \in H^1(\mathbb{R}) : I_c(u) = 0, \|u\|_1 \neq 0\right\},$$

$$(5) \qquad d(c) := \inf_{u \in N_c} J_c(u).$$

Definition. *The solitary wave* $\vec{\varphi}_c(x-ct) = (\varphi_c(x-ct), -c\varphi_c(x-ct))$ *to* (4) *is orbitally stable if for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for* $\vec{u}_0 = (u_0, w_0) \in X$ *with*

$$||\vec{u}_0 - \vec{\varphi}_c||_X < \delta,$$

the solution $\vec{u}(t) = (u(t), w(t))$ *of* (4) *with initial data* $\vec{u}(0) = \vec{u}_0$ *satisfies*

$$\sup_{0 \le t < \infty}\inf_{y \in \mathbb{R}} ||\vec{u}(t) - \vec{\varphi}_c(\cdot + y)||_X < \varepsilon.$$

Otherwise, $\vec{\varphi}(x - ct)$ *is orbitally unstable.*

We recall the following result in [4] which gives a direct proof of orbital stability of the solitary waves of (1) without application of Grillakis, Shatah and Strauss's theory.

Theorem 2.2. [4] *Suppose* $h_1 > 0$, $h_2 > 0$, $a > 0$ *and the function* $d(c)$ *defined in* (5) *is twice differentiable and strictly convex in an interval* $[\xi_1, \xi_2] \subset \left(0, \min\left(1, \frac{h_1}{h_2}\right)\right)$, $\xi_1 < \xi_2$. *Then for every* $c^2 \in (\xi_1, \xi_2)$ *the solitary wave* $\vec{\varphi}_c$ *to* (4) *is orbitally stable in the norm of* X.

3. Main results

We formulate the main result in the paper.

Theorem 3.1. *Suppose* $a > 0$, $h_1 > 0$, $h_2 > 0$ *and* φ_c *is the solitary wave given in* (3) *and defined for* $c^2 \in \left[0, \min\left(\frac{h_1}{h_2}, 1\right)\right)$. *Then:*

(*i*) *if* $\frac{h_1}{h_2} \geq 1$, *then there exists a constant* $\sigma_1 \in [0, 1)$ *defined in* (6) *such that* $d''(c) < 0$ *whenever* $c^2 \in (0, \sigma_1)$ *and* $d''(c) > 0$ *whenever* $c^2 \in (\sigma_1, 1)$ *The solitary wave* $\vec{\varphi}_c$ *with velocity* $c^2 \in (\sigma_1, 1)$ *is orbitally stable. For* $h_1 = h_2$ *we have* $\sigma_1 = \frac{1}{5}$.

(*ii*) *If* $\frac{h_1}{h_2} < 1$, *then there exists a constant* $h_* \approx 0.3499468$ *defined in* (7) *such that for* $\frac{h_1}{h_2} < h_*$ *and* $c^2 \in \left(0, \frac{h_1}{h_2}\right)$ *the inequality* $d''(c) < 0$ *is true.*
If $1 > \frac{h_1}{h_2} > h_*$, *then there exist constants* σ_2 *and* σ_3, $0 < \sigma_2 < \sigma_3 < \frac{h_1}{h_2}$ *defined in* (9) *such that* $d''(c) < 0$ *whenever* $c^2 \in (0, \sigma_2) \cup \left(\sigma_3, \frac{h_1}{h_2}\right)$, *and* $d''(c) > 0$ *whenever* $c^2 \in (\sigma_2, \sigma_3)$. *The solitary wave* $\vec{\varphi}_c$ *with velocity* $c^2 \in (\sigma_2, \sigma_3)$ *is orbitally stable.*

Proof. From Corollary 1 in [3] we find the following formula for $d(c)$

$$d(c) = \frac{6\sqrt{h_2}}{5a^2}(c^2 - 1)^2 \sqrt{(c^2 - 1)\left(c^2 - \frac{h_1}{h_2}\right)}.$$

For $d''(c)$ we obtain the expression

$$d''(c) = \frac{6\sqrt{h_2}}{5a^2} \left\{ (c^2 - 1)\left(c^2 - \frac{h_1}{h_2}\right)^{-3} \right\}^{\frac{1}{2}} R(c),$$

$$R(c) = 30c^6 - 3\left(5 + 17\frac{h_1}{h_2}\right)c^4 + 2\frac{h_1}{h_2}\left(11 + 10\frac{h_1}{h_2}\right)c^2 - \frac{h_1}{h_2}\left(1 + 5\frac{h_1}{h_2}\right).$$

We change the variable $c^2 = z$ in $R(c)$ and get the third order polynomial $R_1(z)$

$$R_1(z) = 30z^3 - 3\left(5 + 17\frac{h_1}{h_2}\right)z^2 + 2\frac{h_1}{h_2}\left(11 + 10\frac{h_1}{h_2}\right)z - \frac{h_1}{h_2}\left(1 + 5\frac{h_1}{h_2}\right).$$

The sign of $d''(c)$ coincides with the sign of the polynomial $R_1(z)$. Therefore the sign of $R_1(z)$ on several subintervals of the real line is of high importance for our study. We apply the Budan-Fourier theorem (see p.246 in [5]): the number of the roots of $R_1(z) = 0$ in the interval (α, β) is equal to $W(\alpha) - W(\beta)$, or smaller by an even nonnegative number. Here $W(\gamma)$ denotes the number of sign changes in the sequence $R_1(\gamma)$, $R_1'(\gamma)$, $R_1''(\gamma)$, $R_1'''(\gamma)$.

(i) Let $\dfrac{h_1}{h_2} > 1$. Then solitary waves exist for $c^2 < 1$, i.e. for $z < 1$. We use the Budan-Fourier theorem and by direct evaluations conclude that $W(0) = 3$ and $W(1) = 2$. Hence there exists one zero σ_1 of the polynomial $R_1(z) = 0$ in $[0, 1)$, i.e.

$$R_1(\sigma_1) = 0, \qquad \sigma_1 \in (0, 1). \tag{6}$$

Since $R_1(0) < 0$, we conclude that $R_1(z) < 0$ for $0 < z < \sigma_1$ and $R_1(z) > 0$ for $\sigma_1 < z < 1$, which concludes the proof of the statement (i).

For $\dfrac{h_1}{h_2} = 1$ from

$$R_1(z) = 5z^3 - 11z^2 + 7z - 1 = (5z - 1)(z - 1)^2,$$

we conclude that statement (i) holds with $\sigma_1 = \dfrac{1}{5}$.

(ii) If $\dfrac{h_1}{h_2} < 1$, then solitary waves exist for $c^2 < \dfrac{h_1}{h_2}$, i.e. for $z < \dfrac{h_1}{h_2}$. We have $W(0) = 3$, $W\left(\dfrac{h_1}{h_2}\right) = 1$ and the number of roots of $R_1(z) = 0$ in $\left(0, \dfrac{h_1}{h_2}\right)$ is zero or two depending on $\dfrac{h_1}{h_2}$. To find the exact number of roots of $R_1(z)$ we evaluate the discriminant $\widehat{\mathfrak{D}}\left(\dfrac{h_1}{h_2}\right)$ of $R_1(z)$ and get

$$\widehat{\mathfrak{D}}\left(\frac{h_1}{h_2}\right) = 60\frac{h_1}{h_2}\left(\frac{h_1}{h_2} - 1\right)^2\left(1340\left(\frac{h_1}{h_2}\right)^3 - 89\left(\frac{h_1}{h_2}\right)^2 + 510\left(\frac{h_1}{h_2}\right) - 225\right).$$

The sign of $\widehat{\mathfrak{D}}\left(\frac{h_1}{h_2}\right)$ depends on the sign of the polynomial $\widehat{\mathfrak{D}}_1\left(\frac{h_1}{h_2}\right)$ given by

$$\widehat{\mathfrak{D}}_1\left(\frac{h_1}{h_2}\right) = 1340\left(\frac{h_1}{h_2}\right)^3 - 89\left(\frac{h_1}{h_2}\right)^2 + 510\left(\frac{h_1}{h_2}\right) - 225.$$

Since $\widehat{\mathfrak{D}}_1\left(\frac{h_1}{h_2}\right)$ is a monotone function of $\frac{h_1}{h_2}$ and $\widehat{\mathfrak{D}}_1(-\infty) = -\infty$, $\widehat{\mathfrak{D}}_1(\infty) = \infty$, it follows that $\widehat{\mathfrak{D}}_1\left(\frac{h_1}{h_2}\right) = 0$ has one real root h_*,

$$\widehat{\mathfrak{D}}_1(h_*) = 0, \tag{7}$$

and $h_* \approx 0.3499468$. Moreover, $\widehat{\mathfrak{D}}_1\left(\frac{h_1}{h_2}\right) < 0$ whenever $\frac{h_1}{h_2} < h_*$ and $\widehat{\mathfrak{D}}_1\left(\frac{h_1}{h_2}\right) > 0$ whenever $1 > \frac{h_1}{h_2} > h_*$

For $\frac{h_1}{h_2} < 1$ we have the following numbers of sign changes for $R_1(z)$:

$$W(-\infty) = 3, \quad W(0) = 3, \quad W\left(\frac{h_1}{h_2}\right) = 1, \quad W(\infty) = 0. \tag{8}$$

Hence, from the Budan Fourier theorem the polynomial $R_1(z)$ has zero roots in $(-\infty, 0)$ and one root in $\left(\frac{h_1}{h_2}, \infty\right)$.

For $\frac{h_1}{h_2} < h_*$ the discriminant of R_1 is negative and according to theorems in [5], the equation $R_1(z) = 0$ has one real and two complex roots in $(-\infty, \infty)$. From (8) we obtain that $R_1(z) = 0$ has zero real roots in $\left(0, \frac{h_1}{h_2}\right)$. Since $R_1(0) < 0$, we conclude that for $\frac{h_1}{h_2} < h_*$ we have $R_1(z) < 0$ for all $z \in \left(0, \frac{h_1}{h_2}\right)$, i.e. $d''(c) < 0$ for $c^2 < \frac{h_1}{h_2}$.

In the other case, i.e. $\frac{h_1}{h_2} > h_*$, the discriminant of R_1 is positive. Therefore from [5] the equation $R_1(z) = 0$ has three real roots in $(-\infty, \infty)$. But $R_1(z)$ has one root in $\left(\frac{h_1}{h_2}, \infty\right)$ and no roots in $(-\infty, 0)$, thus $R_1(z)$ has two real roots in $\left[0, \frac{h_1}{h_2}\right)$, i.e. there exist σ_2 and σ_3 such that $0 < \sigma_2 < \sigma_3 < \frac{h_1}{h_2}$ and

$$R_1(\sigma_2) = 0, \quad R_1(\sigma_3) = 0. \tag{9}$$

Moreover, $R_1(z) < 0$ for $z < \sigma_2$ and $z > \sigma_3$; $R_1(z) > 0$ for $\sigma_2 < z < \sigma_3$. Thus $d''(c) < 0$ for $c^2 < \sigma_2$ and $c^2 > \sigma_3$; and $d''(c) > 0$ for $c^2 \in (\sigma_2, \sigma_3)$. Statement (*ii*) in Theorem 3.1 is proved, which completes the proof of Theorem 3.1. □

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